Free form lens design for general radiant fields

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joint work with Ahmad Sabra

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Background

- Snell law: σ a surface separating two materials with refractive indices n_1, n_2; $\kappa = n_2/n_1$; x=incident direction at a point P on σ, ν unit normal at P, m=refracted or transmitted direction, then

$$x - \kappa\, m = \lambda\, \nu$$

where $\lambda = \Phi(x \cdot \nu, \kappa)$. If $\kappa < 1$ and total reflection occurs; so we need $x \cdot \nu \geq \sqrt{1 - \kappa^2}$.

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- a surface is optically inactive if each incident ray is refracted into the same direction. Examples: sphere around a point source and plane perpendicular to a collimated beam.
- A lens is an homogeneous material, tipically glass, sandwiched between two surfaces.

- We introduced and solved the following far field problem (C.E.G. and Q. Huang, ARMA 2009): given two domains in the sphere Ω, Ω^*, an integrable function $I(x) \geq 0$ in Ω, a Radon measure η in Ω^*, with $\int_\Omega I = \eta(\Omega^*)$, one point source surrounded by medium n_1, then there exists a surface σ separating n_1 and n_2 refracting all rays with directions in Ω into Ω^* preserving energy:

$$\int_{\mathcal{T}_\sigma(E)} I(x)\, dx = \eta(E) \qquad \forall E \subset \Omega^*$$

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- cutting a spherical region around the source (optically inactive) we obtain a lens
- No control on the region to cut; disadvantages: may lead to a bulky lens. Flexibility with the region cut is useful for imaging.

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- QUESTION: can we do the same for an extended source and when the rays emanate with an arbitrary pattern of directions?

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- a function $I(x)$ in Ω and a Radon measure η in Ω^* with $\int_\Omega I = \eta(\Omega^*)$

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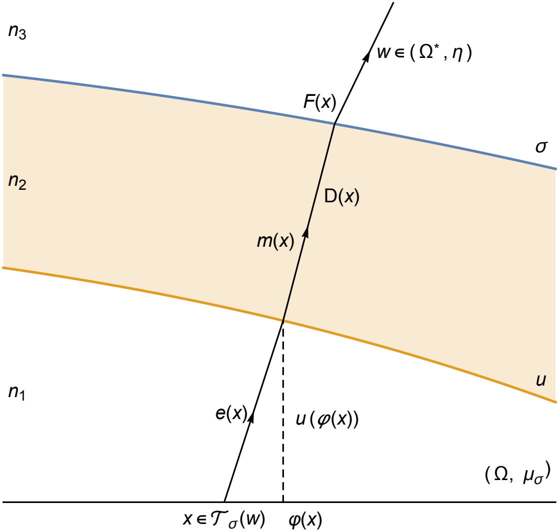
- to find a surface σ_2, the top face of the lens, so that all rays emanating from Ω are refracted by the lens sandwiched by σ_1 and σ_2 into rays with directions in Ω^*, and

$$\int_{\mathcal{T}(E)} I(x)\,dx = \eta(E) \qquad \forall E \subset \Omega^*$$

where $\mathcal{T}(E) = \{x \in \Omega : \text{the lens refracts the ray from } x \text{ into } E\}$.

Material configuration:

below σ_1 the material has refractive index n_1,
between σ_1 and σ_2 the material has refractive index n_2,
above σ_2 the material has refractive index n_3.
$n_2 > n_1, n_3$, let $\kappa_1 = n_2/n_1, \kappa_2 = n_3/n_2$



n_3
$w \in (\Omega^*, \eta)$
$F(x)$
σ
n_2
$D(x)$
$m(x)$
u
n_1
$e(x)$
$u(\varphi(x))$
(Ω, μ_σ)
$x \in \mathcal{T}_\sigma(w)$
$\varphi(x)$

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- From the Snell law, the ray is refracted at $P(x)$ into the direction $m_1(x)$ with

$$e(x) - \kappa_1 \, m_1(x) = \lambda_1 \, \nu_1(x);$$

and since $\kappa_1 > 1$, $\lambda_1 = e(x) \cdot \nu_1 - \kappa_1 \sqrt{1 - \dfrac{1}{\kappa_1^2} (1 - (e(x) \cdot \nu_1)^2)} < 0$.

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$$\lambda_1 \nu_1 \cdot w \leq e(x) \cdot w - \kappa_1 \kappa_2$$
- For example, if $e(x) = w = (0, 0, 1)$, $\kappa_1 \kappa_2 \leq 1$, this condition holds.

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- Since the tangent vectors to f are f_{x_1} and f_{x_2}, we get the system

$$(e(x) - \lambda_1 \nu_1 - (n_2/n_1)(n_3/n_2)\, w) \cdot f_{x_1} = 0$$
$$(e(x) - \lambda_1 \nu_1 - (n_2/n_1)(n_3/n_2)\, w) \cdot f_{x_2} = 0$$

- The only unknown in this system is $d(x)$.

By calculation it can be shown that d satisfies the system

$$[(\kappa_1 - \kappa_2\, w \cdot (e - \lambda_1\, v_1))\, d]_{x_i} = -(e - \kappa_1\, \kappa_2\, w) \cdot (\varphi, u(\varphi))_{x_i}, \quad i = 1, 2$$

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- the vector w is constant

Lens refracting into a fixed direction w

Theorem

We are given a C^2 surface σ_1 given by $(x, u(x))$, a C^1 unit field $e(x) = (e'(x), e_3(x))$, and a unit direction w. Then a lens (σ_1, σ_2), $\sigma_2 \in C^2$, refracting rays with direction $e(x)$ into w exists if and only if

1. $\lambda_1 v_1 \cdot w \leq e(x) \cdot w - \kappa_1 \kappa_2$ *(i.e., $m_1 \cdot w \geq \kappa_2$) and*
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2. $\operatorname{curl} e'(x) = 0$.

Moreover, $\nabla h(x) = e'(x)$ for some h, and σ_2 is parametrized by

$$f(x, C, w) = (\varphi(x), u(\varphi(x))) + d(x, C, w)\, m_1(x)$$

where $m_1(x) = \dfrac{1}{\kappa_1}(e(x) - \lambda_1 \nu_1)$ *and*

$$d(x, C, w) = \frac{C - h(x) + e(x) \cdot (x, 0) - (e(x) - \kappa_1 \kappa_2\, w) \cdot (\varphi(x), u(\varphi(x)))}{\kappa_1 - \kappa_2\, w \cdot (e(x) - \lambda_1 \nu_1(x))}$$

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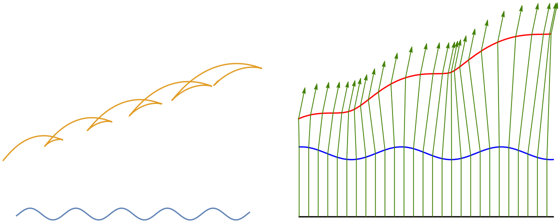
- we then obtain a one parameter family of surfaces $f(x, C, w)$ as the top surface of the desired lens
- $d(x, C, w) > 0$ for $C \geq C^*(\kappa_1, \kappa_2, \Omega, h)$.
- since σ_2 is given parametrically it might have singular points and self intersections. Therefore for some values of C it might not be physically realizable.



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- For example, if u is concave and $e' = \nabla h$ with h convex, then $f(x, C, w)$ has no singular points.
- more precisely, assuming for simplicity we are in the collimated case, i.e., $e(x) = (0, 0, 1)$, we have the following two theorems.

Theorem

Suppose $e(x) = e_3 = (0, 0, 1)$, $w = (w', w_3)$. *If the Lipschitz constants of* u *and* Du, *and* $|w'|$ *are all sufficiently small, then there is an interval* $[-\alpha, \alpha]$ *depending only on these values and* κ_1 *and* κ_2 *such that the parametric surface* $f(x, C, w)$ *has no self-intersections for all* $C \in [-\alpha, \alpha]$.

Theorem

Let $C > \max_\Omega\{(e_3 - \kappa_1\kappa_2 w) \cdot (x, u(x))\}$ *and let* $\mu(y)$ *be the maximum eigenvalue of* $D^2u(y)$. *If* $\mu(y) \leq 0$ *or if* $\mu(y) > 0$ *and*

$$C < \frac{\kappa_1^2(1-\kappa_2)\sqrt{1+|Du(y)|^2}}{\mu(y)\sqrt{\kappa_1^2-1}} + \min_\Omega\{(e_3 - \kappa_1\kappa_2 w) \cdot (c, u(x))\},$$

then the point y *is a regular point for the surface* $f(x, C, w)$.

As a conclusion, when the Lipschitz constants of u, Du and $|w'|$ are all sufficiently small, there is an interval

$$J = [\tau_1, \tau_2]$$

such that the surface parametrized by $f(x, C, w)$ with $C \in J$ has normal for each $x \in \Omega$ and has no self intersections.

As a conclusion, when the Lipschitz constants of u, Du and $|w'|$ are all sufficiently small, there is an interval

$$J = [\tau_1, \tau_2]$$

such that the surface parametrized by $f(x, C, w)$ with $C \in J$ has normal for each $x \in \Omega$ and has no self intersections.
This is consequence of the following Lipschitz estimate of the distance function $d(x, C, w)$:

$$\begin{aligned} |d(x,C,w) - d(y,C,w)| \leq (|C| + M_1) \left(L_e + L_{Du}\, L_\varphi\right) |x-y| \\ + \|e'\|_\infty \, |x-y| \\ + \max |e'(x) - \kappa_1\kappa_2\, w'|\, L_\varphi\, |x-y| \\ + L_u\, L_\varphi\, |x-y| \end{aligned}$$

modulo a multiplicative constant $C(\kappa_1, \kappa_2)$ and with M_1 depending only on Ω, $\kappa_1, \kappa_2, \|e\|_\infty$, and $\|u\|_\infty$.

- We seek a surface solution σ parametrized by

$$F(x) = (\varphi(x), u(\varphi(x))) + d(x)\, m(x)$$

where u is given, $m(x)$ is determined by the normal to u at $(\varphi(x), u(\varphi(x)))$ and the function $d(x)$ is the unknown.

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- The function $d(x)$ is so that at each point $x_0 \in \Omega$ there are $C \in J$ and $w \in \Omega^*$ such that $d(x) \leq d(x, C, w)$ for all $x \in \Omega$ with equality at $x = x_0$.

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- Therefore the normal mapping of σ is given by

$$\mathcal{N}_\sigma(x_0) = \{w \in \Omega^* : \exists\, C \in J \text{ such that } d(x, C, w) \text{ supports } d \text{ at } x = x_0\}$$

and the tracing mapping

$$\mathcal{T}_\sigma(w) = \{x \in \Omega : w \in \mathcal{N}_\sigma(x)\}$$

If σ is defined as above, then we say that the lens (u, σ) refracts Ω into Ω^*. It can be proved that

1. d and F are both uniformly Lipschitz continuous in Ω
2. the surface σ has no self intersections
3. σ has normal at $x \in \Omega \setminus N$ with $|N| = 0$
4. $\mathcal{N}_\sigma(x)$ is singled valued for $x \in \Omega \setminus N$

If the intensity $I(x) \in L^1(\Omega)$, then

$$\mu_\sigma(E) = \int_{\mathcal{T}_\sigma(E)} I(x)\,dx$$

is a Borel measure in Ω^*.

Given η Radon measure in Ω^*, the lens problem is to find a surface σ such that the lens (u, σ) refracts Ω into Ω^* and $\mu_\sigma = \eta$.

Theorem

If $w_1, \cdots, w_N$ are distinct points in Ω^*, $g_1, \cdots, g_N > 0$ and $\eta = \sum g_i \delta_{w_i}$ with the conservation of energy $\int_\Omega I(x)\, dx = \sum g_i$, then there are constants $C_1, \cdots, C_N \in J$ such that the surface σ parametrized by $F(x) = (\varphi(x), u(\varphi(x))) + d(x)\, m(x)$ with

$$d(x) = \min_{1 \le i \le N} d(x, C_i, w_i)$$

is such that the lens (u, σ) refracts Ω into Ω^* and

$$\int_{\mathcal{T}_\sigma(w_i)} I(x)\, dx = g_i, \quad 1 \le i \le N.$$

Theorem

If η is a Radon measure in Ω^* with $\int_\Omega I(x)\, dx = \eta(\Omega^*)$, then there is a lens (u, σ) refracting Ω into Ω^* with $\mu_\sigma = \eta$.

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In the collimated case $\rho = u$, $\mathcal{A}$ depends only u and its der. up to order two and $\mathcal{B}$ depends only on der. of u up to order three but not on u.

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Using the previous construction we solve the following:

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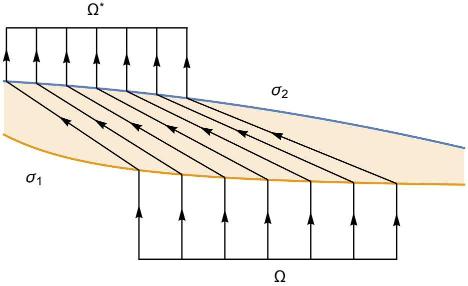

Notice that

- The rays will strike σ_1 at the point $(x, u(x))$, and are then refracted with direction m_1 into the point $f(x) = (x, u(x)) + d(x)\, m_1 \in \sigma_2$.

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- Each ray leaves $f(x)$ with direction e_3 and strikes into the point (Tx, a).
- Then $Tx = (f_1(x), f_2(x))$.

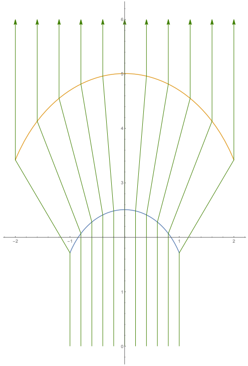

Figure: $Tx = 2x, a = 6, n_1 = n_3 = 1, n_2 = 1.52$

PDE satisfied by $u(x)$

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PDE satisfied by $u(x)$

The explicit formula obtained for distance function $d(x)$ allows to write the surface σ_2 in terms of u and its gradient.
After calculation with the formula obtained for f, we get that u satisfies the following 1st order system:

$$\frac{(1-\kappa_1\kappa_2)u(x)+C}{(\kappa_1^2-\kappa_1\kappa_2)\sqrt{\kappa_1^2+(\kappa_1^2-1)|\nabla u(x)|^2}+\kappa_1^2(1-\kappa_1\kappa_2)}\nabla u(x) = \frac{Tx-x}{\kappa_1^2-1}$$

recall $\kappa_1 = \dfrac{n_2}{n_1}$ and $\kappa_2 = \dfrac{n_3}{n_2}$

Case $n_1 = n_3$

The corresponding PDE is

$$\frac{\nabla u(x)}{\sqrt{\kappa_1^2 + (\kappa_1^2 - 1)|\nabla u(x)|^2}} = \frac{Tx - x}{C}$$

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- Taking absolute values in the pde, squaring both sides, and solving yields $|\nabla u(x)| = \dfrac{\kappa_1 |Tx - x|}{\sqrt{C^2 - (\kappa_1^2 - 1)|Tx - x|^2}}$
- We replace $|\nabla u(x)|$ in the PDE obtaining

$$\nabla u(x) = -\frac{\kappa_1(Tx - x)}{\sqrt{C^2 - (\kappa_1^2 - 1)|Tx - x|^2}} := F(x) = (F_1(x), F_2(x)) \tag{1}$$

If $u \in C^2$ solves the PDE then $\partial_{x_2} F_1 = \partial_{x_1} F_2$

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If $u \in C^2$ solves the PDE then $\partial_{x_2} F_1 = \partial_{x_1} F_2$ and therefore

$$u(x) = u(x_0) + \int_\gamma F(x) \cdot dr \qquad \gamma \text{ joins } x_0 \text{ and } x$$

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Theorem

Letting $Sx = (S_1 x, S_2 x) = Tx - x$, *we have that* (1) *has a solution if and only if*

$$C^2\left(\frac{\partial S_2}{\partial x_1} - \frac{\partial S_1}{\partial x_2}\right) + (\kappa_1^2 - 1)\left(S_1 S_2\left(\frac{\partial S_1}{\partial x_1} - \frac{\partial S_2}{\partial x_2}\right) + S_2^2\frac{\partial S_1}{\partial x_2} - S_1^2\frac{\partial S_2}{\partial x_1}\right) = 0$$

Once u is found, we obtain the top face of the lens from the construction in the first part.

Example: $Tx = (1+\alpha)x$

$$\nabla u(x) = -\frac{\kappa_1 \alpha x}{\sqrt{C^2 - (\kappa_1^2 - 1)\alpha^2 |x|^2}}$$

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Note that the graph of u is then contained in the ellipsoid of equation

$$(z - A)^2 + \kappa_1^2 |x|^2 = \left(\frac{C\kappa_1}{\alpha(\kappa_1^2 - 1)} \right)^2 .$$

Case $n_3 < n_1$

The pde in this case is more complicated because $\kappa_1 \, \kappa_2 \neq 1$

$$\frac{(1-\kappa_1\kappa_2)u(x)+C}{(\kappa_1^2-\kappa_1\kappa_2)\sqrt{\kappa_1^2+(\kappa_1^2-1)|\nabla u(x)|^2}+\kappa_1^2(1-\kappa_1\kappa_2)}\nabla u(x)=\frac{Tx-x}{\kappa_1^2-1}$$

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Set

- $v(x) = \left(u(x) + \dfrac{C}{1-\kappa_1\kappa_2}\right)(\kappa_1-\kappa_2)\sqrt{\kappa_1^2-1} < 0$

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Set

- $v(x)=\left(u(x)+\frac{C}{1-\kappa_1\kappa_2}\right)(\kappa_1-\kappa_2)\sqrt{\kappa_1^2-1}<0$
- $Sx=\frac{\kappa_1(\kappa_1-\kappa_2)^2(Tx-x)}{1-\kappa_1\kappa_2}$

Case $n_3 < n_1$ continued

So the equation can be rewritten as

$$\frac{v(x)\nabla v(x)}{\sqrt{\kappa_1^2(\kappa_1 - \kappa_2)^2 + |\nabla v(x)|^2} + \kappa_1(1 - \kappa_1\kappa_2)} = Sx.$$

Case $n_3 < n_1$ continued

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- $|Sx| < |v(x)|$

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- $|Sx| < |v(x)|$
- We let $t(x) = \sqrt{\kappa_1^2(\kappa_1-\kappa_2)^2+|\nabla v(x)|^2}$, $|\nabla v(x)|^2 = t^2(x) - \kappa_1^2(\kappa_1-\kappa_2)^2$.

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- Take absolute values in (2), square, solve for $t(x)$ obtaining a function of v and S, and replace back in (2) to obtain

$$\nabla v(x) = F(x, v(x)) = (F_1(x, v(x)), F_2(x, v(x))) \quad (3)$$

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- $F(x, v(x)) = G\left(\frac{Sx}{v(x)}\right)$

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- $G(x) = \left(\dfrac{\kappa_1(1-\kappa_1\kappa_2)|x|^2 + \kappa_1\sqrt{(\kappa_1-\kappa_2)^2 - (1-\kappa_2^2)(\kappa_1^2-1)|x|^2}}{1-|x|^2} + \kappa_1(1-\kappa_1\kappa_2)\right)x$

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- if (3) has a C^2 solution, then

$$\partial_{x_2}F_1(x, v_1(x)) + \partial_z F(x, v(x))F_2(x, v(x)) = \partial_{x_1}F_2(x, v(x)) + \partial_z F_2(x, v(x))F_1(x, v(x)).$$

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- Conversely, from existence results for solutions of 1st order systems of pdes: if

  $$\partial_{x_2}F_1(x, z) + \partial_z F_1(x, z))F_2(x, z) = \partial_{x_1}F_2(x, z) + \partial_z F_2(x, z)F_1(x, z).$$

  on an open set O then for every $(x_0, z_0) \in O$, there exists a unique solution v to (3) satisfying $v(x_0) = z_0$ defined on a neighborhood of x_0.

Case $n_3 < n_1$ continued

By calculation using the form of F_1 and F_2, it can be shown:

Theorem

The partial differential equation (3) *has a local solution if*

$$\operatorname{curl} S = 0$$
$$S \times \nabla |S|^2 = 0.$$

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- these conditions mean $\exists w$ such that $S = (w_{x_1}, w_{x_2})$ and

$$w_{x_1 x_2} \left((w_{x_1})^2 - (w_{x_2})^2 \right) + w_{x_1} w_{x_2} \left(w_{x_2 x_2} - w_{x_1 x_1} \right) = 0.$$

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- This equation can be solved for a large class of initial data, for example, given two plane analytic curves $\gamma(s)$ and $\Gamma(s)$, satisfying a non characteristic condition, and a function $z(s)$ $\exists!$ w solving the equation with $w(\gamma) = z$, and $Dw(\gamma) = \Gamma$. So we can construct, S satisfying the conditions in the theorem and mapping γ into Γ.

- This gives local existence of lenses.
- By reversibility of optical paths, if $\kappa_1\kappa_2 > 1$, then the problem has a local solution when T^{-1} verifies the condition in the above theorem.
- Similar results also hold for systems of two reflectors (simpler).

References

C. E. G., *Aspherical lens design*, JOSA A **30** (2013), no. 9, 1719–1726.

C. E. G. and A. Sabra, *Design of pairs of reflectors*, JOSA A **31** (2014), no. 4, 891–899.

C. E. G. and A. Sabra, *Aspherical lens design and imaging*, SIAM Journal on Imaging Sciences, Vol. 9, No.1, pp. 386-411, 2016.

C. E. G. and A. Sabra, *Freeform Lens Design for Scattering Data With General Radiant Fields*, preprint (2017).

C. E. G. and Qingbo Huang, *The refractor problem in reshaping light beams*, Arch. Rational Mech. Anal. **193** (2009), no. 2, 423–443.

, *The near field refractor*, Annales de l'Institut Henri Poincaré (C) Analyse Non Linéaire **31** (2014), no. 4, 655–684.

C. E. G. and H. Mawi, *The far field refractor with loss of energy*, Nonlinear Analysis: Theory, Methods & Applications **82** (2013), 12–46.

C. E. G. and F. Tournier, *Regularity for the near field parallel refractor and reflector problems*, Calc. Var. PDEs (2015).

C. E. G. and A. Sabra, *The reflector problem and the inverse square law*, Nonlinear Analysis: Theory, Methods & Applications **96** (2014), 109–133.

R. De Leo, C. E. G. and H. Mawi, *On the Numerical Solution of the Far Field Refractor Problem*, Nonlinear Analysis: Theory, Methods & Applications, to appear.

Thank you!