LATTICE PATH ENUMERATION AND ANALYTIC COMBINATORICS IN SEVERAL VARIABLES

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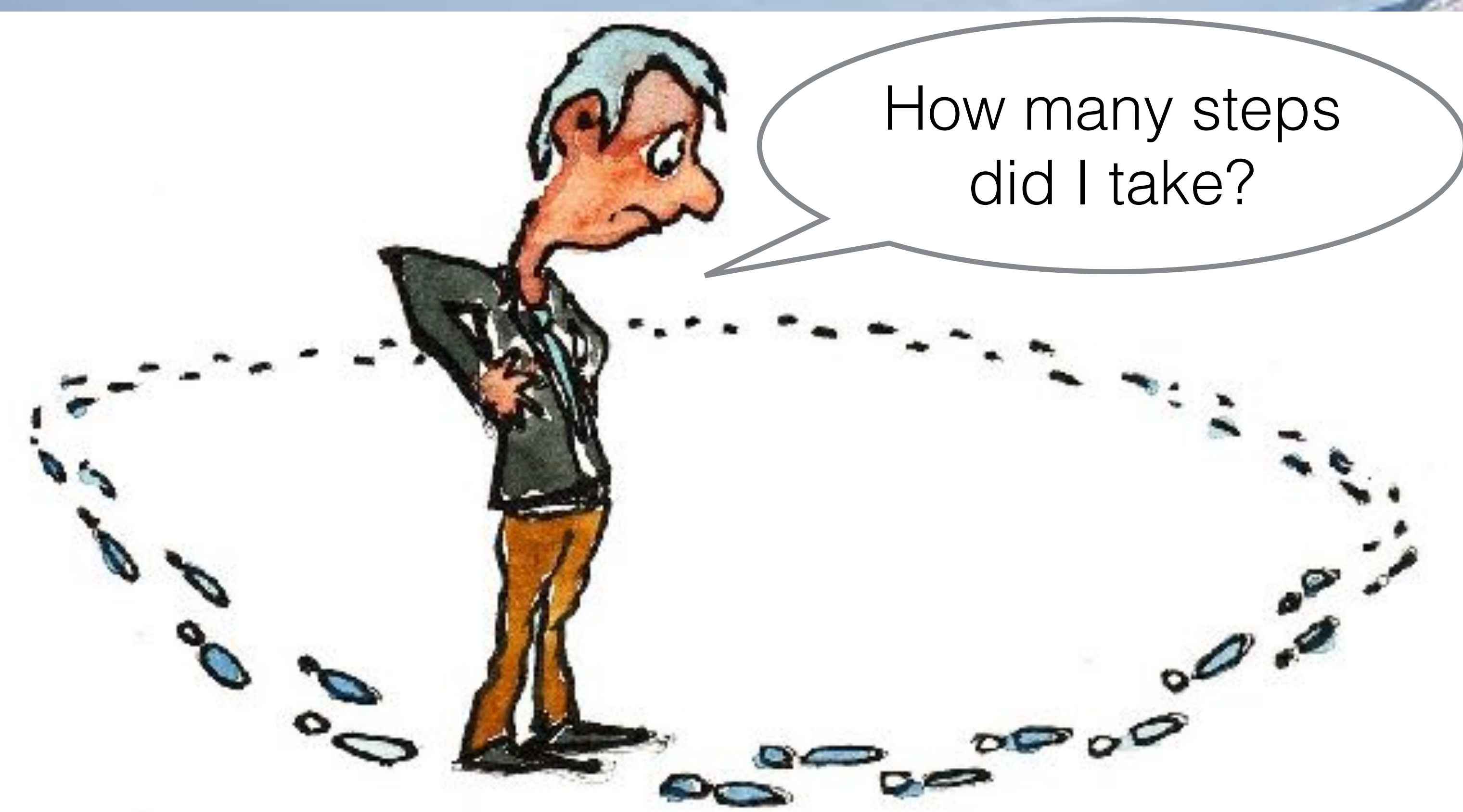


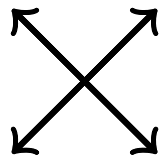
Joint work with Alin Bostan, Mireille Bousquet-Mélou, Julien Courtiel, Manuel Kauers, Marni Mishna, Kilian Raschel, and Mark Wilson

Counting Lattice Paths in a Quadrant

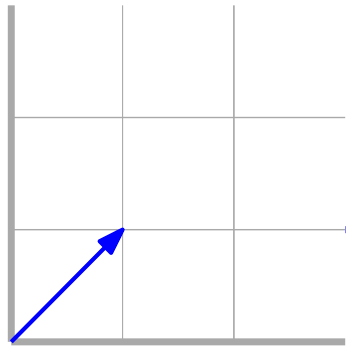
Given: A finite set of steps / directions $\mathcal{S} \subset \{\pm 1, 0\}^2$

Goal: $c_n = \#$ of walks staying in $\mathbb{N}^2$, starting at the origin

For instance, given $\mathcal{S} = \{(1,-1),(1,1),(-1,1),(-1,-1)\} \subset \mathbb{Z}^2$



we have:



$$c_1 = 1$$

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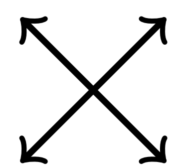

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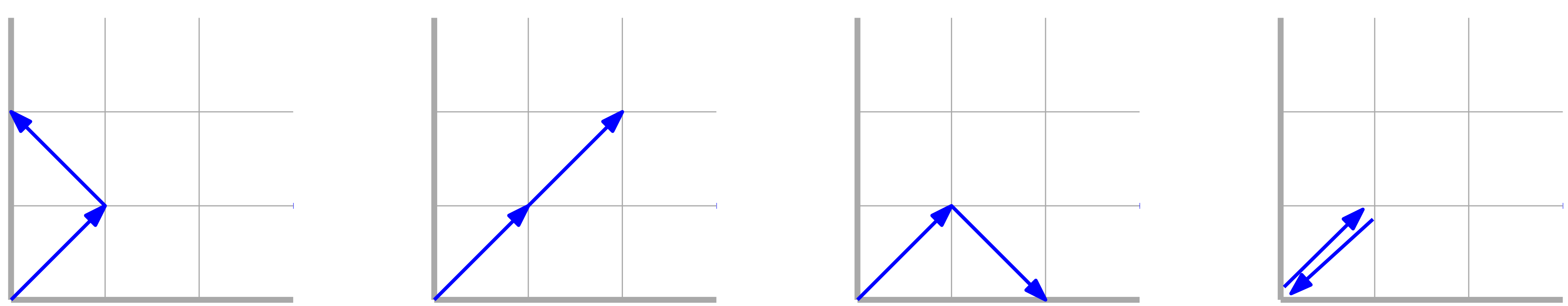

$$c_2 = 4$$

Counting Lattice Paths in a Quadrant

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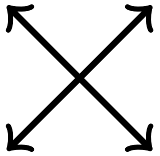

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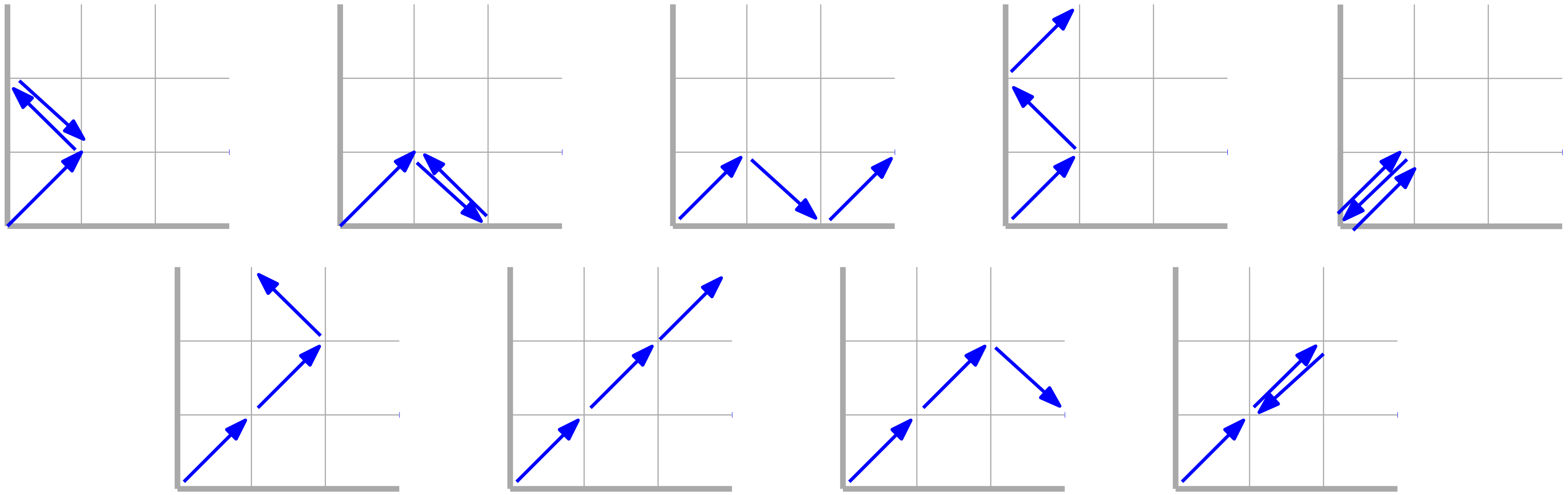


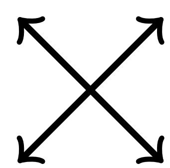
$c_3 = 9$

Counting Lattice Paths in an Orthant

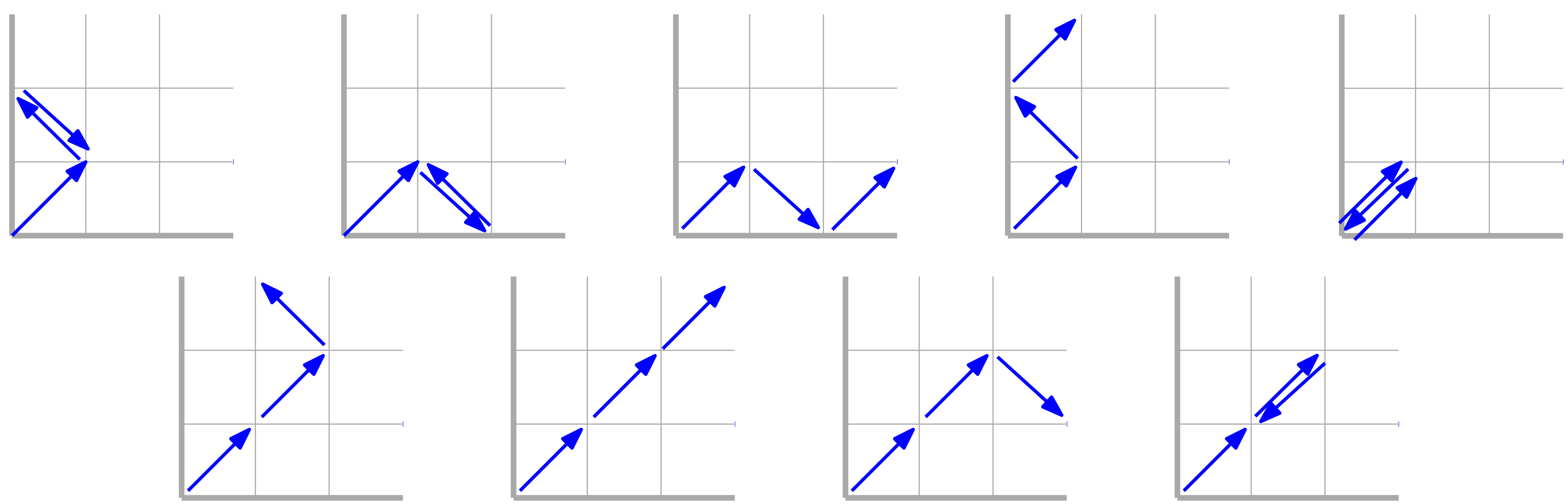
Given: A finite set of steps / directions $\mathcal{S} \subset \{\pm 1, 0\}^2$

Goal: $c_n = \#$ of walks staying in $\mathbb{N}^d$, starting at the origin

For instance, given $\mathcal{S} = \{(1,-1),(1,1),(-1,1),(-1,-1)\} \subset \mathbb{Z}^2$



we have:



$$c_3 = 9$$

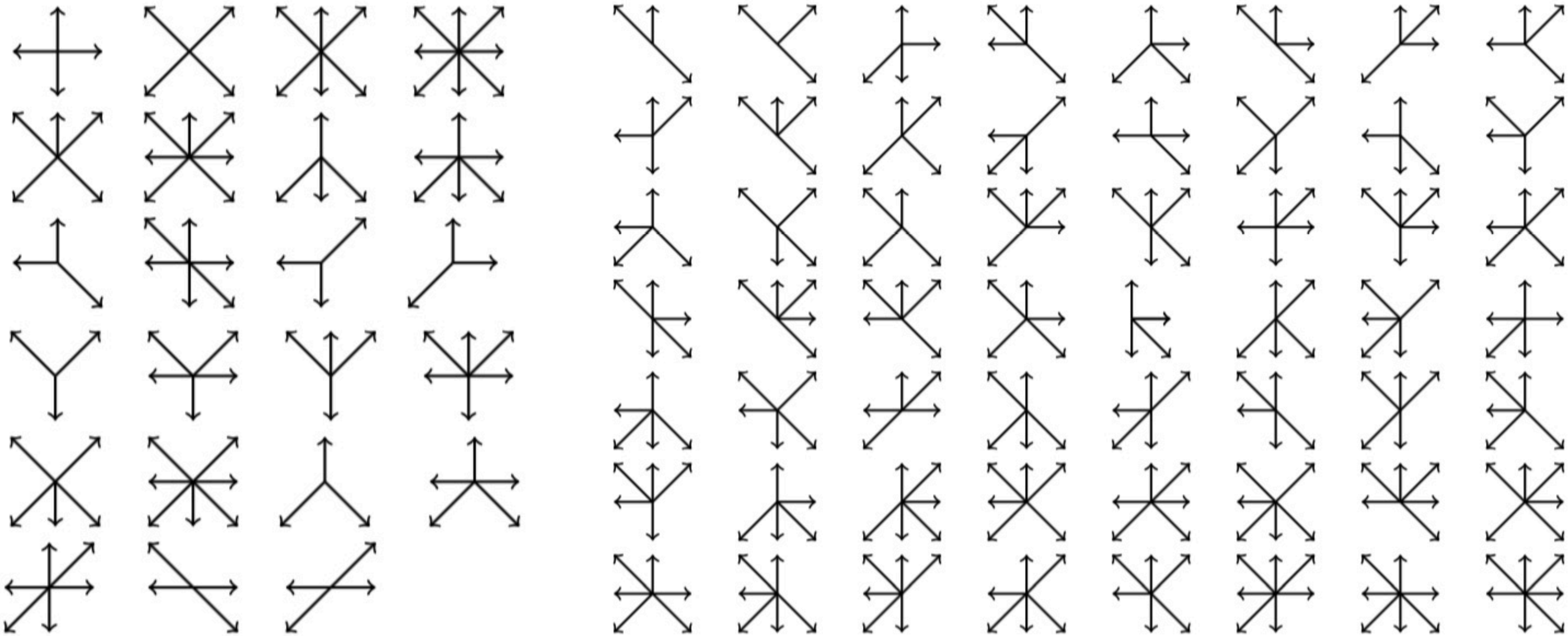
Importance

Applications to:

- Statistical mechanics (polymers in solution, Ising model, ...)
- Queueing theory / operations research
- Other Discrete Structures (trees, words, plane partitions, ...)
- Probability Theory (random walks, branching processes, ...)

Useful toolbox for developing methods for generating functions

Two Dimensional Quadrant Models



Bousquet-Mélou and Mishna (2010) showed there are **79** non-isomorphic two dimensional models.

The Kernel Method

Bousquet-Mélou and Mishna (2010) were able to show that many of these walks have a ***D-finite*** generating function.

Let

$$C(x, y, t) = \sum_{i,j,n \geq 0} c_{i,j,n} x^i y^j t^n$$

where $c_{i,j,n}$ is the number of walks of length n staying in the quarter plane and ending at the point (i, j).

Then

$$\begin{aligned}
C(1, 1, t) &= \text{GF of all walks} \\
C(1, 0, t) &= \text{GF of walks ending at } y = 0 \\
C(0, 1, t) &= \text{GF of walks ending at } x = 0 \\
C(0, 0, t) &= \text{GF of walks ending at } (0, 0)
\end{aligned}$$

The Kernel Method

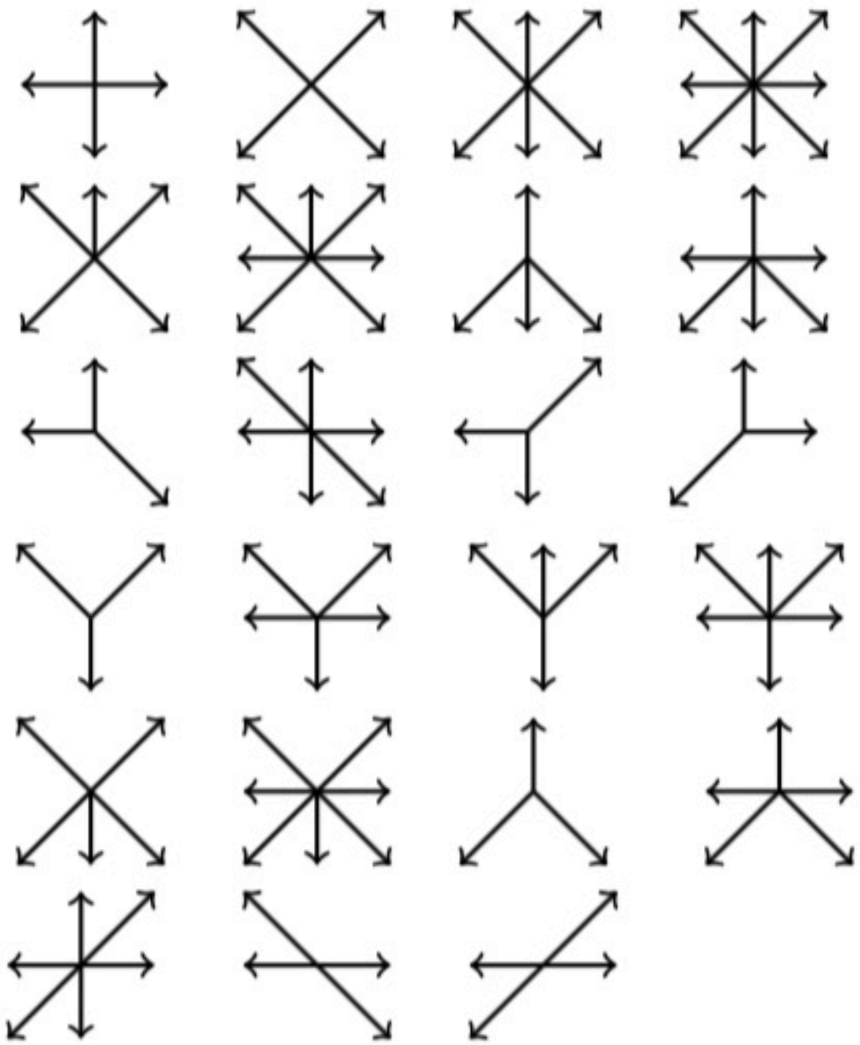
The recursive nature of a walk of length n ending at (i, j) implies that $C(x, y, t)$ satisfies a functional equation of the form

$$K(x, y, t) \cdot C(x, y, t) = 1 + A(x) \cdot C(x, 0, t) + B(y) \cdot C(0, y, t)$$

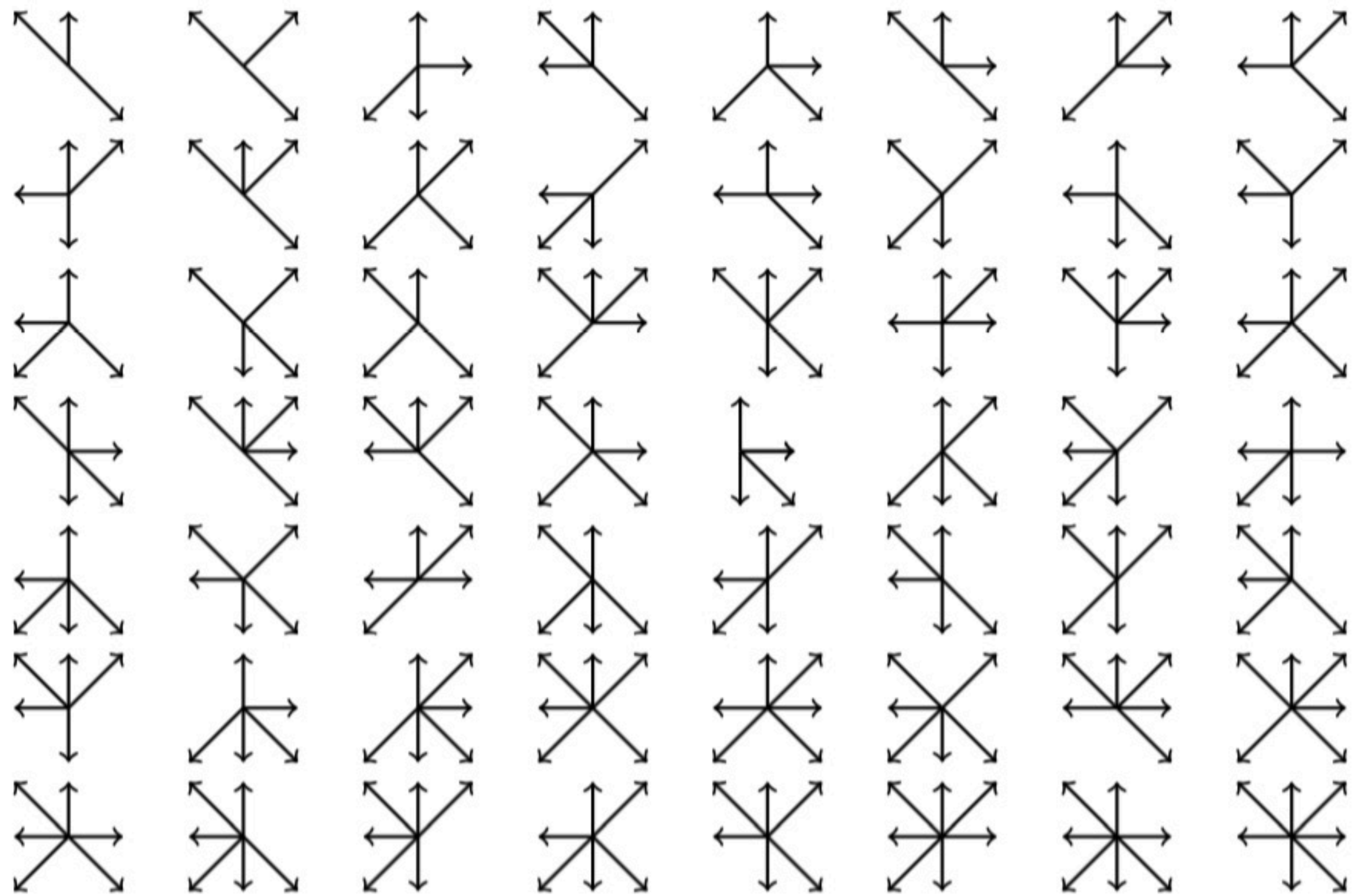
Bousquet-Mélou and Mishna (following Fayolle, Iasnogorodski, Malyshev) use a group $\mathcal{G}$ of bi-rational transformations of the plane associated to this model.

When the group is finite it can **usually** be combined with the functional equation to give a nice representation of $C(1, 1, t)$.

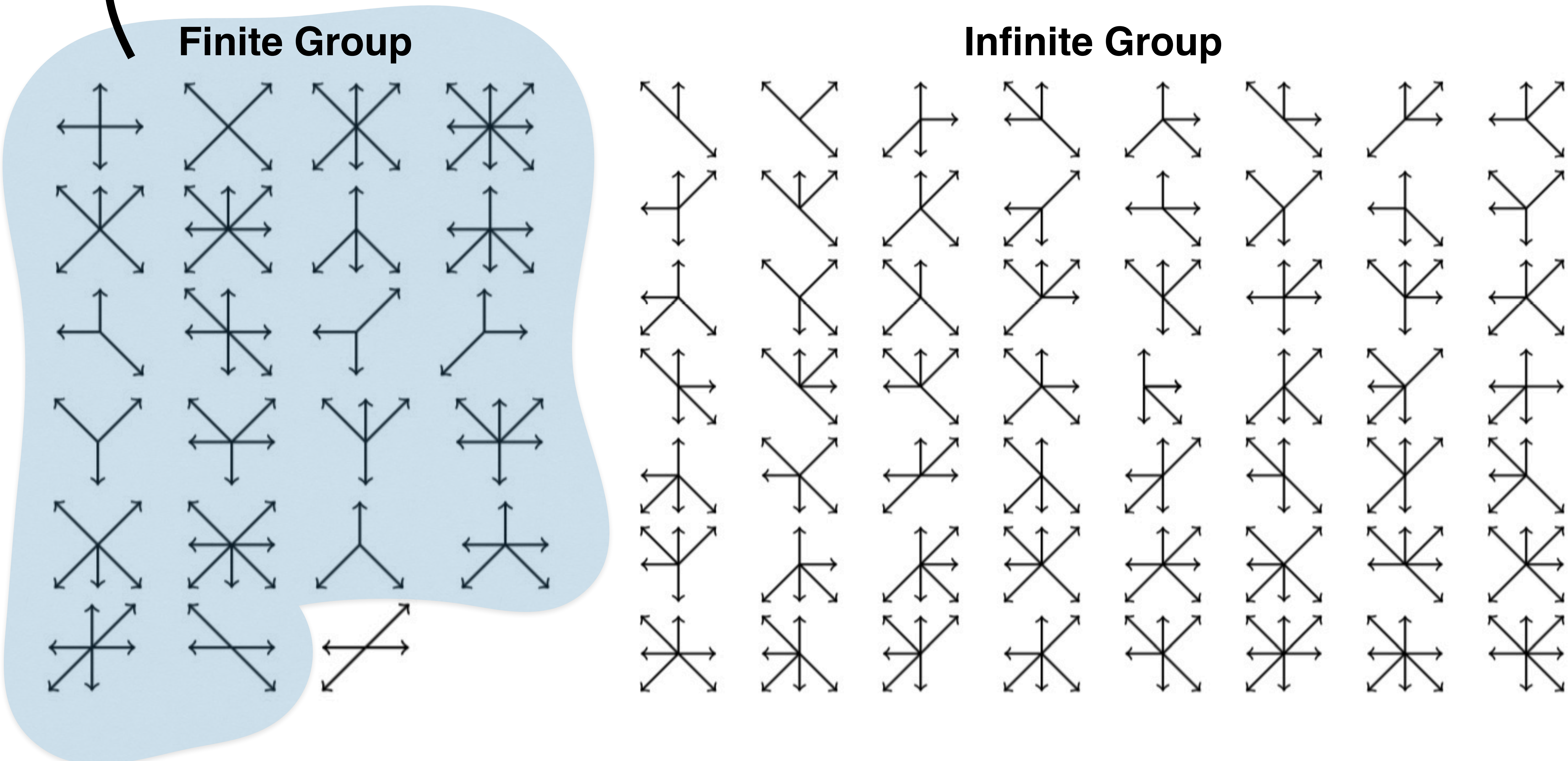
Finite Group



Infinite Group



$C(x, y, t)$ is D-Finite (Bousquet-Mélou & Mishna 2010)



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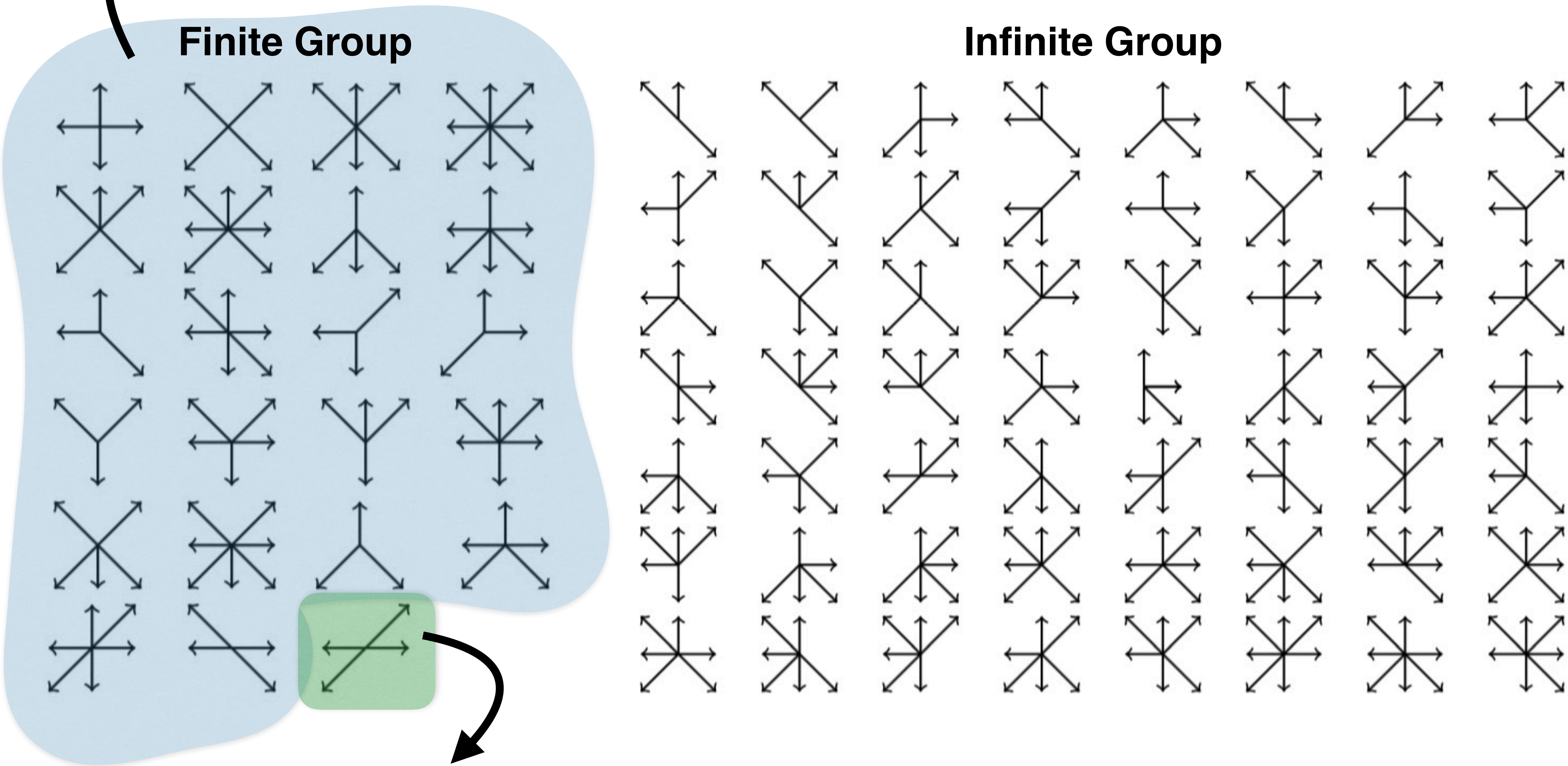
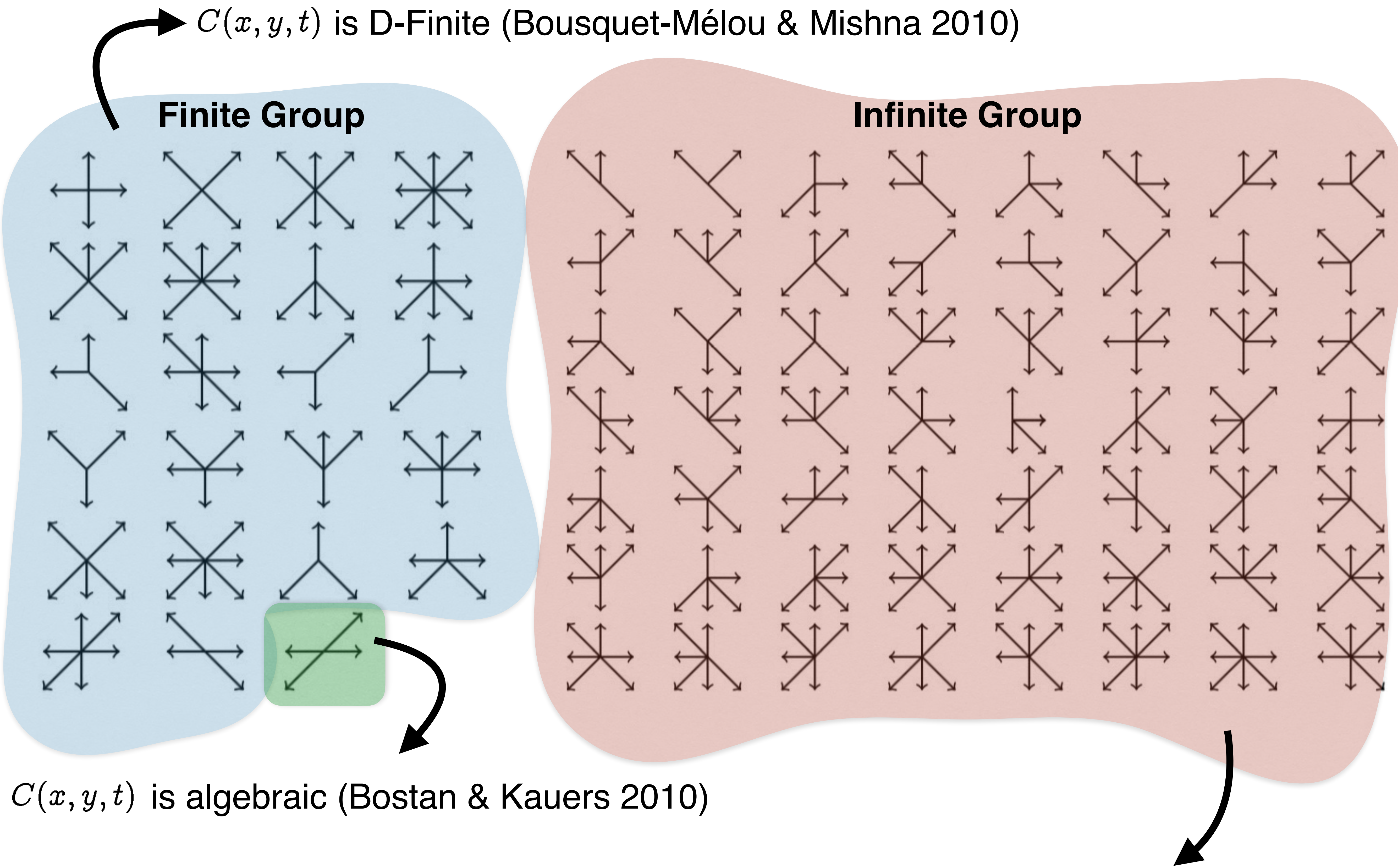


$C(x,y,t)$ is algebraic (Bostan & Kauers 2010)

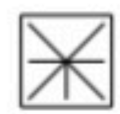
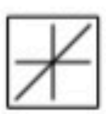
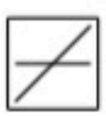
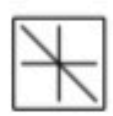
$C(x,y,t)$ is D-Finite (Bousquet-Mélou & Mishna 2010)



$C(x,y,t)$ is algebraic (Bostan & Kauers 2010)

$C(x,y,t)$ is not D-Finite (Kurkova & Raschel 2012)
(Mishna and Rechnitzer 2009 / M. & Mishna 2014)

Bostan and Kauers (2009) **Guessed** Asymptotics

n	$\mathcal{S}$	Asymptotics	n	$\mathcal{S}$	Asymptotics	n	$\mathcal{S}$	Asymptotics
1		$\frac{4}{\pi}\cdot\frac{4^n}{n}$	9		$\frac{\sqrt{3}}{2\sqrt{\pi}}\cdot\frac{3^n}{\sqrt{n}}$	17		$\frac{4\cdot A_n}{\pi}\cdot\frac{(2\sqrt{2})^n}{n^2}$
2		$\frac{2}{\pi}\cdot\frac{4^n}{n}$	10		$\frac{4}{3\sqrt{\pi}}\cdot\frac{4^n}{\sqrt{n}}$	18		$\frac{3\sqrt{3}\cdot B_n}{\pi}\cdot\frac{(2\sqrt{3})^n}{n^2}$
3		$\frac{\sqrt{6}}{\pi}\cdot\frac{6^n}{n}$	11		$\frac{\sqrt{5}}{2\sqrt{2\pi}}\cdot\frac{5^n}{\sqrt{n}}$	19		$\frac{\sqrt{8}(1+\sqrt{2})^{7/2}}{\pi}\cdot\frac{(2+2\sqrt{2})^n}{n^2}$
4		$\frac{8}{3\pi}\cdot\frac{8^n}{n}$	12		$\frac{\sqrt{5}}{3\sqrt{2\pi}}\cdot\frac{5^n}{\sqrt{n}}$	20		$\frac{6C_n}{\pi}\cdot\frac{(2\sqrt{6})^n}{n^2}$
5		$\frac{2\sqrt{2}}{\Gamma(1/4)}\cdot\frac{3^n}{n^{3/4}}$	13		$\frac{2\sqrt{3}}{3\sqrt{\pi}}\cdot\frac{6^n}{\sqrt{n}}$	21		$\frac{\sqrt{3}(1+\sqrt{3})^{7/2}}{2\pi}\cdot\frac{(2+2\sqrt{3})^n}{n^2}$
6		$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\cdot\frac{3^n}{n^{3/4}}$	14		$\frac{\sqrt{7}}{3\sqrt{3\pi}}\cdot\frac{7^n}{\sqrt{n}}$	22		$\frac{\sqrt{6(379+156\sqrt{6})(1+\sqrt{6})^7}}{5\sqrt{95}\pi}\cdot\frac{(2+2\sqrt{6})^n}{n^2}$
7		$\frac{\sqrt{6\sqrt{3}}}{\Gamma(1/4)}\cdot\frac{6^n}{n^{3/4}}$	15		$\frac{3\sqrt{3}}{2\sqrt{\pi}}\cdot\frac{3^n}{n^{3/2}}$	23		$\frac{8}{\pi}\cdot\frac{4^n}{n^2}$
8		$\frac{4\sqrt{3}}{3\Gamma(1/3)}\cdot\frac{4^n}{n^{2/3}}$	16		$\frac{3\sqrt{3}}{2\sqrt{\pi}}\cdot\frac{6^n}{n^{3/2}}$			

One Can **Prove** (and **Explain**) These Results Using A*nalytic Combinatorics in Several Variables*

Theorem (M. and Wilson 2016)

All of the guessed asymptotics on previous slides are true.

We can "read off" many of these asymptotic properties.

Can also prove conjectures of Bostan et al. 2017 on walks returning to bounding axes and the origin.

Diagonals Give Compact Representations

For an element

$$A(x,y,t) = \sum_{i,j,n\geq 0} a_{i,j,n} x^i y^j t^n \in \mathbb{Q}[[x,y,t]]$$

define the diagonal operator

$$\Delta A(x,y,t) := \sum_{n\geq 0} a_{n,n,n} t^n \in \mathbb{Q}[[t]]$$

Example (Binomial Coefficients)

$$F(x,y) = \frac{1}{1-x-y}$$

$$= 1 + x + y + 2xy + x^2 + y^2 + x^3 + 3x^2y + 3xy^2 + y^3 + 6x^2y^2 + \cdots$$

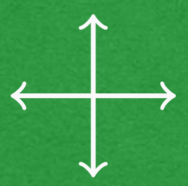
Diagonals Give Compact Representations

Theorem

For the non-algebraic D-finite models there exists an explicit polynomial $P(x,y)$ such that

$$C(1,1,t) = \Delta\left(\frac{P(x,y)}{(1-x)(1-y)(1-txyS(\overline{x},\overline{y}))}\right)$$

Example



When $\mathcal{S} = \{(\pm 1, 0), (0, \pm 1)\}$ then

$$C(1,1,t) = \Delta\left(\frac{(1+x)(1+y)}{1-txy(x+y+1/x+1/y)}\right)$$

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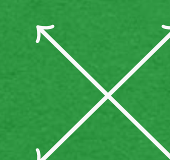

When $\mathcal{S} = \{(\pm 1, 1), (\pm 1, -1)\}$ then

$$C(1,1,t) = \Delta\left(\frac{(1+x)(1+y)}{1-txy(x/y+y/x+xy+1/xy)}\right)$$

Diagonals Give Compact Representations

Theorem

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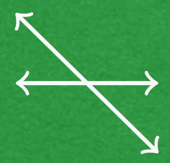
Example



When $\mathcal{S} = \{(-1,1),(-1,0),(1,0),(1,-1)\}$ then

$$C(1,1,t) = \Delta\left(\frac{(x+y)(x-y)(x^2-y)(1+x)}{x^2y(1-txy(x+x/y+y/x+1/x))}\right)$$

Analytic Combinatorics in Several Variables

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}, \qquad f_{n,\ldots,n} \sim C \cdot n^{\alpha} \cdot \rho^{n}$$

Let the domain of convergence of the series be $\mathcal{D}$

Let the singularities of F be $\mathcal{V} = \mathbb{V}(H)$

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Let the domain of convergence of the series be $\mathcal{D}$
Let the singularities of F be $\mathcal{V} = \mathbb{V}(H)$

A **minimal point** is a singularity on the boundary, $\mathbf{w} \in \mathcal{V} \cap \partial\mathcal{D}$
The Cauchy domain of integration can be made arbitrarily close.

Local minimizers of $|z_1 \cdots z_d|^{-1}$ on $\mathcal{V}$ are **critical points**.
These are points where saddle-point approximations can be made.

Analytic Combinatorics in Several Variables

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Minimal critical points, when they exist, typically determine dominant asymptotics. The *type* of these points determines C, α

Analytic Combinatorics in Several Variables

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i}\in\mathbb{N}^d} f \qquad f_{n,\dots,n} \sim C \cdot n^{\alpha} \cdot \rho^{n}$$

Let the do

Let the singular

See Torin's talk for more!

A **minimal point** is a si dary, $\mathbf{w} \in \mathcal{V} \cap \partial\mathcal{D}$

The Cauchy domain of i arbitrarily close.

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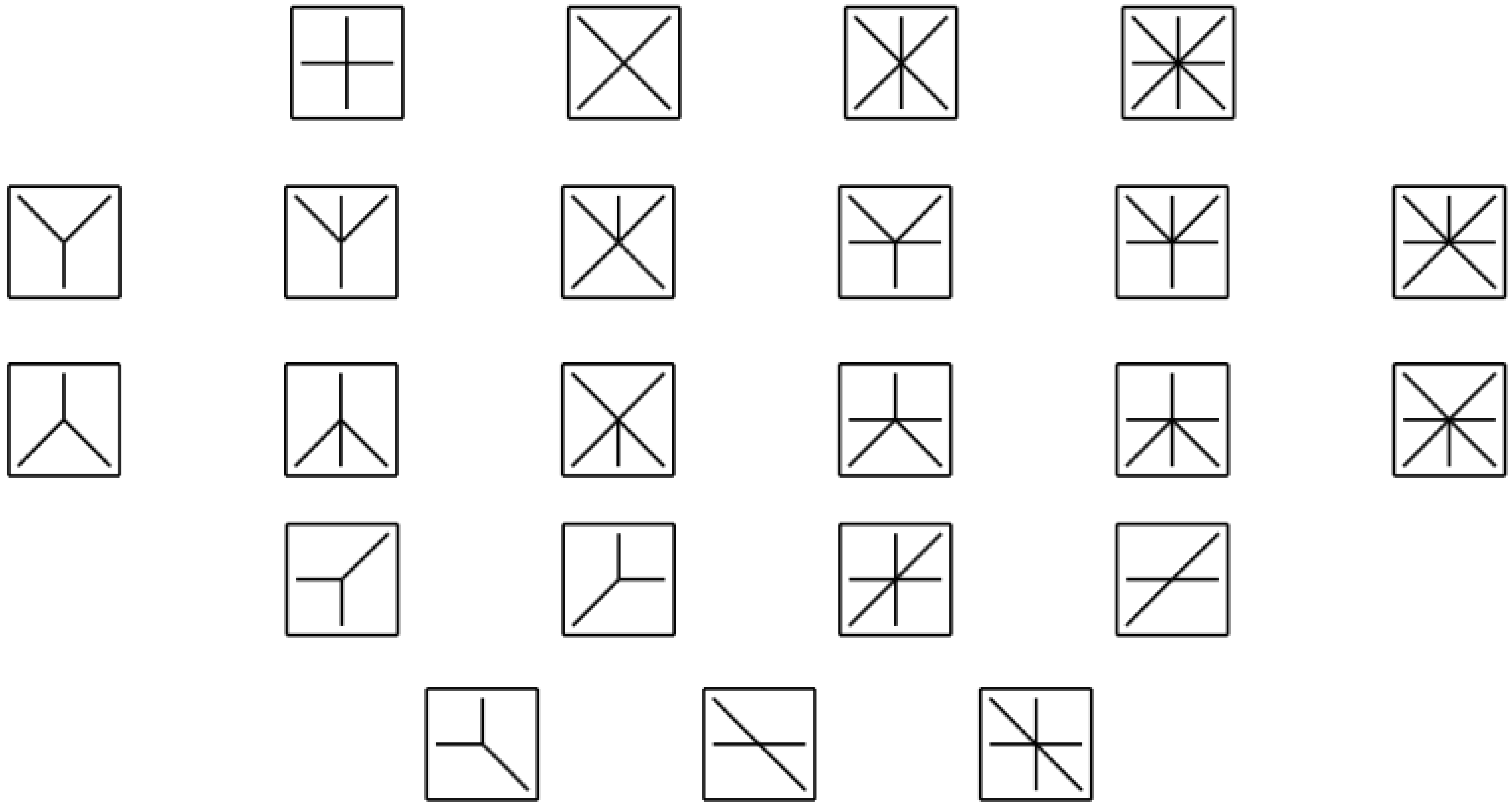
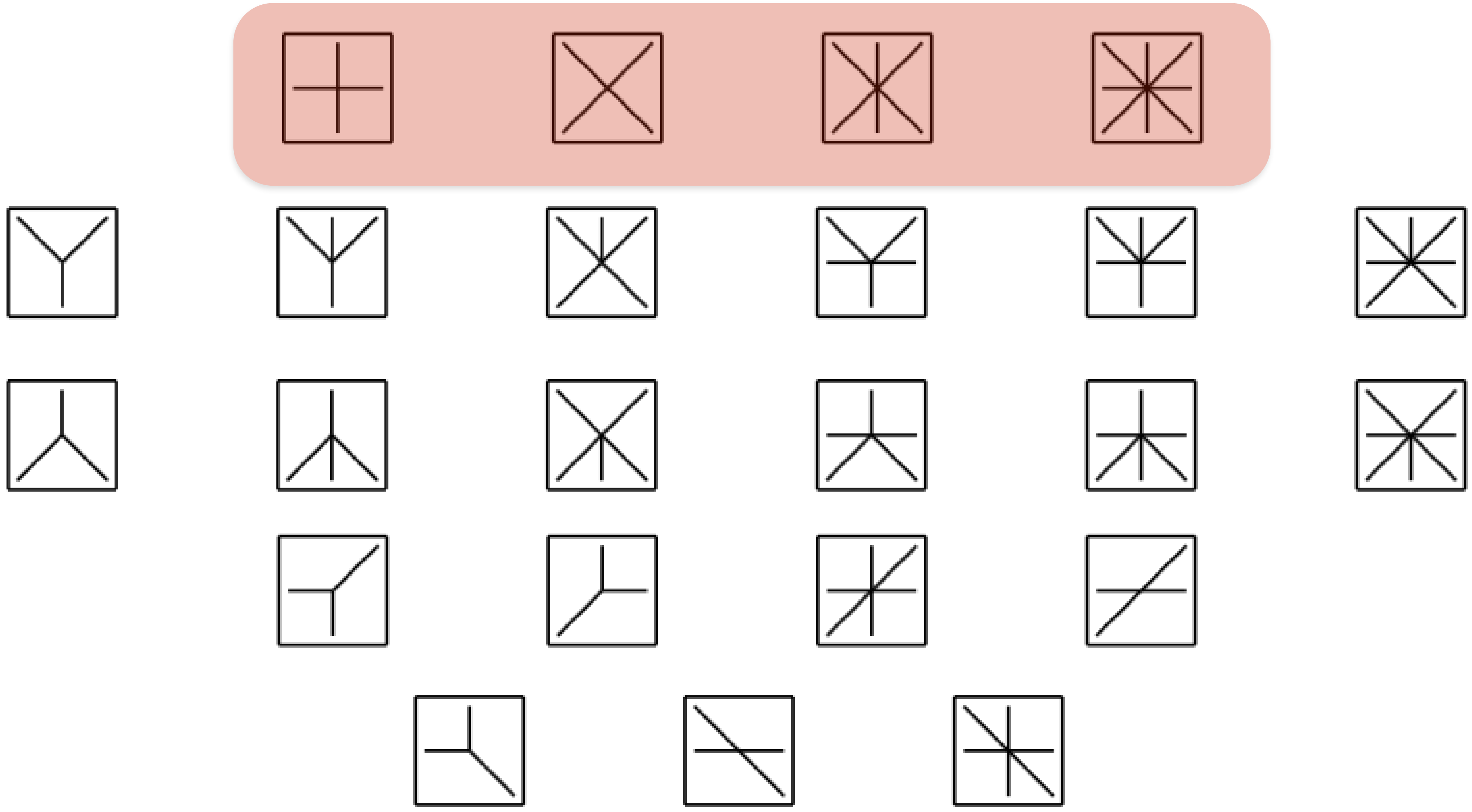
Combinatorial Properties Are Linked to Asymptotics



Combinatorial Properties Are Linked to Asymptotics



Highly Symmetric — M. and Mishna 2014/15

$$c_n \sim C \cdot n^{-1} \cdot |\mathcal{S}|^n$$

Highly Symmetric Models

There is a uniform diagonal expression for walks in an orthant symmetric over every axis.

$$C(t) = \Delta\left(\frac{(1+z_1)\cdots(1+z_d)}{1-t(z_1\cdots z_d)S(\mathbf{z})}\right)$$

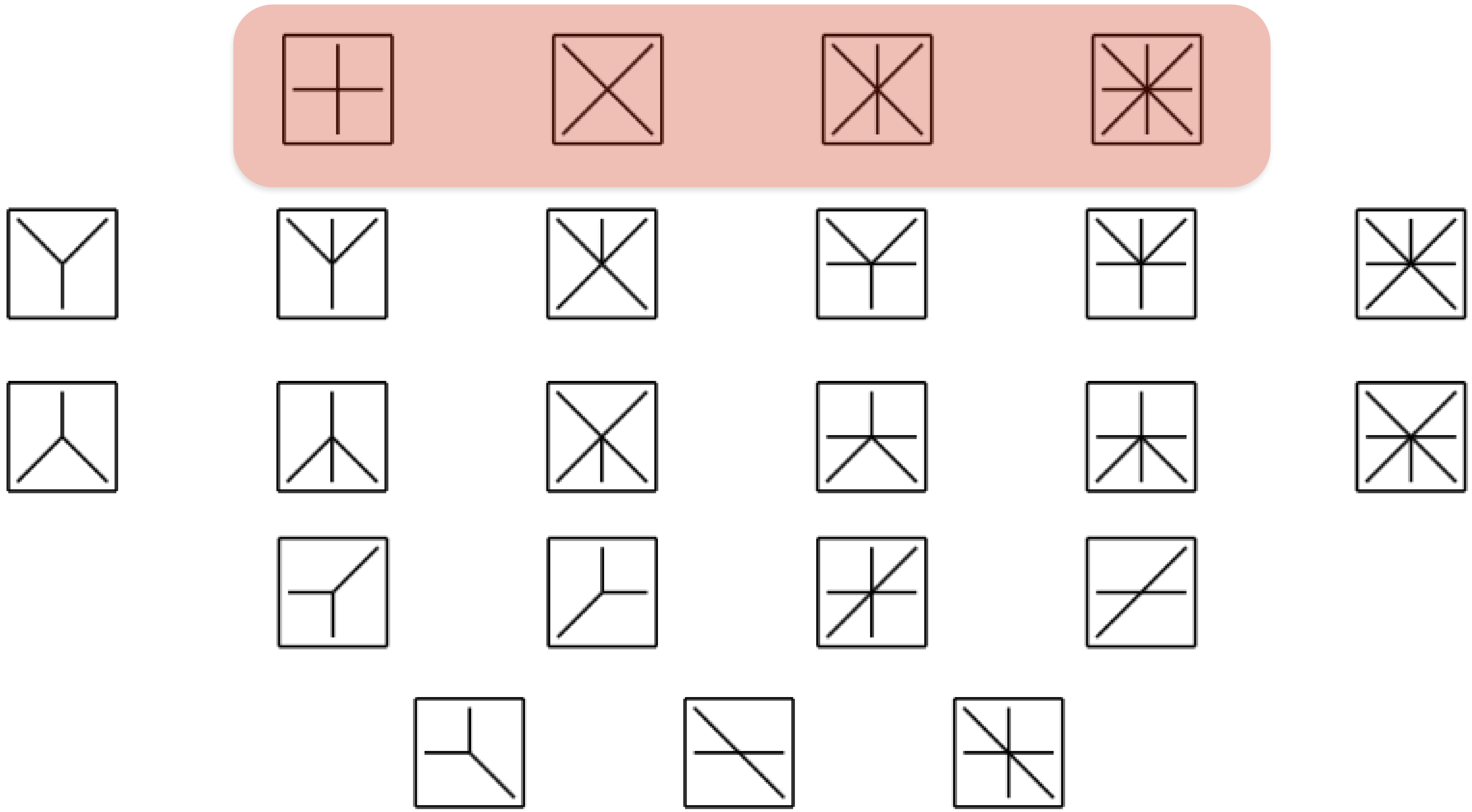
Theorem (M. and Mishna 2016)

Let $\mathcal{S} \subseteq \{-1,0,1\}^d \setminus \{\mathbf{0}\}$ be symmetric with respect to each axis and take a positive step in each direction. Then

$$c_n \sim \left[\left(s^{(1)}\cdots s^{(d)}\right)^{-1/2}\pi^{-d/2}|\mathcal{S}|^{d/2}\right]\cdot n^{-d/2}\cdot|\mathcal{S}|^n,$$

where $s^{(k)} = \#$ of steps which have k^{th} coordinate 1.

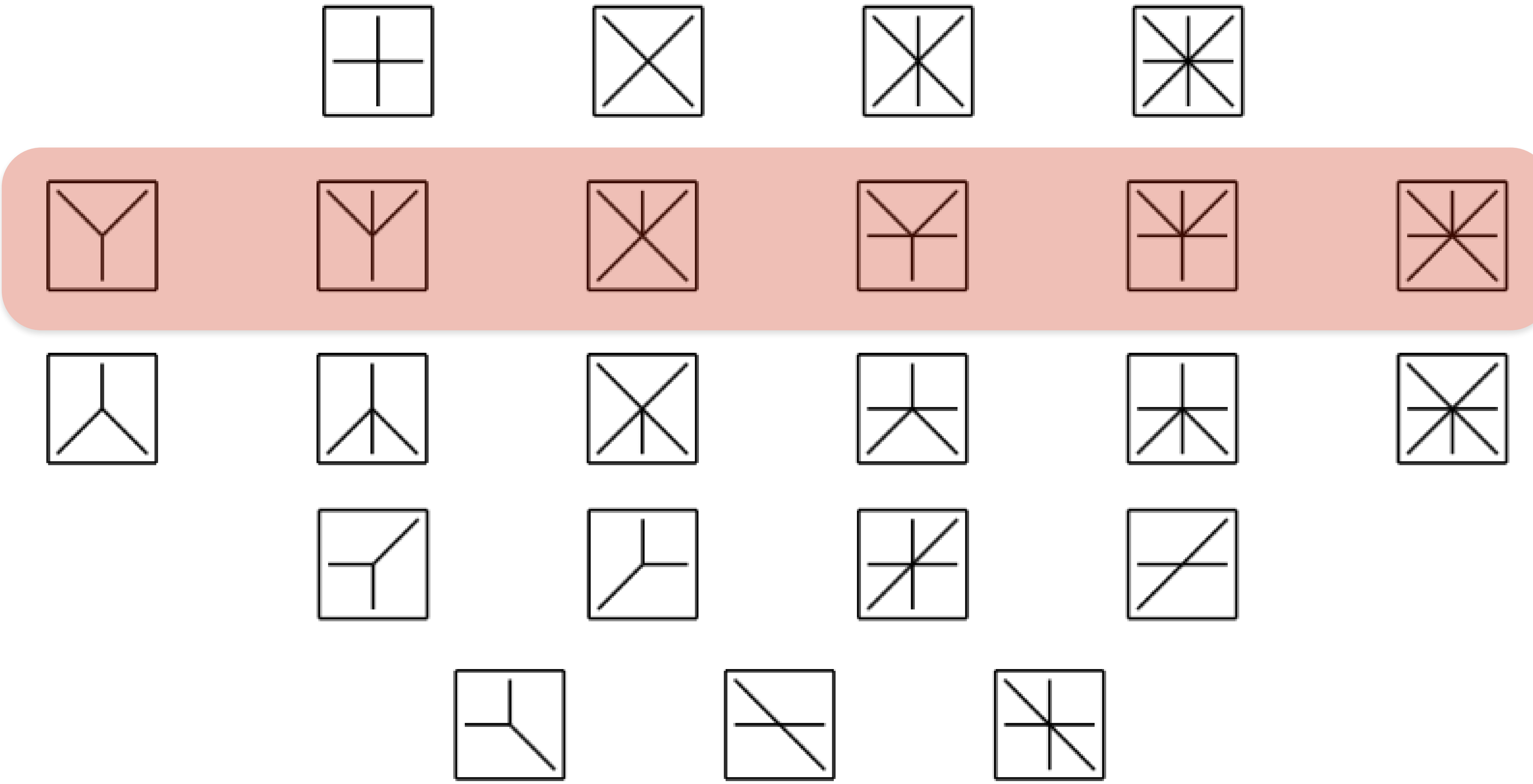
Combinatorial Properties Are Linked to Asymptotics



Highly Symmetric — M. and Mishna 2014/15

$$c_n \sim C \cdot n^{-1} \cdot |\mathcal{S}|^n$$

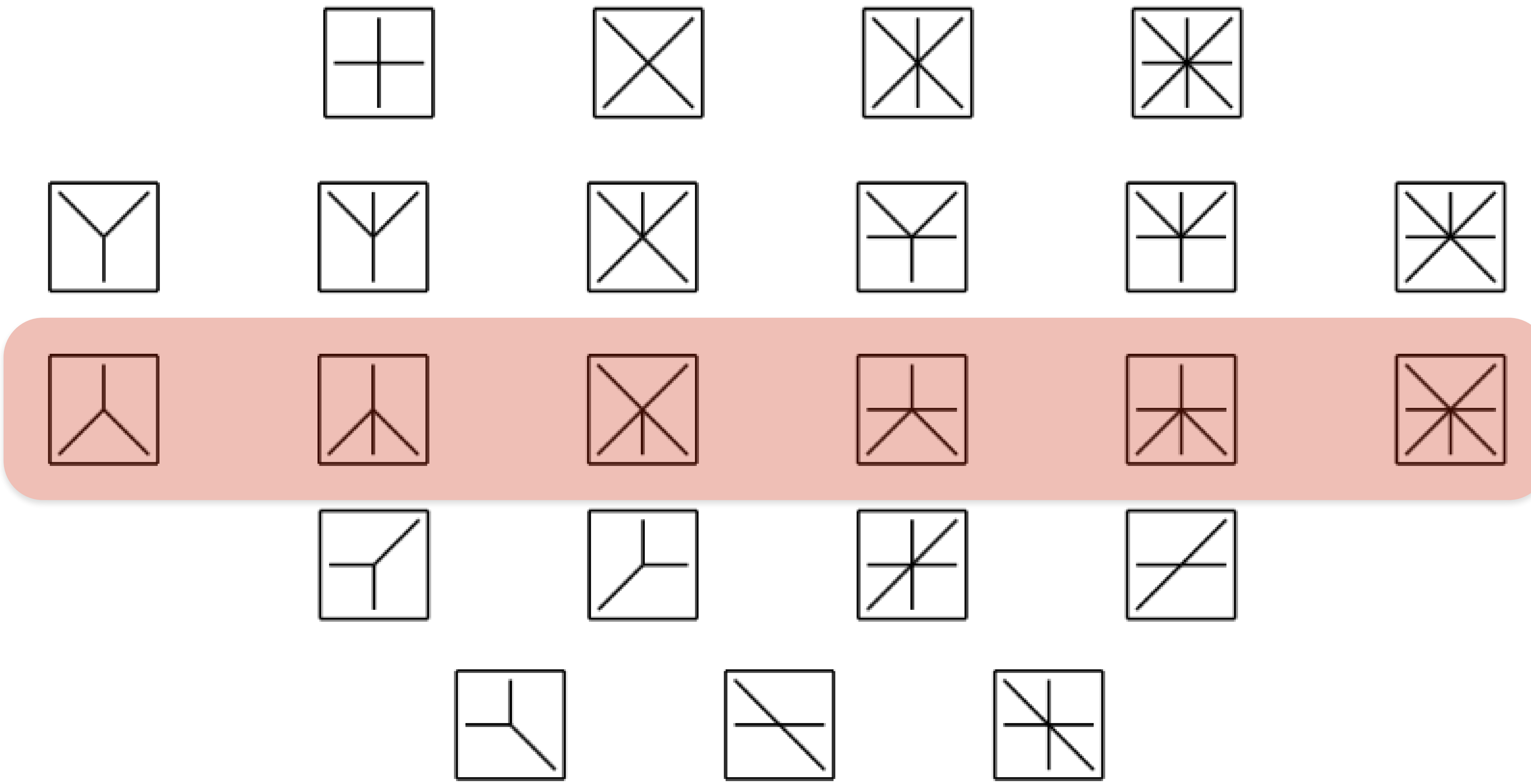
Combinatorial Properties Are Linked to Asymptotics



Positive Drift — M. and Wilson 2016

$$c_n \sim C \cdot n^{-3/2} \cdot |\mathcal{S}|^n$$

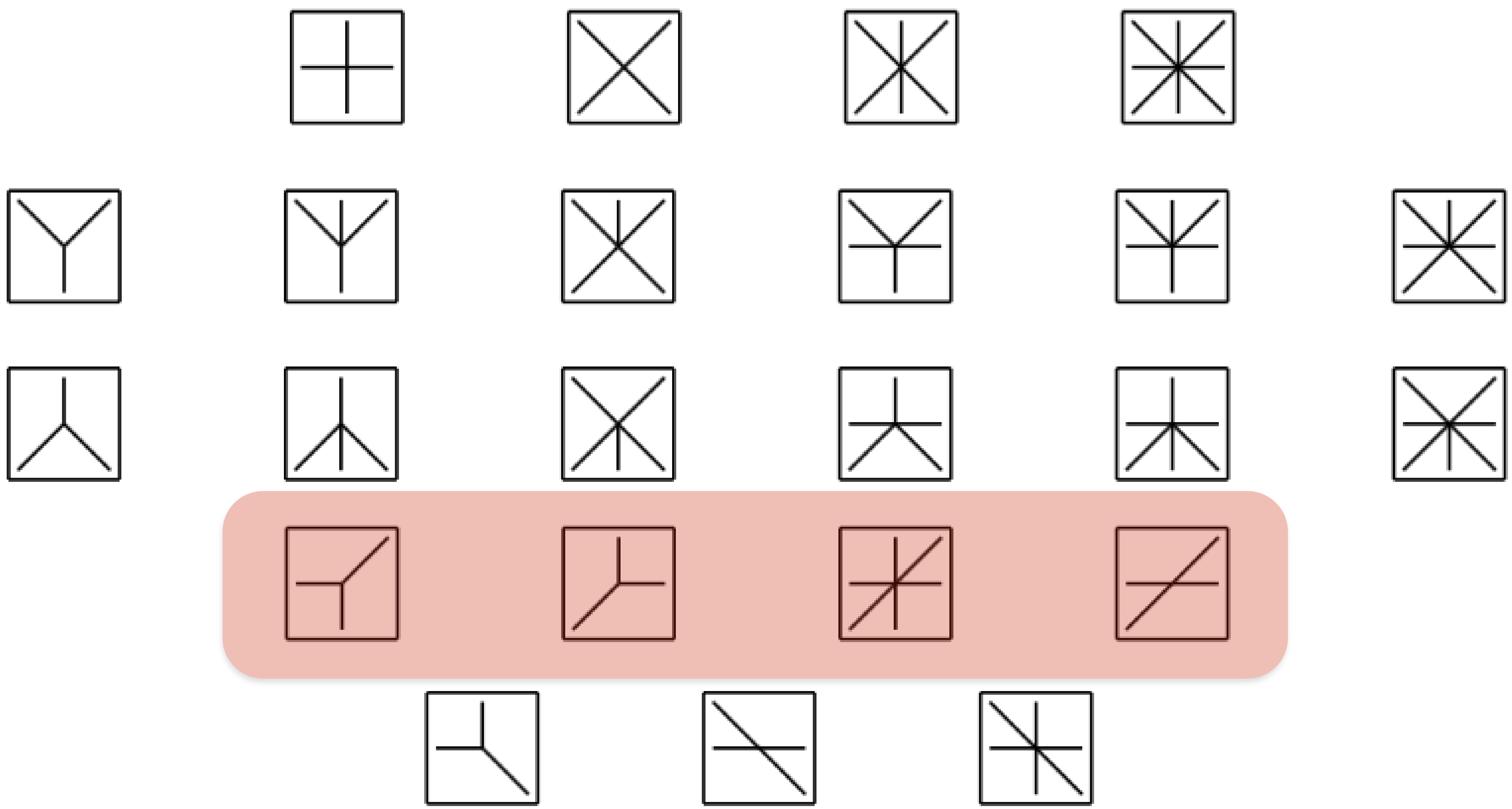
Combinatorial Properties Are Linked to Asymptotics



Negative Drift — M. and Wilson 2016

$$c_n \sim C_n \cdot n^{-2} \cdot S(1,\tau)^n$$

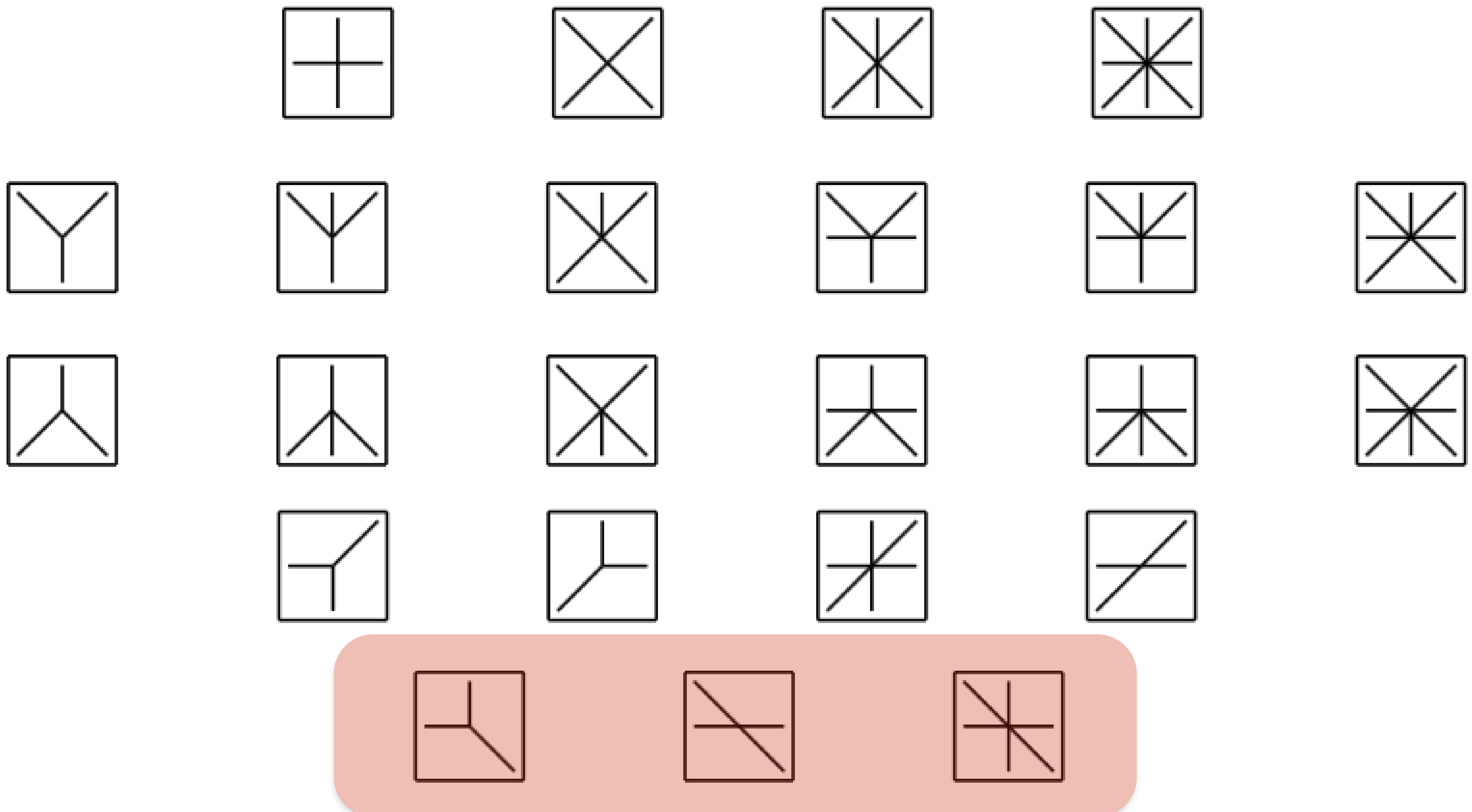
Combinatorial Properties Are Linked to Asymptotics



Algebraic Zero Orbit Sum Cases

Mishna / Bousquet-Mélou and Mishna / Bostan and Kauers

Combinatorial Properties Are Linked to Asymptotics

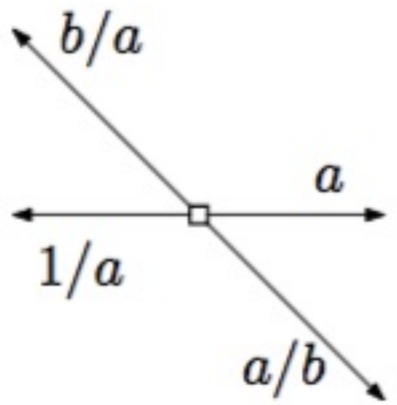


Three Sporadic Cases

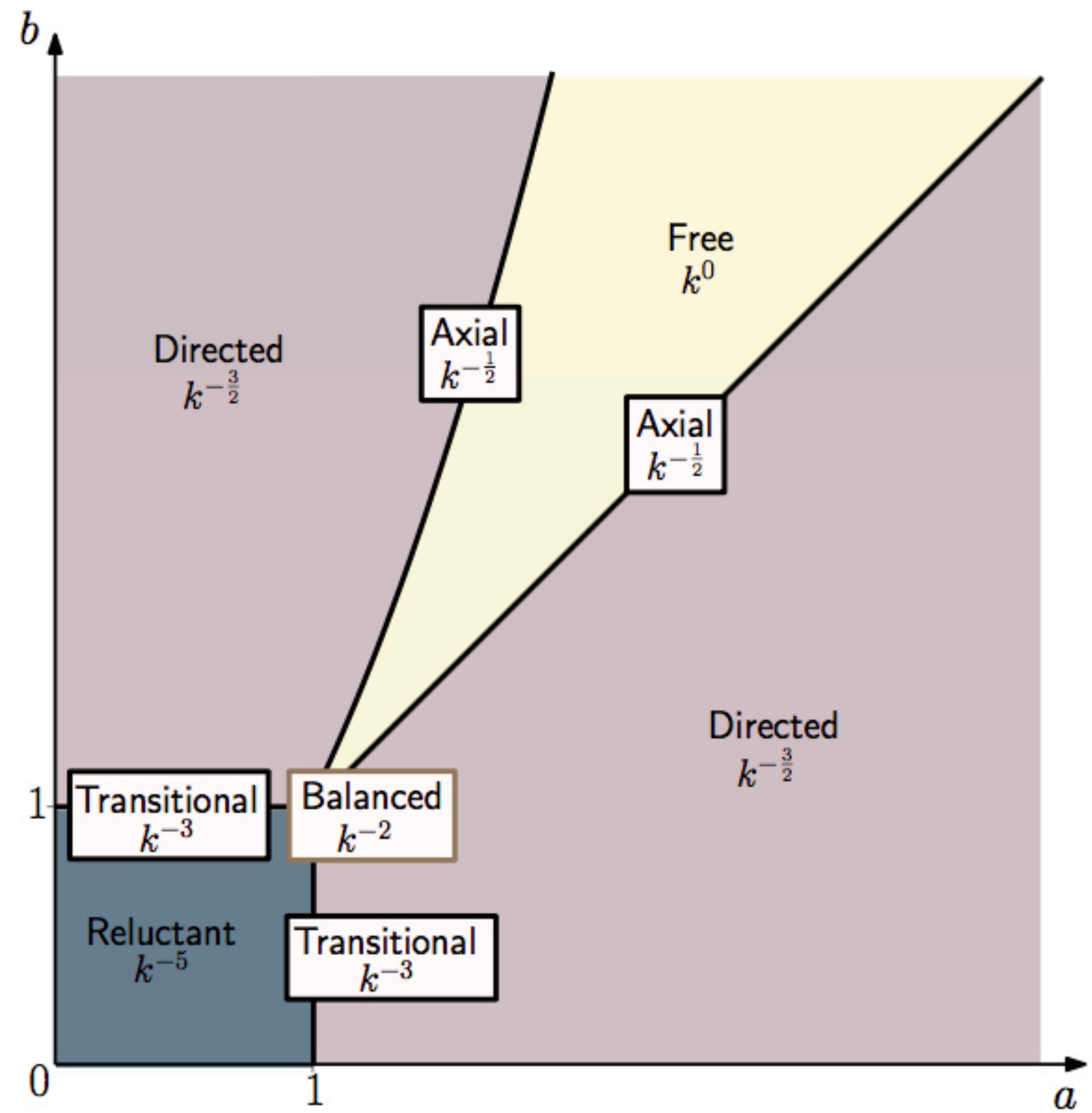
Bousquet-Mélou and Mishna (or through ACSV)

Weighted Lattice Path Models

As a next step, one can use weighted step sets to obtain parametrized diagonal expressions.



$$s_k = C_{[k]} \cdot \rho^k \cdot k^\alpha$$



Weighted Gouyou-Beauchamps Models

Courtiel, M., Mishna, and Raschel 2017 determine asymptotics for this family of models.

$$\Delta\left(\frac{y(y-b)(a-x)(a+x)(a^2y-bx^2)(ay-bx)(ay+bx)}{(1-x)(1-y)(1-txyS_{a,b}(\overline{x},\overline{y}))}\right)$$

Class	Condition	ρ	α
Balanced	$a=b=1$	4	2
Free	$\sqrt{b}<a<b$	$\frac{(1+b)(a^2+b)}{ab}$	0
Reluctant	$a<1$ and $b<1$	4	5
Axial	$b=a^2>1$	$\frac{2(b+1)}{\sqrt{b}}$	$1/2$
	$a=b>1$	$\frac{(1+a)^2}{a}$	$1/2$
Transitional	$a=1, b<1$ or $b=1, a<1$	4	3
Directed	$b>1$ and $\sqrt{b}>a$	$\frac{2(b+1)}{\sqrt{b}}$	$3/2$
	$a>1$ and $a>b$	$\frac{(1+a)^2}{a}$	$3/2$

Centrally weighted highly symmetric models treated in M.'s 2017 thesis.

Weighted Gouyou-Beauchamps Models

Courtiel, M., Mishna, and Rasc[illegible] 2017 determine asymptotics for this family of models.

$$\Delta\left(\frac{y(y-b)(a-x)(a+x\ \cdots\ bx^2)(ay-bx)(ay+bx)}{(1-x)(1\ \cdots\ txyS_{a,b}(\overline{x},\overline{y}))}\right)$$

See Julien's talk for more!

Class	[illegible]	[illegible]	α
Balanced	[illegible]	[illegible]	2
Free	$\sqrt{}$ [illegible]	$\frac{\cdots b)(a^2+b)}{ab}$	0
Reluctant	a [illegible]	4	5
Axial	b [illegible]	$\frac{2(b+1)}{\sqrt{b}}$	$1/2$
	[illegible]	$\frac{(1+a)^2}{a}$	$1/2$
Transitional	$\cdots 1, b<1$ or $b=1, a<1$	4	3
Directed	$b>1$ and $\sqrt{b}>a$	$\frac{2(\cdots+1)}{\sqrt{b}}$	$3/2$
	$a>1$ and $a>b$	$\frac{(1+a)^2}{a}$	$3/2$

Centrally weighted highly symmetric models treated in M.'s 2017 thesis.

Some Ongoing Work

Dealing with longer steps

Almost Symmetric Models

Computational Complexity of ACSV

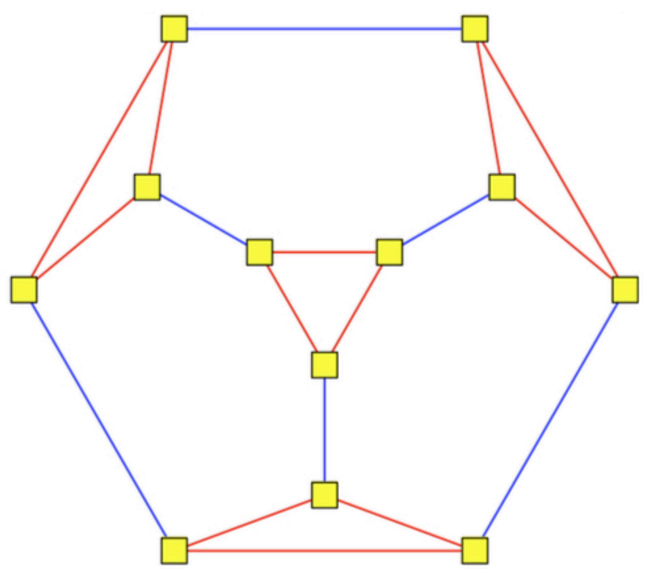
Some Ongoing Work

Dealing with longer steps

Almost Symmetric Models

Computational Complexity of ACSV

More algebraic substitutions fixing the kernel — can still define orbit (but be careful about being a group)



$$\mathcal{S} = \{(1,0),(-1,0),(-2,1),(0,-1)\}.$$

$$K(x,y,t) = 1 - t\sum_{(i,j)\in\mathcal{S}} x^i y^j = 1 - t(x + \overline{x} + \overline{x}^2 y + \overline{y}).$$

Joint work with Alin Bostan and Mireille Bousquet-Mélou

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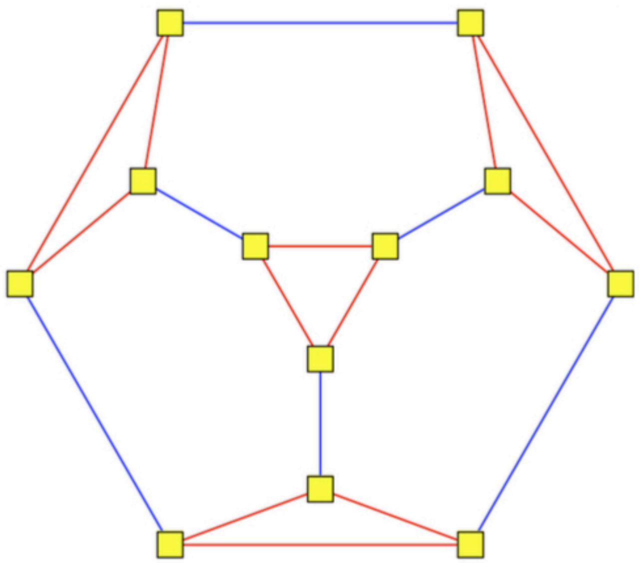

$$\mathcal{S} = \{(1,0),(-1,0),(-2,1),(0,-1)\}.$$

$$Q(1,1;t) = \Delta\left(\frac{(x^2+1)(x^2+2xy-1)(2x^3+x^2y-y)(y^2-x^2)}{x^2y(1-x)(1-y)(1-t(x^3+x^2y+xy^2+y))}\right)$$

$$[t^k]Q(1,1,t) = \frac{(2\sqrt{3})^k}{k^4}\left(C_k + O\left(\frac{1}{k}\right)\right)$$

Joint work with Alin Bostan and Mireille Bousquet-Mélou

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Walks in d dimensions symmetric over all but one axis have a uniform diagonal expression, but may need to take Laurent expansions.

$$F(t) = \Delta \left(\frac{(1+z_1)\cdots(1+z_{d-1})\left(B(\mathbf{z}_{\hat{d}}) - z_d^2 A(\mathbf{z}_{\hat{d}})\right)}{B(\mathbf{z}_{\hat{d}})(1-z_d)(1-tz_1\cdots z_d S(z_1,\ldots,z_{d-1},\overline{z}_d)} \right)$$

$$S(\mathbf{z}) = A(\mathbf{z}_{\hat{d}})\overline{z}_d + Q(\mathbf{z}_{\hat{d}}) + B(\mathbf{z}_{\hat{d}})z_d$$

Joint work with Mark Wilson

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Computational Complexity of ACSV

Walks in d dimensions symmetric over all but one axis have a uniform diagonal expression, but may need to take Laurent expansions.

$$\mathcal{S} = \{(0,1,1),(0,-1,1),(1,0,1),(-1,0,1),(0,0,-1)\}$$

$$F(t) = \Delta\left(\frac{(1+x)(1+y)(-xyz^2+x^2y+xy^2+x+y)}{(x+y)(1+xy)(1-z)(1-txyzS(x,y,\overline{z})}\right)$$

in the ring $\mathcal{R} = \mathbb{Q}((x))((y))((z))[[t]]$

Joint work with Mark Wilson

Some Ongoing Work

Dealing with longer steps

Walks in other regions

Computational Complexity of ACSV

We have the first complexity results and effective algorithms which work on many examples, but want to **relax our assumptions.**

This involves incorporating work on the effective stratification of algebraic varieties, tools from real algebraic geometry (optimization on amoeba complements), and new theoretical results on saddle point integrals.

Joint work with Bruno Salvy

Some Ongoing Work

Dealing with longer steps

Walks in other regions

Computational Complexity of ACSV

Pedagogical Intro + Computer Tools for ACSV

(See PhD thesis for gentle introduction to ACSV, effective methods, connection to other GF classes, lattice paths, other broad applications, and **preliminary code**)

Conclusion

- **Kernel method** gives generating functions as rational diagonals
- Rational diagonals provide **compact encodings**
- ACSV methods are often **effective** and give strong asymptotic results
- We **prove** (and re-discover) **guessed** lattice path **asymptotics**
- Techniques **link combinatorial** and **analytic** properties
- **Flexible** enough to generalize to a wide variety of (D-finite) problems

Thank You

Analytic Combinatorics in Several Variables: Effective Algorithms and Lattice Path Enumeration. S. Melczer. PhD Thesis, University of Waterloo and ENS Lyon. 259 pages. arXiv:1709.05051

Asymptotic lattice path enumeration using diagonals.
S. Melczer and M. Mishna.
Algorithmica, 2016. 30 pages.

Asymptotics of lattice walks via analytic combinatorics in several variables.
S. Melczer and M. C. Wilson.
DMTCS Proceedings of FPSAC 2016. 12 pages.

Weighted Lattice Walks and Universality Classes.
J. Courtiel, S. Melczer, M. Mishna, and K. Raschel.
Journal of Combinatorial Theory, Series A, 2017. 48 pages.

Hypergeometric Integrals

Bostan et al. determine dominant asymptotics of

$$\mathcal{S} = \{(0,-1),(-1,1),(1,1)\}$$

to be

$$\frac{\sqrt{3}}{2\sqrt{\pi}} \cdot I \cdot 3^k k^{-1/2}$$

where

$$I := \int_0^{1/3} \left\{ \frac{(1-3v)^{1/2}}{v^3(1+v^2)^{1/2}} \left[1 + (1-10v^3) \cdot {}_2F_1\left(\left. \begin{matrix} 3/4, 5/4 \\ 1 \end{matrix} \right| 64v^4 \right) \right. \right.$$
$$\left. \left. + 6v^3(3-8v+14v^2) \cdot {}_2F_1\left(\left. \begin{matrix} 5/4, 7/4 \\ 2 \end{matrix} \right| 64v^4 \right) \right] - \frac{2}{v^3} + \frac{4}{v^2} \right\} dv$$

Our results imply $I = 1$.

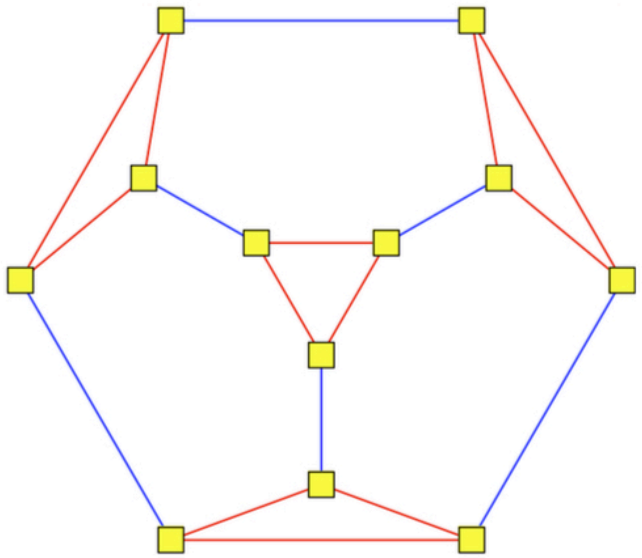
Longer Steps

$$\mathcal{S} = \{(1,0), (-1,0), (-2,1), (0,-1)\}.$$

$$K(x,y,t)Q(x,y,t) = 1 - t\overline{x}(1+\overline{x}y)Q(0,y,t) - t\overline{x}yQ_1(y,t) - t\overline{y}Q(x,0,t),$$

$$K(x,y,t) = 1 - t\sum_{(i,j)\in\mathcal{S}} x^i y^j = 1 - t(x + \overline{x} + \overline{x}^2 y + \overline{y}).$$

$$Q(1,1;t) = \Delta\left(\frac{(x^2+1)(x^2+2xy-1)(2x^3+x^2y-y)(y^2-x^2)}{x^2y(1-x)(1-y)(1-t(x^3+x^2y+xy^2+y))}\right)$$



$$[t^k]Q(1,1,t) = \frac{(2\sqrt{3})^k}{k^4}\left(C_k + O\left(\frac{1}{k}\right)\right)$$

$$C_k = \begin{cases} \frac{5616\sqrt{3}}{\pi} & : k \text{ even} \\ \frac{9720}{\pi} & : k \text{ odd} \end{cases}.$$

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Gessel and Zeilberger (1992) show how to get rational diagonal representations for walks in *Weyl chambers* satisfying certain conditions

Grabiner and Magyar (1993) characterize the step sets which can be analyzed

Highly symmetric walks in $\mathbb{N}^d$ are the walks in the chamber A_1^d