The Implementation Duality

Georg Nöldeke and Larry Samuelson

Universität Basel and Yale University

April 11, 2017

1. Introduction

- We examine an abstract implementation problem, with matching and principal-agent problems as leading special cases.
- Duality plays an important role in studying implementation with quasilinear (transferable) utility.
- In the absence of quasilinearity much of the relevant structure is lost, but not all ....
- Our analysis centers around a pair of maps that we refer to as implementation maps. We show that
  - these maps constitute a duality, that
  - under natural conditions exhibits particulary nice properties.
- The result is a characterization of implementability. We show how this characterization can be used in matching and principal-agent problems.

2. Model
Basic Ingredients

- Compact metric spaces X and Y.
- $\phi : X \times Y \times \mathbb{R} \to \mathbb{R}$, which is
  - continuous,
  - strictly decreasing in its third argument,
  - and satisfies $\phi(x, y, \mathbb{R}) = \mathbb{R}$.

2. Model

Looking from the Other Side

- Compact metric spaces X and Y.
- $\phi : X \times Y \times \mathbb{R} \to \mathbb{R}$, which is
  - continuous,
  - strictly decreasing in its third argument,
  - and satisfies $\phi(x, y, \mathbb{R}) = \mathbb{R}$.
- $\psi : Y \times X \times \mathbb{R} \to \mathbb{R}$ is defined as the inverse of ϕ with respect to the third argument,

$$u = \phi(x, y, \psi(y, x, u)),$$

  and inherits its properties: ψ is
  - continuous,
  - strictly decreasing in its third argument,
  - satisfies $\psi(y, x, \mathbb{R}) = \mathbb{R}$.

2. Model

Interpretation

- In the matching context
  - $\phi(x,y,v)$ is the maximal utility an agent of type $x \in X$ can obtain when matched with an agent of type $y \in Y$ who obtains utility v.
  - $\psi(y,x,u)$ is the maximal utility an agent of type $y \in Y$ can obtain when matched with an agent of type $x \in X$ who obtains utility u.
  - We later specify measures of X and Y and reservation utilities for all agents.
- In the principal-agent context
  - $\phi(x,y,v)$ is the utility of an agent of type $x \in X$ when choosing decision $y \in Y$ and making transfer $v \in \mathbb{R}$ to the principal.
  - $\psi(y,x,u)$ specifies the transfer that provides utility u to an agent of type x who chooses decision y.
  - We later specify a utility function for the principal, a measure over X, describing the distribution of agent types, and reservation utilities for the agent.

2. Model

A Return to the Assumptions

- Compact metric spaces X and Y.
- $\phi : X \times Y \times \mathbb{R} \to \mathbb{R}$, which is
  - continuous,
  - strictly decreasing in its third argument,
  - and satisfies $\phi(x, y, \mathbb{R}) = \mathbb{R}$.

2. Model

Profiles and Assignments

- Let
  - $\mathbf{B}(X)$ be the set of bounded functions $X \to \mathbb{R}$ and $\mathbf{B}(Y)$ the set of bounded functions $Y \to \mathbb{R}$;
  - Y^X be the set of functions $X \to Y$ and X^Y the set of functions $Y \to X$
- $\boldsymbol{u} \in \mathbf{B}(X)$ and $\boldsymbol{v} \in \mathbf{B}(Y)$ are profiles.
- $\boldsymbol{y} \in Y^X$ and $\boldsymbol{x} \in X^Y$ are assignments.
- We endow the sets $\mathbf{B}(X)$ and $\mathbf{B}(Y)$ with the pointwise partial order and the sup norm $\|\cdot\|$
- We show in the paper that the restriction to bounded profiles is without loss of generality.

2. Model

Interpretation

- In the matching model, $\boldsymbol{u}$ and $\boldsymbol{v}$ are profiles of utilities for the buyers and sellers.
- In the principal-agent model, $\boldsymbol{u}$ is a rent function for the agent, giving a utility $\boldsymbol{u}(x)$ for each type x of agent, and $\boldsymbol{v}$ is a tariff function giving the tariff $\boldsymbol{v}(y)$ at which any agent can buy decision y.
- In the matching model, $y = \boldsymbol{y}(x)$ identifies the seller y with whom buyer x matches, and $x = \boldsymbol{x}(y)$ identifies the buyer x with whom seller y matches.
- In the principal-agent model, $\boldsymbol{y}$ is decision assignment; $y = \boldsymbol{y}(x)$ identifies the decision y for agent type x. The function $\boldsymbol{x}$ is a type assignment; $x = \boldsymbol{x}(y)$ identifies the agent x to whom the principal assigns decision y.

2. Model
Implementation

- A profile $\boldsymbol{v} \in \mathbf{B}(Y)$ implements $(\boldsymbol{u}, \boldsymbol{y}) \in \mathbf{B}(X) \times Y^X$ if

$$\begin{aligned} \boldsymbol{u}(x) &= \max_{y \in Y} \phi(x, y, \boldsymbol{v}(y)) \\ \boldsymbol{y}(x) &\in \arg\max_{y \in Y} \phi(x, y, \boldsymbol{v}(y)). \end{aligned}$$

- Similarly, a profile $\boldsymbol{u} \in \mathbf{B}(X)$ implements $(\boldsymbol{v}, \boldsymbol{x}) \in \mathbf{B}(Y) \times X^Y$ if

$$\begin{aligned} \boldsymbol{v}(y) &= \max_{x \in X} \psi(y, x, \boldsymbol{u}(x)) \\ \boldsymbol{x}(y) &\in \arg\max_{x \in X} \psi(y, x, \boldsymbol{u}(x)). \end{aligned}$$

- We let $\boldsymbol{I}(X) \subset \mathbf{B}(X)$ and $\boldsymbol{I}(Y) \subset \mathbf{B}(Y)$ denote the sets of implementable profiles.

2. Model

Interpretation

- Matching interpretation: $\boldsymbol{v}$ implements $(\boldsymbol{u},\boldsymbol{y})$ if, given the seller utility prices given by $\boldsymbol{v}$, every buyer x finds it optimal to select seller $\boldsymbol{y}(x)$ and thereby achieves utility $\boldsymbol{u}(x)$.
- Similarly, $\boldsymbol{u}$ implements $(\boldsymbol{v},\boldsymbol{x})$ if, given the buyer utility prices given by $\boldsymbol{u}$, every seller y finds it optimal to select seller $\boldsymbol{x}(y)$ and thereby achieves utility $\boldsymbol{v}(y)$.
- Principal-agent interpretation: $\boldsymbol{v}$ implements $(\boldsymbol{u},\boldsymbol{y})$ if, given the tariff $\boldsymbol{v}$, every buyer x finds it optimal to select decision $\boldsymbol{y}(x)$ and thereby achieves utility $\boldsymbol{u}(x)$.
- $\boldsymbol{u}$ implements $(\boldsymbol{v},\boldsymbol{x})$ if, given the rent function $\boldsymbol{u}$, for every decision y agent $\boldsymbol{x}(y)$ is the one who can pay the most for decision y and $\boldsymbol{v}(y)$ is the corresponding willingness to pay.

- The implementation maps $\Phi : \mathbf{B}(Y) \to \mathbf{B}(X)$ and $\Psi : \mathbf{B}(X) \to \mathbf{B}(Y)$ are defined by setting

$$\Phi \boldsymbol{v}(x) = \sup_{y \in Y} \phi(x, y, \boldsymbol{v}(y)) \quad \forall x \in X$$
$$\Psi \boldsymbol{u}(y) = \sup_{x \in X} \psi(y, x, \boldsymbol{u}(x)) \quad \forall y \in Y.$$

3. Duality

Some Properties of Implementation Maps

Proposition 1

The implementation maps Φ *and* Ψ

- *are continuous,*
- *map bounded sets into bounded sets,*
- *implement continuous profiles, and*
- *have images that coincide with the set of implementable profiles:*

$$\boldsymbol{I}(X) = \Phi\boldsymbol{B}(X) \subset \boldsymbol{C}(X) \text{ and } \boldsymbol{I}(Y) = \Psi\boldsymbol{B}(Y) \subset \boldsymbol{C}(Y).$$

- It is immediate from the definitions that implementable profiles are contained in the images of the implementation maps.
- The other direction requires an argument using our assumptions on (X, Y, ϕ).

3. Duality

Duality

The implementation maps Φ and Ψ are dualities (in the sense of Penot (2010)), i.e., maps with the property that the image of the infimum of a set is the supremum of the image of the set).

This property is a straightforward implication of:

Proposition 2

The implementation maps Φ and Ψ are a Galois connection. That is,

$$\boldsymbol{u} \;\geq\; \Phi\boldsymbol{v} \;\Longleftrightarrow\; \boldsymbol{v} \;\geq\; \Psi\boldsymbol{u}$$

holds for all $\boldsymbol{u} \in \mathbf{B}(X)$ and $\boldsymbol{v} \in \mathbf{B}(Y)$.

Proof:

$$\begin{aligned}
\boldsymbol{u} \geq \Phi\boldsymbol{v} \;&\Longleftrightarrow\; \boldsymbol{u}(x) \geq \sup_{y\in Y} \phi(x,y,\boldsymbol{v}(y)) \text{ for all } x \in X \\
&\Longleftrightarrow\; \boldsymbol{u}(x) \geq \phi(x,y,\boldsymbol{v}(y)) \text{ for all } x \in X \text{ and } y \in Y \\
&\Longleftrightarrow\; \psi(y,x,\boldsymbol{u}(x)) \leq \boldsymbol{v}(y) \text{ for all } x \in X \text{ and } y \in Y \\
&\Longleftrightarrow\; \boldsymbol{v}(y) \geq \sup_{x\in X} \psi(y,x,\boldsymbol{u}(x)) \text{ for all } y \in Y \\
&\Longleftrightarrow\; \boldsymbol{v} \geq \Psi\boldsymbol{u}.
\end{aligned}$$

Galois connections have many nice properties. For instance:

Corollary 1

The implementation maps Φ and Ψ
[1.1] satisfy the **cancellation rule***, that is, for all $\boldsymbol{u} \in \mathbf{B}(X)$ and $\boldsymbol{v} \in \mathbf{B}(Y)$:*

$$\boldsymbol{v} \geq \Psi\Phi\boldsymbol{v} \text{ and } \boldsymbol{u} \geq \Phi\Psi\boldsymbol{u};$$

[1.2] are **order reversing***, that is, for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathbf{B}(X)$ and $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbf{B}(Y)$:*

$$\boldsymbol{v}_1 \geq \boldsymbol{v}_2 \Rightarrow \Phi\boldsymbol{v}_2 \geq \Phi\boldsymbol{v}_1 \text{ and } \boldsymbol{u}_1 \geq \boldsymbol{u}_2 \Rightarrow \Psi\boldsymbol{u}_2 \geq \Psi\boldsymbol{u}_1;$$

[1.3] and satisfy the **semi-inverse rule***, that is, for all $\boldsymbol{u} \in \mathbf{B}(X)$ and $\boldsymbol{v} \in \mathbf{B}(Y)$:*

$$\Phi\Psi\Phi\boldsymbol{v} = \Phi\boldsymbol{v} \text{ and } \Psi\Phi\Psi\boldsymbol{u} = \Psi\boldsymbol{u}.$$

3. Duality

Characterizing Implementability: Profiles

Proposition 3

[3.1] $\boldsymbol{u} \in \boldsymbol{B}(X)$ is implementable if and only if $\boldsymbol{u} = \Phi\Psi\boldsymbol{u}$.
[3.2] $\boldsymbol{v} \in \boldsymbol{B}(Y)$ is implementable if and only if $\boldsymbol{v} = \Psi\Phi\boldsymbol{v}$.

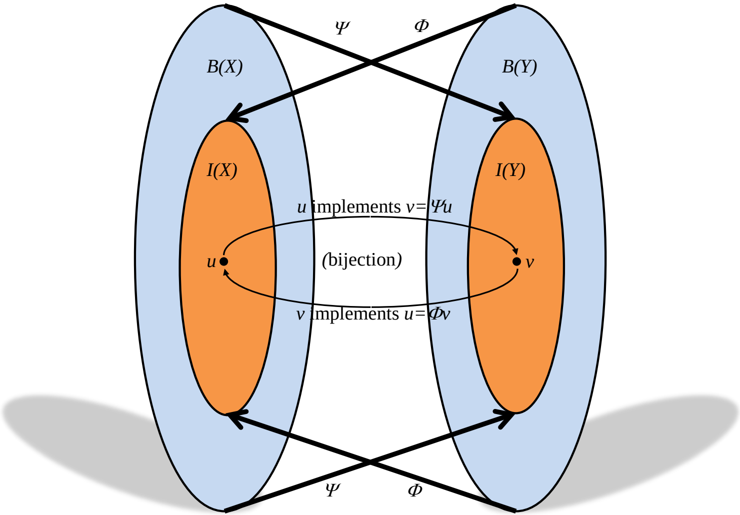
The semi-inverse property of a Galois connection ensures that the image of the implementation maps have such a fixed point characterization. Our assumptions ensure that these images are the implementable profiles. Some implications include:

$$\boldsymbol{I}(X) = \Phi\boldsymbol{I}(Y) \text{ and } \boldsymbol{I}(Y) = \Psi\boldsymbol{I}(X).$$

$$\boldsymbol{u} = \Phi\boldsymbol{v} \iff \boldsymbol{v} = \Psi\boldsymbol{u}, \quad \text{for all } \boldsymbol{u} \in \boldsymbol{I}(X) \text{ and } \boldsymbol{v} \in \boldsymbol{I}(Y).$$

3. Duality

Characterizing Implementability: Illustration



3. Duality

Characterizing Implementability: Assignments

$$\Gamma_{\boldsymbol{u},\boldsymbol{v}} = \{(x,y) \in X \times Y \mid \boldsymbol{u}(x) = \phi(x,y,\boldsymbol{v}(y))\}$$

Corollary 2

[2.1] An assignment $\boldsymbol{y} \in Y^X$ *is implementable if and only if there exist profiles* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *that implement each other with* $\Gamma_{\boldsymbol{u},\boldsymbol{v}}$ *containing the graph of* $\boldsymbol{y}$*, i.e.,*

$$(x,\boldsymbol{y}(x)) \in \Gamma_{\boldsymbol{u},\boldsymbol{v}} \quad \text{for all } x \in X.$$

[2.2] The argmax correspondences $\boldsymbol{X}_{\boldsymbol{u}}$ *and* $\boldsymbol{Y}_{\boldsymbol{v}}$ *are then inverses and their graphs coincide with* $\Gamma_{\boldsymbol{u},\boldsymbol{v}}$*, i.e., they satisfy*

$$\hat{x} \in \boldsymbol{X}_{\boldsymbol{u}}(\hat{y}) \iff \hat{y} \in \boldsymbol{Y}_{\boldsymbol{v}}(\hat{x}) \iff (\hat{x},\hat{y}) \in \Gamma_{\boldsymbol{u},\boldsymbol{v}}.$$

3. Duality

Some Properties of Sets of Implementable Profiles

Corollary 3

- *The sets of implementable profiles* $\boldsymbol{I}(X)$ *and* $\boldsymbol{I}(Y)$ *are closed subsets of* $\boldsymbol{B}(X)$ *and* $\boldsymbol{B}(Y)$*.*
- *Bounded sets of implementable profiles are equicontinuous.*
- *Closed and bounded sets of implementable profiles are compact.*

- A matching problem is given by $(X, Y, \phi, \mu, \nu, \underline{u}, \underline{v})$, where
  - (X, Y, ϕ) are as before,
  - μ and ν are measures on X and Y with full support, and
  - $\underline{u}$ and $\underline{v}$ are continuous reservation utilities.
- A match for a matching problem is a measure λ on $X \times Y$ satisfying the conditions

$$\lambda(\tilde{X} \times Y) \leq \mu(\tilde{X}) \quad (1)$$
$$\lambda(X \times \tilde{Y}) \leq \nu(\tilde{Y}). \quad (2)$$

- An outcome is a triple $(\lambda, \boldsymbol{u}, \boldsymbol{v})$.

4. Matching

Pairwise Stable and Stable Outcomes

- An outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ outcome is feasible if

$$\begin{aligned} \boldsymbol{u}(x) &= \phi(x,y,\boldsymbol{v}(y)) \quad \forall (x,y) \in \text{supp}(\lambda) \\ \boldsymbol{u}(x) &= \underline{\boldsymbol{u}}(x) \quad \forall x \in \text{supp}(\mu - \lambda_X) \\ \boldsymbol{v}(y) &= \underline{\boldsymbol{v}}(y) \quad \forall y \in \text{supp}(\nu - \lambda_Y). \end{aligned}$$

- A feasible outcome is pairwise stable if it satisfies the incentive constraints

$$\boldsymbol{u}(x) \geq \phi(x,y,\boldsymbol{v}(y)) \quad \forall (x,y) \in X \times Y$$

  and is individually rational if it satisfies

$$\begin{aligned} \boldsymbol{u}(x) &\geq \underline{\boldsymbol{u}}(x) \quad \forall x \in X \\ \boldsymbol{v}(y) &\geq \underline{\boldsymbol{v}}(y) \quad \forall y \in Y, \end{aligned}$$

- and it is stable if it is both pairwise stable and individually rational.

We can connect pairwise stability and implementation:

Lemma 1

Let the matching problem $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$ *be balanced, and let* λ *be a full match.*
[1.1] The outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ *is feasible if and only if* $\operatorname{supp} \lambda \subset \Gamma_{\boldsymbol{u},\boldsymbol{v}}$.
[1.2] If the outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ *is feasible, then the following statements are equivalent: (i)* $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ *is pairwise stable, (ii)* $\boldsymbol{v}$ *implements* $\boldsymbol{u}$*, (iii)* $\boldsymbol{u}$ *implements* $\boldsymbol{v}$*, (iv)* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *implement each other.*

4. Matching

Existence of Pairwise Stable Outcomes

Proposition 4

Let the matching problem $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$ be balanced. Then the set of pairwise stable full outcomes satisfying initial condition (y_1, v_1) is nonempty and closed.

The proof follows the same pattern as proof for existence of solutions to an optimal transportation problem:

1. Matching problems with finite numbers of agents have pairwise stable outcomes (e.g., Demange and Gale (1985))
2. Construct sequence of finite matching problems $(X_n, Y_n, \phi_n, \mu_n, \nu_n, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$ converging to $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$
3. Construct an associated bounded sequence of pairwise stable outcomes $(\lambda_n, \boldsymbol{u}_n, \boldsymbol{v}_n)$
4. Extract converging subsequence and show that limit $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ is pairwise stable for $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$.

Proposition 5

There exists a stable outcome $(\lambda, \boldsymbol{u}, \boldsymbol{v})$ *for the matching problem* $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$.

- Pairwise stable outcomes can be constructed for any initial condition of the form $\boldsymbol{u}(x_1) = u_1$ for some $x_1 \in X$ and $u_1 \in \mathbb{R}$.
- Existence of stable outcomes then is an easy corollary to Proposition 4

We are often interested in deterministic matches:

Corollary 4

Let the matching problem $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$ be balanced, and let $\boldsymbol{y} \in Y^X$ be a measure-preserving assignment. Then the associated deterministic match $\lambda_{\boldsymbol{y}}$ is pairwise stable if and only if $\boldsymbol{y}$ is implementable.

The set of pairwise stable outcomes forms a lattice:

Proposition 6

Let the matching problem $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$ *be balanced. Let* $(\lambda_1, \boldsymbol{u}_1, \boldsymbol{v}_1)$ *and* $(\lambda_2, \boldsymbol{u}_2, \boldsymbol{v}_2)$ *be pairwise stable full outcomes. Then there exist pairwise stable full outcomes* $(\lambda_3, \boldsymbol{u}_1 \vee \boldsymbol{u}_2, \boldsymbol{v}_1 \wedge \boldsymbol{v}_2)$ *and* $(\lambda_4, \boldsymbol{u}_1 \wedge \boldsymbol{u}_2, \boldsymbol{v}_1 \vee \boldsymbol{v}_2)$*.*

The proof uses the duality property of the implementation maps and the connection between implementability and pairwise stability.

It then follows easily that:

Corollary 5

The set of stable profiles of the matching problem $(X, Y, \phi, \mu, \nu, \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}})$ *form a complete lattice. In the minimal outcome, the equality* $\boldsymbol{u}(x) = \underline{\boldsymbol{u}}(x)$ *holds for some* $x \in X$.

5. Principal-Agent Problems

Setting the Stage

We have:

- Agent with utility function $\phi(x,y,v)$.
- Principal with utility function $\pi(x,y,v)$.
  - $\pi : X \times Y \times \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing in v and satisfies $\pi(x,y,\mathbb{R}) = \mathbb{R}$.
- Agent's type distributed on X according to μ.
- $\underline{\boldsymbol{u}} \in \boldsymbol{C}(X)$: reservation utility profile for the agent.

The principal's problem can be formulated as:

$$\max_{\{\boldsymbol{v}\in \boldsymbol{I}(Y):\ \boldsymbol{v}\leq \Psi \underline{\boldsymbol{u}}\}} \int_{x\in X} \max_{y\in \boldsymbol{Y}_{\boldsymbol{v}}} \pi(x,y,\boldsymbol{v}(y))d\mu(x) = \max_{\{\boldsymbol{v}\in \boldsymbol{I}(Y):\ \boldsymbol{v}\leq \Psi \underline{\boldsymbol{u}}\}} \Pi(\boldsymbol{v}).$$

Straightforward arguments ensure that the integral exists.

5. Principal-Agent Problems

Existence Result

Proposition 7

A solution to the principal's problem exist.

Proof:

- Check that Π is upper semicontinuous.
- Show that there is no loss of generality in imposing a lower bound on the feasible tariffs to obtain a compact choice set.
- Apply Weierstrass.

Will the participation constraint bind?

- This is a triviality with quasilinear utility.
- In general, a solution to the principal's problem need not cause the participation constraint to bind. We offer three sufficient conditions:
  - Private values.
  - "Uniform" income effects.
  - Single crossing.
- The last two are special cases of a "strong implementability" condition.

6. Further Results

Single Crossing

With $X = [\underline{x}, \bar{x}] \subset \mathbb{R}$, $Y = [\underline{y}, \bar{y}] \subset \mathbb{R}$ the single-crossing condition

$$\phi(x_1, y_2, v_2) \geq \phi(x_1, y_1, v_1) \Rightarrow \phi(x_2, y_2, v_2) > \phi(x_2, y_1, v_1)$$

for all $x_1 < x_2 \in X$, $y_1 < y_2 \in Y$, and $v_1, v_2 \in \mathbb{R}$ implies

- all increasing decision functions are implementable, and
- stable outcomes with deterministic matchings exist.

6. Further Results

Extensions

We can extend the analysis to incorporate stochastic contracts and moral hazard in the principal-agent problem.

7. Discussion

THANK YOU