Generalisations of the 15-Puzzle
(Sliding Tokens on Graphs)

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A classical puzzle: the 15-Puzzle

1 2 3 4
5 6 7 8
9 10 11 12
13 14 15

1 2 3 12
9 11 1 10
6 4 14
15 8 7 5

1 2 3 4
5 6 7 8
9 10 11 12
13 14 15
A classical puzzle: the 15-Puzzle

Can you always solve it?
Sliding token puzzles

we can interpret the 15-puzzle as a problem involving moving tokens on a given graph:
What if we would play on a different graph?
And maybe more empty spaces and/or repeated tokens?
Sliding token puzzles

- for a given graph $G$ on $n$ vertices, define $\text{puz}(G)$ as the graph that has:
  - **nodes:** all possible placements of $n - 1$ different tokens on $G$
  - **adjacency:** sliding one token along an edge of $G$ to an empty vertex
Sliding token puzzles

- for a given graph $G$ on $n$ vertices, define $\text{puz}(G)$ as the graph that has:
  - nodes: all possible placements of $n - 1$ different tokens on $G$
  - adjacency: sliding one token along an edge of $G$ to an empty vertex

- and our standard decision problems become:
  - are two token configurations in one component of $\text{puz}(G)$?
  - is $\text{puz}(G)$ connected?
**Sliding token puzzles**

**Theorem** (Wilson, 1974)

- if $G$ is a 2-connected graph, then $\text{puz}(G)$ is connected, except if:
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- if $G$ is a 2-connected graph, then $\text{puz}(G)$ is connected, except if:
  - $G$ is a cycle on $n \geq 4$ vertices
    (then $\text{puz}(G)$ has $(n - 2)!$ components)
**Theorem**  (Wilson, 1974)

- if \( G \) is a 2-connected graph, then \( \text{puz}(G) \) is connected, except if:
  - \( G \) is a cycle on \( n \geq 4 \) vertices
    (then \( \text{puz}(G) \) has \((n - 2)!\) components)
  - \( G \) is bipartite different from a cycle
    (then \( \text{puz}(G) \) has 2 components)
Sliding token puzzles

**Theorem** (Wilson, 1974)

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  - \( G \) is a cycle on \( n \geq 4 \) vertices
    - (then \( \text{puz}(G) \) has \( (n-2)! \) components)
  - \( G \) is bipartite different from a cycle
    - (then \( \text{puz}(G) \) has 2 components)
  - \( G \) is the exceptional graph \( \Theta_0 \)
    - (\( \text{puz}(\Theta_0) \) has 6 components)
Why does Wilson only consider 2-connected graphs?
Why does Wilson only consider 2-connected graphs?

- since \( \text{puz}(G) \) is never connected if \( G \) has connectivity below 2:
**Generalised sliding token puzzles**

- what would happen if:
  - we have fewer than $n - 1$ tokens (i.e. more empty vertices)?
  - and/or not all tokens are the same?
Generalised sliding token puzzles

- what would happen if:
  - we have fewer than $n - 1$ tokens (i.e. more empty vertices)?
  - and/or not all tokens are the same?

- so suppose we have a set $(k_1, k_2, \ldots, k_p)$ of labelled tokens
  - meaning: $k_1$ tokens with label 1, $k_2$ tokens with label 2, etc.
  - tokens with the same label are indistinguishable
  - we can assume that $k_1 \geq k_2 \geq \cdots \geq k_p$
    and their sum is at most $n - 1$
Generalised sliding token puzzles

what would happen if:

- we have fewer than \( n - 1 \) tokens (i.e. more empty vertices)?
- and/or not all tokens are the same?

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  and their sum is at most \( n - 1 \)

the corresponding graph of all token configurations on \( G \) is denoted by \( \text{puz}(G; k_1, \ldots, k_p) \)
Generalised sliding token puzzles

**Theorem** (Brightwell, vdH & Trakultraipruk, 2013)

- $G$ a graph on $n$ vertices, $(k_1, k_2, \ldots, k_p)$ a token set, then $\text{puz}(G; k_1, \ldots, k_p)$ is connected, except if:
Generalised sliding token puzzles

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**Generalised sliding token puzzles**

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  - $G$ is a path and $p \geq 2$
Generalised sliding token puzzles

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  - $G$ is not connected
  - $G$ is a path and $p \geq 2$
  - $G$ is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
**Generalised sliding token puzzles**

**Theorem** (Brightwell, vdH & Trakultraipruk, 2013)

- **$G$** a graph on $n$ vertices, $(k_1, k_2, \ldots, k_p)$ a token set, then $\text{puz}(G; k_1, \ldots, k_p)$ is connected, except if:
  - $G$ is not connected
  - $G$ is a path and $p \geq 2$
  - $G$ is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
  - $G$ is a 2-connected, bipartite graph with token set $(1^{n-1})$
**Generalised sliding token puzzles**

**Theorem**  (Brightwell, vdH & Trakultraipruk, 2013)

- \( G \) a graph on \( n \) vertices, \((k_1, k_2, \ldots, k_p)\) a token set,
  then \( \text{puz}(G; k_1, \ldots, k_p) \) is connected, except if:
  - \( G \) is not connected
  - \( G \) is a path and \( p \geq 2 \)
  - \( G \) is a cycle, and \( p \geq 3 \), or \( p = 2 \) and \( k_2 \geq 2 \)
  - \( G \) is a 2-connected, bipartite graph with token set \((1^{n-1})\)
  - \( G \) is the exceptional graph \( \Theta_0 \) with token set \((2, 2, 2), (2, 2, 1, 1), (2, 1, 1, 1, 1) \) or \((1, 1, 1, 1, 1, 1)\)
Generalised sliding token puzzles

**Theorem**  (Brightwell, vdH & Trakultraipruk, 2013)

- $G$ a graph on $n$ vertices, $(k_1, k_2, \ldots, k_p)$ a token set, then $\text{puz}(G; k_1, \ldots, k_p)$ is connected, except if:
  - $G$ is not connected
  - $G$ is a path and $p \geq 2$
  - $G$ is a cycle, and $p \geq 3$, or $p = 2$ and $k_2 \geq 2$
  - $G$ is a 2-connected, bipartite graph with token set $(1^{(n-1)})$
  - $G$ is the exceptional graph $\Theta_0$ with some “bad” token sets
  - $G$ has connectivity $1$, $p \geq 2$ and there is a “separating path preventing tokens from moving between blocks”
Generalised sliding token puzzles

“separating paths” in graphs of connectivity one:

bad:

\[
\begin{array}{ccc}
2 & & 1 \\
1 & & 1 \\
1 & & 1 \\
\end{array}
\]

good:

\[
\begin{array}{ccc}
2 & & 1 \\
1 & & 1 \\
1 & & 1 \\
\end{array}
\]
Generalised sliding token puzzles

“separating paths” in graphs of connectivity one:

- **bad**
- **good**
The Structure of $T(\theta_0; (2, 1, 1, 1, 1))$

The following are the three groups of standard token configurations in the labelled token graph $T(\theta_0; (2, 1, 1, 1, 1))$.

Figure A.8: Part 1 of Group $B_1$ in $T(\theta_0; (2, 1, 1, 1, 1))$
Figure A.9: Part 2 of Group $B_1$ in $T(\theta_0; (2, 1, 1, 1, 1))$
Figure A.10: Part 1 of Group $B_2$ in $T(\theta_0; (2, 1, 1, 1))$
Figure A.11: Part 2 of Group $B_2$ in $T(\theta_0; (2, 1, 1, 1))$
Figure A.12: Part 1 of Group $B_3$ in $T(\theta_0; (2, 1, 1, 1))$
Figure A.13: Part 2 of Group $B_3$ in $T(\theta_0; (2, 1, 1, 1, 1))$
Generalised sliding token puzzles

we can also characterise:

- given a graph $G$, token set $(k_1, \ldots, k_p)$, and two token configurations on $G$, are the two configurations in the same component of $\text{puz}(G; k_1, \ldots, k_p)$?
configuration $\alpha$, let $\alpha_i$ be a token configuration obtained from $\alpha$ by moving some tokens (if necessary) to make all the vertices on $P_i$ unoccupied.

Let $G$ be a connected graph with connectivity 1, $n(G) - (k_1 + k_2 + \cdots + k_p) = 1$, and $B$ a block in $G$. Then $B$ contains at least one cut-vertex of $G$. Let $v_B$ be one of these cut-vertices. Given a token configuration $\alpha$, let $\alpha_{v_B}$ be a token configuration obtained from $\alpha$ by moving some tokens (if necessary) to make $v_B$ unoccupied.

We denote the multiset of all the tokens used in a token configuration $\alpha$ by $\tau(\alpha)$. For example, if $\alpha$ is any of the token configurations in Figure 2.4, then $\tau(\alpha) = \{1, 1, 2, 2, 3, 3\} = (2, 2, 2)$.

**Theorem 2.3**

Let $G$ be a connected graph with $n(G) \geq 3$, $k_1 \geq k_2 \geq \cdots \geq k_p$ positive integers for some integer $p \geq 2$, and $k_1 + k_2 + \cdots + k_p \leq n(G) - 1$. Then two token configurations $\alpha$ and $\beta$ are in the same component of $T(G; (k_1, k_2, \ldots, k_p))$ if and only if at least one of the following conditions holds:

1. $T(G; (k_1, k_2, \ldots, k_p))$ is connected;
2. $G$ is a path, and the orders of tokens on $G$ of $\alpha$ and $\beta$ are the same;
3. $G$ is a cycle, and the cyclic orders of tokens on $G$ of $\alpha$ and $\beta$ are the same;
4. $G$ is the graph $\theta_0$, and
   
   (a) $(k_1, k_2, \ldots, k_p) = (2, 2, 2)$ or $(2, 2, 1, 1)$, and for any $(1,1)$-standard token configurations $\alpha'$ and $\beta'$ which can be reached from $\alpha$ and $\beta$, respectively, we have that $\alpha'$ and $\beta'$ are in the same group from the following two groups:
Group $a_1$: $(1,1)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,2,s,t)$, where $s,t \in \{3,4\}$. I.e., token configurations which have the following forms:

\[
\begin{array}{c}
\begin{array}{cc}
1 & 1 \\
2 & t
\end{array}
\quad
\begin{array}{cc}
1 & 1 \\
2 & s
\end{array}
\quad
\begin{array}{cc}
1 & 1 \\
2 & s
\end{array}
\quad
\begin{array}{cc}
1 & 1 \\
2 & t
\end{array}
\end{array}
\]

Group $a_2$: $(1,1)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,s,2,t)$, where $s,t \in \{3,4\}$,

(b) $(k_1,k_2,\ldots,k_p) = (2,1,1,1,1)$, and for any $(1,1)$-standard token configurations $\alpha'$ and $\beta'$ which can be reached from $\alpha$ and $\beta$, respectively, we have $\alpha'$ and $\beta'$ are in the same group from the following three groups:

Group $b_1$: $(1,1)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,3,4,5)$ or $(2,5,4,3)$;

Group $b_2$: $(1,1)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,4,3,5)$ or $(2,5,3,4)$;

Group $b_3$: $(1,1)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,3,5,4)$ or $(2,4,5,3)$;

(c) $(k_1,k_2,\ldots,k_p) = (1,1,1,1,1,1)$, and for any $(1,6)$-standard token configurations $\alpha'$ and $\beta'$ which can be reached from $\alpha$ and $\beta$, respectively, we have $\alpha'$ and $\beta'$ are in the same group from the following six groups:

Group $c_1$: $(1,6)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,3,4,5)$;

Group $c_2$: $(1,6)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,5,4,3)$;

Group $c_3$: $(1,6)$-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,4,3,5)$;
Group $c_4$: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,5,3,4)$;

Group $c_5$: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,3,5,4)$;

Group $c_6$: (1,6)-standard token configurations of which the cyclic order of tokens on the lower 5-cycle is $(2,4,5,3)$.

5. $G$ is a 2-connected bipartite graph other than a cycle, there are $n(G) - 1$ different tokens, and one of the following holds:

(a) $\alpha$ and $\beta$ have their unoccupied vertices at even distance in $G$, and $\alpha\beta^{-1}$ is an even permutation;

(b) $\alpha$ and $\beta$ have their unoccupied vertices at odd distance in $G$, and $\alpha\beta^{-1}$ is an odd permutation.

6. $G$ is a connected graph with connectivity 1 other than a path, $n(G) - (k_1 + k_2 + \cdots + k_p) = l \geq 2$, $P_1, P_2, \ldots, P_m$ are all the separating paths of size $l$ in $G$, and $\tau(\alpha_i|_{G_{i,1}}) = \tau(\beta_i|_{G_{i,1}})$ and $\tau(\alpha_i|_{G_{i,2}}) = \tau(\beta_i|_{G_{i,2}})$ for all $i = 1, 2, \ldots, m$.

7. $G$ is a connected graph with connectivity 1 other than a path, $n(G) - (k_1 + k_2 + \cdots + k_p) = 1$, for each block $B$ in $G$, $\tau(\alpha_{v_B}|_B) = \tau(\beta_{v_B}|_B)$, and at least one of the following conditions holds:

(a) $T(B; \tau(\alpha_{v_B}|_B))$ is connected;

(b) $B$ is a cycle, and the cyclic orders of tokens of $\alpha_{v_B}|_B$ and $\beta_{v_B}|_B$ are the same;

(c) $B$ is the graph $\theta_0$, and $\alpha_{v_B}|_B$ and $\beta_{v_B}|_B$ satisfy 4(a), 4(b), or 4(c) above;

(d) $B$ is a 2-connected bipartite graph other than a cycle, there are $n(B) - 1$ different tokens in $\alpha_{v_B}|_B$ and $\beta_{v_B}|_B$, and $\alpha_{v_B}|_B \cdot (\beta_{v_B}|_B)^{-1}$ is an even permutation.
Generalised sliding token puzzles

- we can also characterise:
  - given a graph $G$, token set $(k_1, \ldots, k_p)$, and two token configurations on $G$,
  - are the two configurations in the same component of $\text{puz}(G; k_1, \ldots, k_p)$?

- so recognising connectivity properties of $\text{puz}(G; k_1, \ldots, k_p)$ is easy

- can we say something about the number of steps we would need?
The length of sliding token paths

**Shortest-A-to-B-Token-Moves**

*Input:* a graph $G$, a token set $(k_1, \ldots, k_p)$, two token configurations $A$ and $B$ on $G$, and a positive integer $N$

*Question:* can we go from $A$ to $B$ in at most $N$ steps?
The length of sliding token paths

**Theorem** (Goldreich, 1984-2011)

- restricted to the case that there are \( n - 1 \) different tokens,

  \texttt{SHORTEST-A-TO-B-TOKEN-MOVES} is \texttt{NP-complete}
The length of sliding token paths

**Theorem** (Goldreich, 1984-2011)

- restricted to the case that there are \( n - 1 \) different tokens,

  \( \text{SHORTEST-A-TO-B-TOKEN-MOVES} \) is **NP-complete**

**Theorem** (vdH & Trakultraipruk, 2013; probably others earlier)

- restricted to the case that all tokens are the same,

  \( \text{SHORTEST-A-TO-B-TOKEN-MOVES} \) is in **P**
The length of sliding token paths

Theorem (Goldreich, 1984-2011)
- restricted to the case that there are \( n - 1 \) different tokens,
  \( \text{SHORTEST-A-TO-B-TOKEN-MOVES} \) is \( \text{NP-complete} \)

Theorem (vdH & Trakultraipruk, 2013; probably others earlier)
- restricted to the case that all tokens are the same,
  \( \text{SHORTEST-A-TO-B-TOKEN-MOVES} \) is in \( \text{P} \)

Theorem (vdH & Trakultraipruk, 2013)
- restricted to the case that there is just one special token
  and all others are the same:
  \( \text{SHORTEST-A-TO-B-TOKEN-MOVES} \) is already \( \text{NP-complete} \)
Robot motion

- the proof of that last result uses ideas of the proof of

**Theorem** (Papadimitriou, Raghavan, Sudan & Tamaki, 1994)

- **Shortest-Robot-Motion-with-One-Robot** is NP-complete
Robot motion

- the proof of that last result uses ideas of the proof of

**Theorem** (Papadimitriou, Raghavan, Sudan & Tamaki, 1994)

- **SHORTEST-ROBOT-MOTION-WITH-ONE-ROBOT** is NP-complete

- **Robot Motion** problems on graphs are *sliding token* problems,
  - with some *special tokens* (the robots)
    - that have to *end in specified positions*
  - all *other tokens* are just *obstacles*
    - and it is *not important where those are at the end*