How to prove algorithmically the transcendence of D-finite power series

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Inria informatics mathematics

♦ Lattice walks ♦ Banff ♦ September 19, 2017 ♦

Algebraic and transcendental power series

In contrast with the "hard" theory of arithmetic transcendence, it is usually "easy" to establish transcendence of functions.

[Flajolet, Sedgewick, 2009]

▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *algebraic* if it is a root of some algebraic equation $P(t, f(t)) = 0$, where $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$.

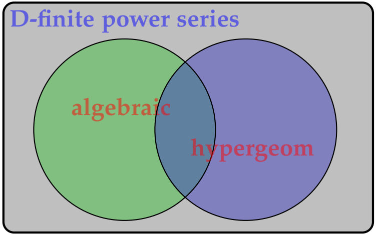
Otherwise, f is called *transcendental*.

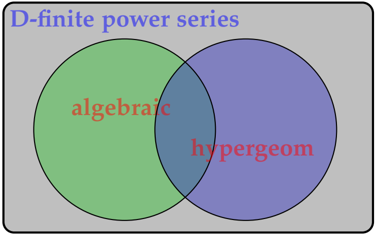
▷ Goal: Given $f \in \mathbb{Q}[[t]]$, either in explicit form (by a formula), or in implicit form (by a functional equation), determine its *algebraicity* or *transcendence*.

Motivations

- Number theory: first step towards proving the transcendence of a complex number is to prove that some power series is transcendental
- Combinatorics: nature of generating functions may reveal strong underlying structures
- Computer science: are algebraic power series (intrinsically) easier to manipulate?

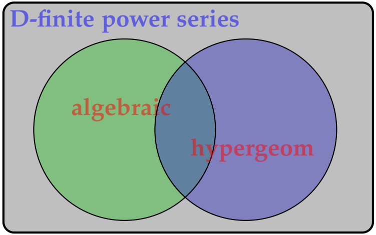
An important particular case: transcendence of hypergeometric series





$f(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Q}[[t]]$ is

▷ *algebraic* if $P(t, f(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$



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▷ *D-finite* if $c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero

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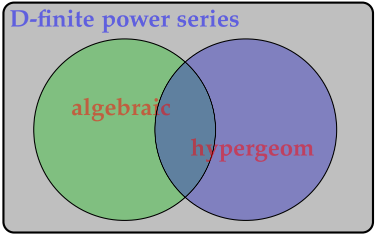

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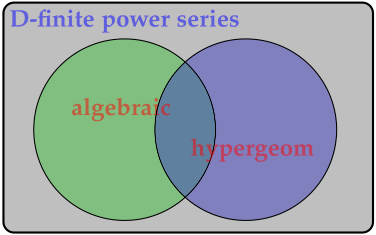

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Theorem [Schwarz, 1873; Beukers, Heckman, 1989]

Characterization of { *hypergeom* } $\cap$ { *algebraic* } $\longrightarrow$ *nice transcendence test*

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —represented by a linear differential equation with polynomial coefficients and suitable initial conditions— is transcendental, or not.

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E.g.,

$$f = \ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6} - \cdots$$

is D-finite and can be represented by the second-order equation

$$\left((t-1)\partial_t^2 + \partial_t\right)(f) = 0, \quad f(0) = 0, f'(0) = -1.$$

The algorithm should recognize that f is transcendental.

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▷ Notation: For a D-finite series f, we write $L_f^{\min}$ for its *differential resolvent*, i.e. the least order monic differential operator in $\mathbb{Q}(t)\langle\partial_t\rangle$ that cancels f.

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▷ Warning: $L_f^{\min}$ is not known a priori; only some multiple L of it is given.

▷ Difficulty: $L_f^{\min}$ might not be irreducible. E.g., $L_{\ln(1-t)}^{\min} = \left(\partial_t + \frac{1}{t-1}\right)\partial_t$.

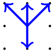
Three examples

(A) Apéry's power series [Apéry, 1978] (used in his proof of $\zeta(3) \notin \mathbb{Q}$)

$$\sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = 1 + 5\,t + 73\,t^2 + 1445\,t^3 + 33001\,t^4 + \cdots$$

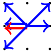
(B) GF of trident walks in the quarter plane

$$\sum_n a_n t^n = 1 + 2\,t + 7\,t^2 + 23\,t^3 + 84\,t^4 + 301\,t^5 + 1127\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

(C) GF of a quadrant model with repeated steps

$$\sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + 520\,t^6 + \cdots,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

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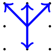
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

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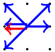
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

Question: *How to prove that these three power series are transcendental?*

Main properties of algebraic series

If $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$ is algebraic, then

- [Algebraic prop.]
  f is D-finite; $L_f^{\min}$ has a basis of algebraic solutions [Abel, 1827; Tannery, 1875]

- [Arithmetic prop.]
  f is globally bounded [Eisenstein, 1852]
  $$\exists C \in \mathbb{N}^* \text{ with } a_n C^n \in \mathbb{Z} \text{ for } n \geq 1$$

- [Analytic prop.]
  $(a_n)_n$ has "nice" asymptotics [Puiseux, 1850; Flajolet, 1987]
  Typically, $a_n \sim \kappa \, \rho^n \, n^\alpha$ with $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ and $\rho \in \overline{\mathbb{Q}}$ and $\kappa \cdot \Gamma(\alpha + 1) \in \overline{\mathbb{Q}}$

For $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$, if one of the following holds

- f is not D-finite $$\prod_n \frac{1}{1-t^n}$$
- f is not globally bounded $$\sum_n \frac{1}{n} t^n$$
- $(a_n)_n$ has incompatible asymptotics $$\sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n \ (\dagger)$$

then f is transcendental

(†) $a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}$ and $\frac{\Gamma(-1/2)}{\pi^{3/2}} = -\frac{2}{\pi} \notin \overline{\mathbb{Q}}$

Problem: Decide if *all* solutions of a given equation L of order n are algebraic

• Starting point [Jordan, 1878]: If so, then for some solution y of L, $u = y'/y$ has alg. degree at most $(49n)^{n^2}$ and satisfies a Riccati equation of order $n-1$

Algorithm (L irreducible) [Painlevé, 1887], [Boulanger, 1898], [Singer, 1979]

1. Decide if the Riccati equation has an algebraic solution u of degree at most $(49n)^{n^2}$ — degree bounds + algebraic elimination
2. (Abel's problem) Given an algebraic u, decide whether $y'/y = u$ has an algebraic solution y — [Risch 1970], [Baldassarri & Dwork 1979]

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▷ [Singer, 2014]: computation of L^{alg}, the factor of L whose solution space is spanned by all algebraic solutions of L $\longrightarrow$ requires ODE factoring

Problem: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by a differential equation $L(f) = 0$ and sufficiently many initial terms, is transcendental.

1. Compute L^{alg} [Singer, 2014]
2. Decide if L^{alg} annihilates f

▷ Benefit: Solves (in principle) Stanley's problem.

▷ Drawbacks: Step 1 involves *impractical bounds* & *requires ODE factorization*

▷ ODE factorization is effective
[Schlesinger, 1897], [Singer, 1981], [Grigoriev, 1990], [van Hoeij, 1997]

▷ …but possibly extremely costly [Grigoriev, 1990] $\exp\left((\text{bitsize}(L)2^n)^{2^n}\right)$

New method: the basic idea

Problem: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by a differential equation $L(f) = 0$ and sufficiently many initial terms, is transcendental.

Basic remark: If $L_f^{\min}$ has a logarithmic singularity, then f is transcendental.

$\triangleright$ Pros and cons: Avoids factorization of L, but requires to compute $L_f^{\min}$.

Ex. (A): Apéry's power series

$$f(t) = \sum_n a_n t^n, \quad \text{where } a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

▷ Creative telescoping:

$$(n+1)^3 a_n - (2n+3)(17n^2+51n+39)a_{n+1} + (n+2)^3 a_{n+2} = 0, \quad a_0 = 1,\ a_1 = 5$$

▷ Conversion from recurrence to differential equation $L(f) = 0$, where

$$L = (t^4 - 34t^3 + t^2)\partial_t^3 + (6t^3 - 153t^2 + 3t)\partial_t^2 + (7t^2 - 112t + 1)\partial_t + t - 5$$

▷ $L_f^{\min} = \frac{1}{t^4 - 34t^3 + t^2} L$ using L irreducible, or cf. new algorithm

▷ Basis of formal solutions of $L_f^{\min}$ at $t = 0$:

$$\left\{ 1 + 5t + O(t^2),\ \ln(t) + (5\ln(t) + 12)t + O(t^2),\ \ln(t)^2 + (5\ln(t)^2 + 24\ln(t))t + O(t^2) \right\}$$

▷ Conclusion: f is transcendental

	OEIS	$\mathfrak{S}$	nature	ODE size		OEIS	$\mathfrak{S}$	nature	ODE size
1	A005566		T	(3, 4)	13	A151275		T	(5, 24)
2	A018224		T	(3, 5)	14	A151314		T	(5, 24)
3	A151312		T	(3, 8)	15	A151255		T	(4, 16)
4	A151331		T	(3, 6)	16	A151287		T	(5, 19)
5	A151266		T	(5, 16)	17	A001006		A	(2, 3)
6	A151307		T	(5, 20)	18	A129400		A	(2, 3)
7	A151291		T	(5, 15)	19	A005558		T	(3, 5)
8	A151326		T	(5, 18)					
9	A151302		T	(5, 24)	20	A151265		A	(4, 9)
10	A151329		T	(5, 24)	21	A151278		A	(4, 12)
11	A151261		T	(4, 15)	22	A151323		A	(2, 3)
12	A151297		T	(5, 18)	23	A060900		A	(3, 5)

▷ Computer-driven discovery and proof; no human proof yet

▷ Proof uses creative telescoping, ODE factorization, Singer's algorithm

▷ For models 5–10, asymptotics do not conclude. E.g. $a_n \sim \frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$

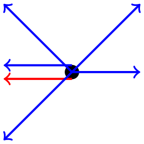
Ex. (B): D-Finite quadrant models [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

	OEIS	$\mathfrak{S}$	nature	asympt		OEIS	$\mathfrak{S}$	nature	asympt
1	A005566		T	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		T	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		T	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		T	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		T	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		T	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		T	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		T	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		T	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		A	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		T	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		A	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		T	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		T	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		T	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
9	A151302		T	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		A	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		T	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		A	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		T	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		A	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		T	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		A	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

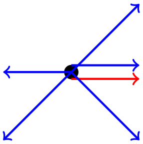
$$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

▷ Asymptotics guessed by [B., Kauers '09], proved by [Melczer, Wilson '15]

Ex. (C): two difficult quadrant models with repeated steps



Case A



Case B

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

- GF is D-finite and transcendental in Case A.
- GF is algebraic in Case B.

▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses Guess'n'Prove and new algorithm for transcendence.
▷ All other criteria and algorithms fail or do not terminate.

The new method: a first version

Input: $f(t) \in \mathbb{Q}[[t]]$, given as the generating function of a binomial sum
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

1. Compute an ODE L for $f(t)$ — Creative telescoping
2. Compute $L_f^{\min}$ — degree bounds + diff. Hermite-Padé
3. Decide if $L_f^{\min}$ has only algebraic solutions; if so return A, else return T. [Singer, 1979]

▷ Drawback: Step 3 can be very costly in practice.

The new method: an efficient version

Input: $f(t) \in \mathbb{Q}[[t]]$, given as the generating function of a binomial sum
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

1. Compute an ODE L for $f(t)$ — Creative telescoping
2. Compute $L_f^{\min}$ — degree bounds + diff. Hermite-Padé
3. If $L_f^{\min}$ has a logarithmic singularity, return T; otherwise return A

▷ This algorithm is always correct when it returns T; *conjecturally*, it is also always correct when it returns A

▷ Using p-curvatures and the Grothendieck-Katz conjecture (proved by [Katz, 1972] for *Picard-Fuchs systems*) yields an *unconditional* algorithm.

Central sub-task: computation of $L_f^{\min}$

Problem: Given a D-finite power series $f \in \mathbb{Q}[[t]]$ by a differential equation $L(f) = 0$ and sufficiently many initial terms, compute its resolvent $L_f^{\min}$.

▷ Why isn't this easy? After all, it is just a differential analogue of:

Given an algebraic power series $f \in \mathbb{Q}[[t]]$ by an algebraic equation $P(t, f) = 0$ and sufficiently many initial terms, compute its minimal polynomial P_f^{min}.

▷ $L_f^{\min}$ is a factor of L, but contrary to the commutative case:

- factorization of diff. operators is not unique $\quad \partial_t^2 = (\partial_t + \frac{1}{t-c})(\partial_t - \frac{1}{t-c})$
- ... and it is difficult to compute
- $\deg_t L_f^{\min} \gg \deg_t L$, due to apparent singularities $\quad t\partial_t - N \mid \partial_t^{N+1}$

Central sub-task: computation of $L_f^{\min}$

▷ Strategy (inspired by the approach in [van Hoeij, 1997], itself based on ideas from [Chudnovsky, 1980], [Bertrand & Beukers, 1982], [Ohtsuki, 1982])

1. $L_f^{\min}$ is Fuchsian, so it can be written

$$L_f^{\min} = \partial_t^n + \frac{a_{n-1}(t)}{A(t)}\partial_t^{n-1} + \cdots + \frac{a_0(t)}{A(t)^n}, \qquad n \leq \operatorname{ord}(L)$$

   with $A(t)$ squarefree and $\deg(a_{n-i}) \leq \deg(A^i) - i$.

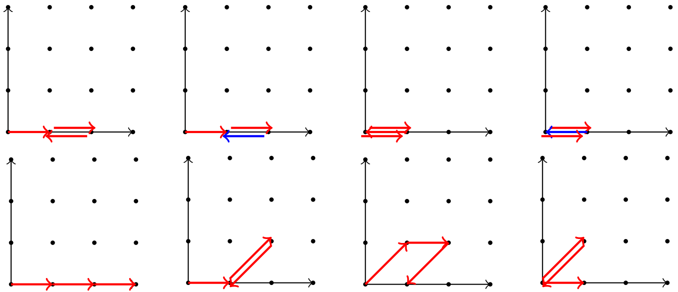
2. $\deg(A)$ can be bounded in terms of n and (local) data of L (via *apparent singularities* and *Fuchs' relation*)

3. Guess and Prove: For $n = 1, 2, \ldots,$
   1. Guess differential equation of order n for f (use bounds and linear algebra)
   2. Once found a nontrivial candidate, certify it, or go to previous step.

Ex. (C): a difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\left\{ \text{—walks of length } n \text{ in } \mathbb{N}^2 \text{ from } (0,0) \text{ to } (\star,0) \right\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.



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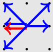
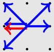
Let $a_n = \#$ $\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)$ $\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation L for $f(t)$ of order 11 and degree 73 — high-tech Guess'n'Prove
2. If $\mathrm{ord}(L_f^{\min}) \leq 10$, then $\deg_t(L_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $L_f^{\min} = L$
5. L has a log singularity at $t = 0$, and so f is transcendental $\square$

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

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3. Rule out this possibility [Beckermann, Labahn, 1994]
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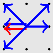
Conclusions

- Simple, efficient and robust algorithmic method for transcendence
- Central sub-task: computation of $L_f^{\min}$ $\longrightarrow$ useful in other contexts!
- Basic theoretical tool: Fuchs' relation
- Basic algorithmic tool: Guess'n'Prove via Hermite-Padé approximants + efficient computer algebra
- Brute-force / naive algorithms = hopeless on combinatorial examples

Open problem

Find a *human proof* for the following statement

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \# \Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(0,0)\Big\}$

$$(a_n)_{n \geq 0} = (1,\ 0,\ 3,\ 0,\ 26,\ 0,\ 323,\ 0,\ 4830,\ 0,\ 80910, \ldots)$$

Then

$$a_{2n} = \frac{6(6n+1)!(2n+1)!}{(3n)!(4n+3)!(n+1)!}.$$

Thanks for your attention!