Counting planar Eulerian orientations.

Tony Guttmann – joint work with Andrew Elvey Price

School of Mathematics and Statistics
The University of Melbourne, Australia

Banff, BIRS, September 2017

MAPS, EULERIAN MAPS AND ORIENTATIONS

- A *planar map* is a proper embedding of a connected graph on the surface of a sphere.
- An *Eulerian map* is a planar map with all vertices of even degree.
- An *Eulerian orientation* is an oriented Eulerian map in which each vertex has equal in-degree and out-degree
- A *4-valent Eulerian orientation* has each vertex obeying the ice-rule, so is in the 6-vertex model universality class
- This problem was studied by Kostov and Zinn-Justin (2000). Kostov gives

$$\log Z \sim \frac{c(T-T_c)^2}{\log(T-T_c)}.$$

- For Eulerian maps the o.g.f. corresponds to the derivative of $\log Z$, so should behave as $(T-T_c)/\log(T-T_c)$.

Maps, Eulerian maps and orientations

- A *planar map* is a proper embedding of a connected graph on the surface of a sphere.
- An *Eulerian map* is a planar map with all vertices of even degree.
- An *Eulerian orientation* is an oriented Eulerian map in which each vertex has equal in-degree and out-degree
- A *4-valent Eulerian orientation* has each vertex obeying the ice-rule, so is in the 6-vertex model universality class
- This problem was studied by Kostov and Zinn-Justin (2000). Kostov gives

$$\log Z \sim \frac{c(T-T_c)^2}{\log(T-T_c)}.$$

- For Eulerian maps the o.g.f. corresponds to the derivative of $\log Z$, so should behave as $(T-T_c)/\log(T-T_c)$.

Maps, Eulerian maps and orientations

- A *planar map* is a proper embedding of a connected graph on the surface of a sphere.
- An *Eulerian map* is a planar map with all vertices of even degree.
- An *Eulerian orientation* is an oriented Eulerian map in which each vertex has equal in-degree and out-degree
- A *4-valent Eulerian orientation* has each vertex obeying the ice-rule, so is in the 6-vertex model universality class
- This problem was studied by Kostov and Zinn-Justin (2000). Kostov gives

$$\log Z \sim \frac{c(T-T_c)^2}{\log(T-T_c)}.$$

- For Eulerian maps the o.g.f. corresponds to the derivative of $\log Z$, so should behave as $(T-T_c)/\log(T-T_c)$.

Maps, Eulerian maps and orientations

- A *planar map* is a proper embedding of a connected graph on the surface of a sphere.
- An *Eulerian map* is a planar map with all vertices of even degree.
- An *Eulerian orientation* is an oriented Eulerian map in which each vertex has equal in-degree and out-degree
- A *4-valent Eulerian orientation* has each vertex obeying the ice-rule, so is in the 6-vertex model universality class
- This problem was studied by Kostov and Zinn-Justin (2000). Kostov gives

$$\log Z \sim \frac{c(T-T_c)^2}{\log(T-T_c)}.$$

- For Eulerian maps the o.g.f. corresponds to the derivative of $\log Z$, so should behave as $(T-T_c)/\log(T-T_c)$.

Maps, Eulerian maps and orientations

- A *planar map* is a proper embedding of a connected graph on the surface of a sphere.
- An *Eulerian map* is a planar map with all vertices of even degree.
- An *Eulerian orientation* is an oriented Eulerian map in which each vertex has equal in-degree and out-degree
- A *4-valent Eulerian orientation* has each vertex obeying the ice-rule, so is in the 6-vertex model universality class
- This problem was studied by Kostov and Zinn-Justin (2000). Kostov gives

$$\log Z \sim \frac{c(T-T_c)^2}{\log(T-T_c)}.$$

- For Eulerian maps the o.g.f. corresponds to the derivative of $\log Z$, so should behave as $(T-T_c)/\log(T-T_c)$.

Maps, Eulerian maps and orientations

- A *planar map* is a proper embedding of a connected graph on the surface of a sphere.
- An *Eulerian map* is a planar map with all vertices of even degree.
- An *Eulerian orientation* is an oriented Eulerian map in which each vertex has equal in-degree and out-degree
- A *4-valent Eulerian orientation* has each vertex obeying the ice-rule, so is in the 6-vertex model universality class
- This problem was studied by Kostov and Zinn-Justin (2000). Kostov gives

$$\log Z \sim \frac{c(T-T_c)^2}{\log(T-T_c)}.$$

- For Eulerian maps the o.g.f. corresponds to the derivative of $\log Z$, so should behave as $(T-T_c)/\log(T-T_c)$.

Eulerian orientations

- The generating function for the number of rooted planar Eulerian maps is just (Tutte, 1963)

$$M(t) = \frac{8t^2 + 12t - 1 + (1-8t)^{3/2}}{32t^2}.$$

- Planar maps with additional structure are much studied in both enumerative combinatorics and mathematical physics.
- Eulerian orientations have recently been studied by Bonichon, Bousquet-Mélou, Dorbec and Pennarun (Europ. J. Combin. 65 (2017) 59-91.)
- They focus on Eulerian orientations counted by edges, and on 4-valent Eulerian orientations counted by vertices.

EULERIAN ORIENTATIONS

- The generating function for the number of rooted planar Eulerian maps is just (Tutte, 1963)

$$M(t) = \frac{8t^2 + 12t - 1 + (1-8t)^{3/2}}{32t^2}.$$

- Planar maps with additional structure are much studied in both enumerative combinatorics and mathematical physics.
- Eulerian orientations have recently been studied by Bonichon, Bousquet-Mélou, Dorbec and Pennarun (Europ. J. Combin. 65 (2017) 59-91.)
- They focus on Eulerian orientations counted by edges, and on 4-valent Eulerian orientations counted by vertices.

Eulerian orientations

- The generating function for the number of rooted planar Eulerian maps is just (Tutte, 1963)

$$M(t) = \frac{8t^2 + 12t - 1 + (1-8t)^{3/2}}{32t^2}.$$

- Planar maps with additional structure are much studied in both enumerative combinatorics and mathematical physics.
- Eulerian orientations have recently been studied by Bonichon, Bousquet-Mélou, Dorbec and Pennarun (Europ. J. Combin. 65 (2017) 59-91.)
- They focus on Eulerian orientations counted by edges, and on 4-valent Eulerian orientations counted by vertices.

Eulerian orientations

- The generating function for the number of rooted planar Eulerian maps is just (Tutte, 1963)

$$M(t) = \frac{8t^2 + 12t - 1 + (1-8t)^{3/2}}{32t^2}.$$

- Planar maps with additional structure are much studied in both enumerative combinatorics and mathematical physics.
- Eulerian orientations have recently been studied by Bonichon, Bousquet-Mélou, Dorbec and Pennarun (Europ. J. Combin. 65 (2017) 59-91.)
- They focus on Eulerian orientations counted by edges, and on 4-valent Eulerian orientations counted by vertices.

Figure shows a rooted Eulerian map and a rooted Eulerian orientation.

Approach of Bonichon et al.

- They count by generating a sequence of subsets and supersets, indexed by k, each of which have algebraic o.g.fs.
- At order k each subset gives coefficients correct to order $O(z^{k+2})$, and in this way they calculate the first 15 terms.
- This approach is of exponential complexity, and they found the growth constant $11.56 \leq \mu \leq 13.005$ and is around 12.5.
- We have developed a polynomial-time algorithm, and used this to generate 100 terms in the generating functions.
- Analysing these series, we conjecture that the growth constant for Eulerian orientations counted by edges is exactly 4π, and for 4-valent Eulerian orientations counted by vertices is $4\sqrt{3}\pi$.
- Both behave as $A(1-\mu z)/\log(1-\mu z)$.

Approach of Bonichon et al.

- They count by generating a sequence of subsets and supersets, indexed by k, each of which have algebraic o.g.fs.
- At order k each subset gives coefficients correct to order $O(z^{k+2})$, and in this way they calculate the first 15 terms.
- This approach is of exponential complexity, and they found the growth constant $11.56 \leq \mu \leq 13.005$ and is around 12.5.
- We have developed a polynomial-time algorithm, and used this to generate 100 terms in the generating functions.
- Analysing these series, we conjecture that the growth constant for Eulerian orientations counted by edges is exactly 4π, and for 4-valent Eulerian orientations counted by vertices is $4\sqrt{3}\pi$.
- Both behave as $A(1-\mu z)/\log(1-\mu z)$.

Approach of Bonichon et al.

- They count by generating a sequence of subsets and supersets, indexed by k, each of which have algebraic o.g.fs.
- At order k each subset gives coefficients correct to order $O(z^{k+2})$, and in this way they calculate the first 15 terms.
- This approach is of exponential complexity, and they found the growth constant $11.56 \leq \mu \leq 13.005$ and is around 12.5.
- We have developed a polynomial-time algorithm, and used this to generate 100 terms in the generating functions.
- Analysing these series, we conjecture that the growth constant for Eulerian orientations counted by edges is exactly 4π, and for 4-valent Eulerian orientations counted by vertices is $4\sqrt{3}\pi$.
- Both behave as $A(1-\mu z)/\log(1-\mu z)$.

Approach of Bonichon et al.

- They count by generating a sequence of subsets and supersets, indexed by k, each of which have algebraic o.g.fs.
- At order k each subset gives coefficients correct to order $O(z^{k+2})$, and in this way they calculate the first 15 terms.
- This approach is of exponential complexity, and they found the growth constant $11.56 \leq \mu \leq 13.005$ and is around 12.5.
- We have developed a polynomial-time algorithm, and used this to generate 100 terms in the generating functions.
- Analysing these series, we conjecture that the growth constant for Eulerian orientations counted by edges is exactly 4π, and for 4-valent Eulerian orientations counted by vertices is $4\sqrt{3}\pi$.
- Both behave as $A(1-\mu z)/\log(1-\mu z)$.

Approach of Bonichon et al.

- They count by generating a sequence of subsets and supersets, indexed by k, each of which have algebraic o.g.fs.
- At order k each subset gives coefficients correct to order $O(z^{k+2})$, and in this way they calculate the first 15 terms.
- This approach is of exponential complexity, and they found the growth constant $11.56 \leq \mu \leq 13.005$ and is around 12.5.
- We have developed a polynomial-time algorithm, and used this to generate 100 terms in the generating functions.
- Analysing these series, we conjecture that the growth constant for Eulerian orientations counted by edges is exactly 4π, and for 4-valent Eulerian orientations counted by vertices is $4\sqrt{3}\pi$.
- Both behave as $A(1-\mu z)/\log(1-\mu z)$.

Approach of Bonichon et al.

- They count by generating a sequence of subsets and supersets, indexed by k, each of which have algebraic o.g.fs.
- At order k each subset gives coefficients correct to order $O(z^{k+2})$, and in this way they calculate the first 15 terms.
- This approach is of exponential complexity, and they found the growth constant $11.56 \le \mu \le 13.005$ and is around 12.5.
- We have developed a polynomial-time algorithm, and used this to generate 100 terms in the generating functions.
- Analysing these series, we conjecture that the growth constant for Eulerian orientations counted by edges is exactly 4π, and for 4-valent Eulerian orientations counted by vertices is $4\sqrt{3}\pi$.
- Both behave as $A(1-\mu z)/\log(1-\mu z)$.

Approach of Elvey Price and G.

- Let $U(x)$ be the ogf for Eulerian orientations, counted by edges. We find a system of functional equations which gives the generating function $U(x)$.
- Similarly, we find a system of functional equations for the ogf $A(x)$ for 4-valent planar Eulerian orientations, counted by vertices.
- In each case these functional equations give rise to a polynomial time algorithm for computing the coefficients.
- The key idea is to define a class of maps in bijection with Eulerian orientations, and count the maps.

Approach of Elvey Price and G.

- Let $U(x)$ be the ogf for Eulerian orientations, counted by edges. We find a system of functional equations which gives the generating function $U(x)$.
- Similarly, we find a system of functional equations for the ogf $A(x)$ for 4-valent planar Eulerian orientations, counted by vertices.
- In each case these functional equations give rise to a polynomial time algorithm for computing the coefficients.
- The key idea is to define a class of maps in bijection with Eulerian orientations, and count the maps.

Approach of Elvey Price and G.

- Let $U(x)$ be the ogf for Eulerian orientations, counted by edges. We find a system of functional equations which gives the generating function $U(x)$.
- Similarly, we find a system of functional equations for the ogf $A(x)$ for 4-valent planar Eulerian orientations, counted by vertices.
- In each case these functional equations give rise to a polynomial time algorithm for computing the coefficients.
- The key idea is to define a class of maps in bijection with Eulerian orientations, and count the maps.

Approach of Elvey Price and G.

- Let $U(x)$ be the ogf for Eulerian orientations, counted by edges. We find a system of functional equations which gives the generating function $U(x)$.
- Similarly, we find a system of functional equations for the ogf $A(x)$ for 4-valent planar Eulerian orientations, counted by vertices.
- In each case these functional equations give rise to a polynomial time algorithm for computing the coefficients.
- The key idea is to define a class of maps in bijection with Eulerian orientations, and count the maps.

Approach of Elvey Price and G.

- Define a *numbered* rooted planar map (N-map) to be a rooted planar map in which each vertex is labelled by a number such that the root vertex is labelled by 0 and any two vertices which are joined by an edge have labels differing by 1.
- Planar rooted Eulerian orientations with n edges are in bijection with N-maps with n edges, so that we just need to count N-maps by edges.

Proposition

For any positive integer n, the number of N-maps with n edges equals the number of rooted orientations with n edges. Also, the number of N-maps with n edges, where each face has degree 4 is equal to the number of 4-valent rooted orientations with n edges.

Approach of Elvey Price and G.

- Define a *numbered* rooted planar map (N-map) to be a rooted planar map in which each vertex is labelled by a number such that the root vertex is labelled by 0 and any two vertices which are joined by an edge have labels differing by 1.
- Planar rooted Eulerian orientations with n edges are in bijection with N-maps with n edges, so that we just need to count N-maps by edges.

Proposition

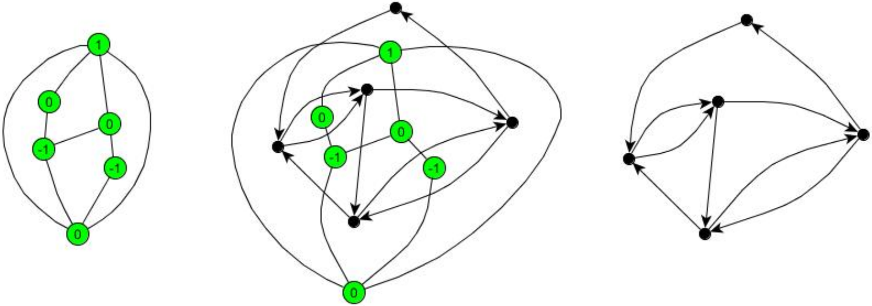
For any positive integer n, the number of N-maps with n edges equals the number of rooted orientations with n edges. Also, the number of N-maps with n edges, where each face has degree 4 is equal to the number of 4-valent rooted orientations with n edges.

Approach of Elvey Price and G.

- Define a *numbered* rooted planar map (N-map) to be a rooted planar map in which each vertex is labelled by a number such that the root vertex is labelled by 0 and any two vertices which are joined by an edge have labels differing by 1.
- Planar rooted Eulerian orientations with n edges are in bijection with N-maps with n edges, so that we just need to count N-maps by edges.

Proposition

For any positive integer n, the number of N-maps with n edges equals the number of rooted orientations with n edges. Also, the number of N-maps with n edges, where each face has degree 4 is equal to the number of 4-valent rooted orientations with n edges.



An example of the transformation between an N-map (left of diagram) and the corresponding Eulerian orientation (right of diagram).

Proof

Proof.

Given an N-map, construct a directed map by orienting each edge from the lower number to the higher number. Around each face, the number of clockwise edges is equal to the number of anticlockwise edges. Hence the dual of this map (where the orientations of the edges are defined by rotating the original edges 90° clockwise) is an Eulerian orientation. Reversing each of these steps shows that that this transformation is bijective. Hence, the number of N-maps with n edges is equal to the number of rooted Eulerian orientations with n edges. Using the same bijection, the number of 4-valent rooted Eulerian orientations with n edges is equal to the number of N-maps with n edges, where each face has degree 4. $\square$

THE 4-VALENT CASE

- We define an operation called a *contraction* on some vertices, some of which are *highlighted*.
- The root-0 vertex, v_0, is the only contracted vertex.
- We need to define three catalytic variables:
- a is conjugate to the number of highlighted corners in those maps in which the root-0 vertex, v_0, is the only contracted vertex.
- b is conjugate to the degree of the adjacent root-1 vertex v_1.
- c is conjugate to the half-degree of the outer face.
- x is conjugate to the number of edges.

The 4-valent case

- We define an operation called a *contraction* on some vertices, some of which are *highlighted.*
- The root-0 vertex, v_0, is the only contracted vertex.
- We need to define three catalytic variables:
- a is conjugate to the number of highlighted corners in those maps in which the root-0 vertex, v_0, is the only contracted vertex.
- b is conjugate to the degree of the adjacent root-1 vertex v_1.
- c is conjugate to the half-degree of the outer face.
- x is conjugate to the number of edges.

The 4-valent case

- We define an operation called a *contraction* on some vertices, some of which are *highlighted.*
- The root-0 vertex, v_0, is the only contracted vertex.
- We need to define three catalytic variables:
- a is conjugate to the number of highlighted corners in those maps in which the root-0 vertex, v_0, is the only contracted vertex.
- b is conjugate to the degree of the adjacent root-1 vertex v_1.
- c is conjugate to the half-degree of the outer face.
- x is conjugate to the number of edges.

THE 4-VALENT CASE

- We define an operation called a *contraction* on some vertices, some of which are *highlighted.*
- The root-0 vertex, v_0, is the only contracted vertex.
- We need to define three catalytic variables:
- a is conjugate to the number of highlighted corners in those maps in which the root-0 vertex, v_0, is the only contracted vertex.
- b is conjugate to the degree of the adjacent root-1 vertex v_1.
- c is conjugate to the half-degree of the outer face.
- x is conjugate to the number of edges.

The 4-valent case

- We define an operation called a *contraction* on some vertices, some of which are *highlighted.*
- The root-0 vertex, v_0, is the only contracted vertex.
- We need to define three catalytic variables:
- a is conjugate to the number of highlighted corners in those maps in which the root-0 vertex, v_0, is the only contracted vertex.
- b is conjugate to the degree of the adjacent root-1 vertex v_1.
- c is conjugate to the half-degree of the outer face.
- x is conjugate to the number of edges.

The 4-valent case

- We define an operation called a *contraction* on some vertices, some of which are *highlighted*.
- The root-0 vertex, v_0, is the only contracted vertex.
- We need to define three catalytic variables:
- a is conjugate to the number of highlighted corners in those maps in which the root-0 vertex, v_0, is the only contracted vertex.
- b is conjugate to the degree of the adjacent root-1 vertex v_1.
- c is conjugate to the half-degree of the outer face.
- x is conjugate to the number of edges.

The 4-valent case

- We define an operation called a *contraction* on some vertices, some of which are *highlighted.*
- The root-0 vertex, v_0, is the only contracted vertex.
- We need to define three catalytic variables:
- a is conjugate to the number of highlighted corners in those maps in which the root-0 vertex, v_0, is the only contracted vertex.
- b is conjugate to the degree of the adjacent root-1 vertex v_1.
- c is conjugate to the half-degree of the outer face.
- x is conjugate to the number of edges.

These series are characterised by the following system of equations:

$$G(x,b,c) = 1 + \Lambda_z(P(x,z,b,c)),$$

$$J(x,c) = G(x,1,c),$$

$$\begin{aligned} P(x,a,b,c) = {} & x^2b^2\frac{P(x,a,b,c) - P(x,a,1,c)}{b-1} \\ & + xbP(x,a,b,c)(a + 2[c^1]G(x,b,c)) \\ & + xbc(1 + P(x,a,1,c))G(x,b,c), \end{aligned}$$

$$\Lambda_z\left(z^n\right) = [c^n]J(x,c) \text{ for } n \geq 0.$$

The generating function $K(x)$ is given by the equation

$$K(x) = \frac{1}{x}[c^1]J(x,c).$$

Since there are only finitely many such maps with any given number of vertices, each of these generating functions is a series in x where each coefficient is a polynomial in the other variables. The first few terms of each series are as follows:

$$J(x,c) = 1 + cx + 2c^2x^2 + (4c + 5c^3)x^3 + \ldots$$

$$G(x,b,c) = 1 + cbx + (bc^2 + b^2c^2)x^2 + (2b^2c + 2b^3c + 2bc^3 + 2b^2c^3 + b^3c^3)x^3 + \ldots$$

$$P(x,a,b,c) = bcx + (ab^2c + bc^2 + b^2c^2)x^2 + x^3(a^2b^3c + ab^3c^2 + ab^2c^2 + abc^2 + b^3c^3 + 2b^3c + 2b^2c^3 + b^2c + 2bc^3)$$

Calculating the coefficients from these equations

- We use a dynamic program to calculate the coefficients in polynomial time.
- For 4-valent Eulerian orientations counted by vertices, if we calculate the coefficient of x^n in each of the functions $P, \; G, \; J$ in that order, for $n = 0, 1, 2, \ldots$, then each coefficient is determined only by values which have been previously calculated.
- The coefficients were calculated *modulo* a prime smaller than 2^{31}, repeated for several different primes, sufficient to calculate the coefficient by use of the Chinese Remainder Theorem.
- We calculated 90 terms of the o.g.f. for Eulerian orientations counted by edges $U(x)$, and 100 terms for the o.g.f. for 4-valent Eulerian orientations counted by vertices, $A(x)$.

Calculating the coefficients from these equations

- We use a dynamic program to calculate the coefficients in polynomial time.
- For 4-valent Eulerian orientations counted by vertices, if we calculate the coefficient of x^n in each of the functions $P, \; G, \; J$ in that order, for $n = 0, 1, 2, \ldots$, then each coefficient is determined only by values which have been previously calculated.
- The coefficients were calculated *modulo* a prime smaller than 2^{31}, repeated for several different primes, sufficient to calculate the coefficient by use of the Chinese Remainder Theorem.
- We calculated 90 terms of the o.g.f. for Eulerian orientations counted by edges $U(x)$, and 100 terms for the o.g.f. for 4-valent Eulerian orientations counted by vertices, $A(x)$.

Calculating the coefficients from these equations

- We use a dynamic program to calculate the coefficients in polynomial time.
- For 4-valent Eulerian orientations counted by vertices, if we calculate the coefficient of x^n in each of the functions $P,\ G,\ J$ in that order, for $n = 0, 1, 2, \dots$, then each coefficient is determined only by values which have been previously calculated.
- The coefficients were calculated *modulo* a prime smaller than 2^{31}, repeated for several different primes, sufficient to calculate the coefficient by use of the Chinese Remainder Theorem.
- We calculated 90 terms of the o.g.f. for Eulerian orientations counted by edges $U(x)$, and 100 terms for the o.g.f. for 4-valent Eulerian orientations counted by vertices, $A(x)$.

Calculating the coefficients from these equations

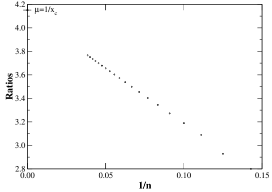
- We use a dynamic program to calculate the coefficients in polynomial time.
- For 4-valent Eulerian orientations counted by vertices, if we calculate the coefficient of x^n in each of the functions $P,\ G,\ J$ in that order, for $n = 0, 1, 2, \ldots,$ then each coefficient is determined only by values which have been previously calculated.
- The coefficients were calculated *modulo* a prime smaller than 2^{31}, repeated for several different primes, sufficient to calculate the coefficient by use of the Chinese Remainder Theorem.
- We calculated 90 terms of the o.g.f. for Eulerian orientations counted by edges $U(x)$, and 100 terms for the o.g.f. for 4-valent Eulerian orientations counted by vertices, $A(x)$.

Series analysis 101. Ratio method.

- $$F(z) = \sum_n c_n \cdot z^n \sim C \cdot (1 - z/z_c)^{-\gamma},$$

- $$c_n \sim \frac{C}{\Gamma(\gamma)} \cdot z_c^{-n} \cdot n^{\gamma - 1}.$$

- $r_n = \frac{c_n}{c_{n-1}} = \frac{1}{z_c}\left(1 + \frac{\gamma - 1}{n} + o(\frac{1}{n})\right).$



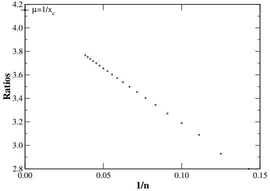
Ratios vs. $1/n$ for triang. SAPs. Inter/grad $1/z_c$, γ

Series analysis 101. Ratio method.

- $$F(z) = \sum_n c_n \cdot z^n \sim C \cdot (1 - z/z_c)^{-\gamma},$$

- $$c_n \sim \frac{C}{\Gamma(\gamma)} \cdot z_c^{-n} \cdot n^{\gamma - 1}.$$

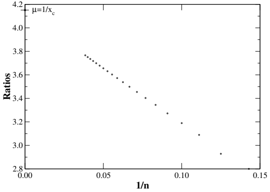
- $r_n = \frac{c_n}{c_{n-1}} = \frac{1}{z_c}\left(1 + \frac{\gamma-1}{n} + \mathrm{o}(\frac{1}{n})\right).$



Ratios vs. $1/n$ for triang. SAPs. Inter/grad $1/z_c$, γ

SERIES ANALYSIS 101. RATIO METHOD.

- $$F(z) = \sum_n c_n \cdot z^n \sim C \cdot (1 - z/z_c)^{-\gamma},$$
- $$c_n \sim \frac{C}{\Gamma(\gamma)} \cdot z_c^{-n} \cdot n^{\gamma-1}.$$
- $r_n = \frac{c_n}{c_{n-1}} = \frac{1}{z_c}\left(1 + \frac{\gamma-1}{n} + \mathrm{o}(\frac{1}{n})\right).$



Ratios vs. $1/n$ for triang. SAPs. Inter/grad $1/z_c$, γ

Series analysis 101. Differential approximants.

- $\sum_{k=0}^{M} Q_k(z)(z\frac{\mathrm{d}}{\mathrm{d}z})^k \tilde{F}(z) = P(z)$
- The singularities of $\tilde{F}(z)$ are approximated by zeros $z_i,\ i = 1, \ldots, N_M$ of $Q_M(z)$.
- Critical exponents γ_i from the indicial equation. If only a single root at z_i,

$$\gamma_i = M - 1 - \frac{Q_{M-1}(z_i)}{z_i Q'_M(z_i)}.$$

Series analysis 101. Differential approximants.

- $\sum_{k=0}^{M} Q_k(z)(z\frac{\mathrm{d}}{\mathrm{d}z})^k \tilde{F}(z) = P(z)$
- The singularities of $\tilde{F}(z)$ are approximated by zeros $z_i,\ i = 1, \ldots, N_M$ of $Q_M(z)$.
- Critical exponents γ_i from the indicial equation. If only a single root at z_i,

$$\gamma_i = M - 1 - \frac{Q_{M-1}(z_i)}{z_i Q'_M(z_i)}.$$

Series analysis 101. Differential approximants.

- $\sum_{k=0}^{M} Q_k(z)(z\frac{\mathrm{d}}{\mathrm{d}z})^k \tilde{F}(z) = P(z)$
- The singularities of $\tilde{F}(z)$ are approximated by zeros $z_i,\ i = 1, \ldots, N_M$ of $Q_M(z)$.
- Critical exponents γ_i from the indicial equation. If only a single root at z_i,
$$\gamma_i = M - 1 - \frac{Q_{M-1}(z_i)}{z_i Q'_M(z_i)}.$$

Example of differential approximants.

Critical point and exponent estimates for self-avoiding polygons. Numbers in parentheses give the uncertainty in the last quoted digits.

L	Second order DA		Third order DA	
	x_c^2	$2-\alpha$	x_c^2	$2-\alpha$
0	0.29289321854(19)	1.50000065(41)	0.29289321865(12)	1.50000040(28)
5	0.29289321875(21)	1.50000010(59)	0.29289321852(48)	1.50000041(99)
10	0.29289321855(23)	1.50000060(48)	0.29289321878(32)	1.49999999(97)
15	0.29289321859(19)	1.50000054(43)	0.29289321861(37)	1.50000035(67)
20	0.29289321866(15)	1.50000038(33)	0.29289321860(21)	1.50000049(43)

ANALYSING EULERIAN ORIENTATION SERIES.

Analysing our series by the method of differential approximants (DAs), we assumed a power-law singularity of the form

$$f(x) \sim C(1 - x/x_c)^{\alpha}.$$

For $U(x)$ we found the closest singularity to the origin to be at $x_c \approx 0.07957736$, with an exponent around $\alpha \approx 1.24$. However there was a second very close singularity at $x \approx 0.0795789$, with an exponent around 2.26, and a third, less precisely located singularity at around $x \approx 0.0798$. $1/4\pi = 0.0795774 \cdots$.
This behaviour, where one has two singularities very close together, with an exponent separated by about 1.0, is known to be characteristic of a confluent singularity involving a logarithmic term.

ANALYSING EULERIAN ORIENTATION SERIES.

Analysing our series by the method of differential approximants (DAs), we assumed a power-law singularity of the form

$$f(x) \sim C(1 - x/x_c)^{\alpha}.$$

For $U(x)$ we found the closest singularity to the origin to be at $x_c \approx 0.07957736$, with an exponent around $\alpha \approx 1.24$. However there was a second very close singularity at $x \approx 0.0795789$, with an exponent around 2.26, and a third, less precisely located singularity at around $x \approx 0.0798$. $1/4\pi = 0.0795774\cdots$.

This behaviour, where one has two singularities very close together, with an exponent separated by about 1.0, is known to be characteristic of a confluent singularity involving a logarithmic term.

Analysing Eulerian orientation series.

Analysing our series by the method of differential approximants (DAs), we assumed a power-law singularity of the form

$$f(x) \sim C(1 - x/x_c)^{\alpha}.$$

For $U(x)$ we found the closest singularity to the origin to be at $x_c \approx 0.07957736$, with an exponent around $\alpha \approx 1.24$. However there was a second very close singularity at $x \approx 0.0795789$, with an exponent around 2.26, and a third, less precisely located singularity at around $x \approx 0.0798$. $1/4\pi = 0.0795774 \cdots$.

This behaviour, where one has two singularities very close together, with an exponent separated by about 1.0, is known to be characteristic of a confluent singularity involving a logarithmic term.

ANALYSING EULERIAN ORIENTATION SERIES.

Analysing our series by the method of differential approximants (DAs), we assumed a power-law singularity of the form

$$f(x) \sim C(1 - x/x_c)^{\alpha}.$$

For $U(x)$ we found the closest singularity to the origin to be at $x_c \approx 0.07957736$, with an exponent around $\alpha \approx 1.24$. However there was a second very close singularity at $x \approx 0.0795789$, with an exponent around 2.26, and a third, less precisely located singularity at around $x \approx 0.0798$. $1/4\pi = 0.0795774 \cdots$.
This behaviour, where one has two singularities very close together, with an exponent separated by about 1.0, is known to be characteristic of a confluent singularity involving a logarithmic term.

SWITCH TO RATIO METHOD

If the generating function behaves as $A(1-\mu x)/\log(1-\mu x)$, then

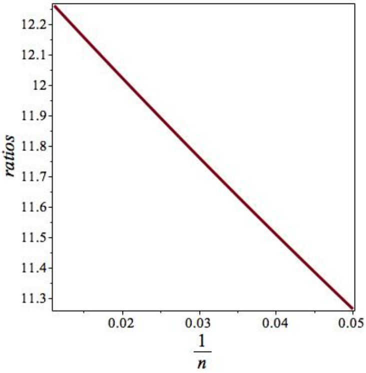
$$[x^n]f(x) = \frac{c\cdot\mu^n}{n^2}\left(\frac{1}{\log^2 n}+\frac{a}{\log^3 n}+\frac{b}{\log^4 n}+\frac{c}{\log^5 n}+o\left(\frac{1}{\log^5 n}\right)\right). \tag{1}$$

Logs make life difficult!
The ratio of successive coefficients is

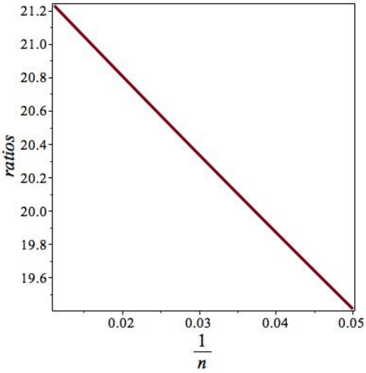
$$r_n = \frac{[x^n]f(x)}{[x^{n-1}]f(x)} = \mu\left(1-\frac{2}{n}-\frac{2}{n\log n}\left(1+\frac{c_1}{\log n}+\frac{c_2}{\log^2 n}+\frac{c_3}{\log^3 n}\right)+o\right.$$

Ratios for $U(x)$ and $A(x)$ plotted against $1/n$ below. Note curvature.

RATIO PLOTS



Ratio plot of coefficients of $U(x)$.

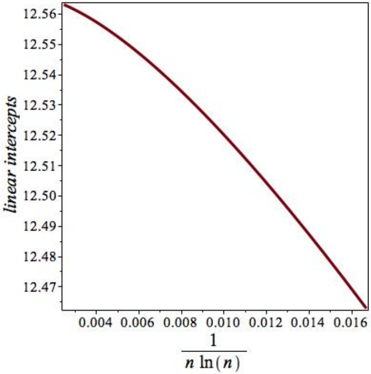


Ratio plot of coefficients of $A(x)$.

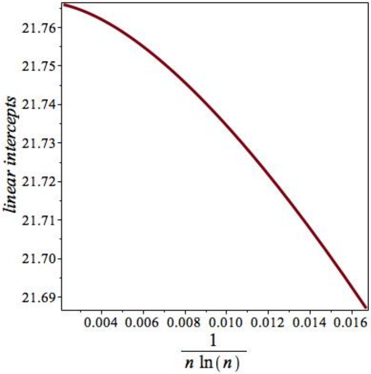
Refined ratios

If we eliminate the O$(1/n)$ term by constructing linear intercepts,

$$l_n = n\cdot r_n - (n-1)\cdot r_{n-1} = \mu\left(1 + \frac{2}{n\log^2 n} + \frac{4c_1}{n\log^3 n} + \frac{6c_2}{n\log^4 n} + \cdots\right)$$



Lin. int. of ratios vs. $1/n\log^2 n$.



Lin. int. of ratios vs. $1/n\log^2 n$.

Getting more ratios

Gradient wrong – we are far from asymtopia. We need more terms!

Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*

The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.

Every differential approximant naturally reproduces exactly all coefficients used in its derivation.

Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.

These subsequent coefficients will usually be approximate.

Getting more ratios

Gradient wrong – we are far from asymtopia. We need more terms!

Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*

The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.

Every differential approximant naturally reproduces exactly all coefficients used in its derivation.

Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.

These subsequent coefficients will usually be approximate.

Getting more ratios

Gradient wrong – we are far from asymtopia. We need more terms!
Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*

The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.

Every differential approximant naturally reproduces exactly all coefficients used in its derivation.

Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.

These subsequent coefficients will usually be approximate.

Getting more ratios

Gradient wrong – we are far from asymtopia. We need more terms!

Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*

The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.

Every differential approximant naturally reproduces exactly all coefficients used in its derivation.

Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.

These subsequent coefficients will usually be approximate.

Getting more ratios

Gradient wrong – we are far from asymtopia. We need more terms!

Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*

The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.

Every differential approximant naturally reproduces exactly all coefficients used in its derivation.

Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.

These subsequent coefficients will usually be approximate.

Getting more ratios

Gradient wrong – we are far from asymtopia. We need more terms!
Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*
The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.
Every differential approximant naturally reproduces exactly all coefficients used in its derivation.
Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.
These subsequent coefficients will usually be approximate.

Getting more ratios

Gradient wrong – we are far from asymtopia. We need more terms!

Not realistic to get vastly more terms exactly, but we can get them approximately with high enough precision for our purposes by using the *method of series extension.*

The idea is simply to use the method of differential approximants *to predict subsequent ratios/terms*.

Every differential approximant naturally reproduces exactly all coefficients used in its derivation.

Being a D-finite differential equation, it implies the value of *all* subsequent coefficients.

These subsequent coefficients will usually be approximate.

Getting more ratios II

The first approximate coefficient will be the most accurate, with accuracy declining with increasing order of predicted coefficients.

In practice we construct many DAs. We then calculate the average of the predicted coefficients (or ratios) across all constructed DAs, as well as their standard deviation.

We have experimentally found the true error to be between 1 and 2 standard deviations.

Getting more ratios II

The first approximate coefficient will be the most accurate, with accuracy declining with increasing order of predicted coefficients.

In practice we construct many DAs. We then calculate the average of the predicted coefficients (or ratios) across all constructed DAs, as well as their standard deviation.

We have experimentally found the true error to be between 1 and 2 standard deviations.

The method of series extension by differential approximants

The number of terms we can predict varies from problem to problem.

In this case we are extremely fortunate, in that the standard deviation of the coefficient estimates increases extremely slowly, and so we are confident in predicting 1000 extra ratios for both series, which we expect to be accurate to more than 10 significant digits.

Using these additional 1000 terms, we reconstruct the above plots.

The locus passes through a maximum, as expected, and the linear intercepts are now decreasing with increasing n, as predicted. We also show the same plot, but with the abscissa restricted to ratios corresponding to $700 \le n \le 1100$. The value of the ordinate at the origin is precisely $4\sqrt{3}\pi$, and the extrapolated locus is convincingly going through the origin.

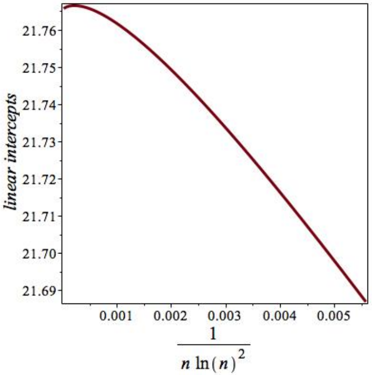
THE METHOD OF SERIES EXTENSION BY DIFFERENTIAL APPROXIMANTS

The number of terms we can predict varies from problem to problem. In this case we are extremely fortunate, in that the standard deviation of the coefficient estimates increases extremely slowly, and so we are confident in predicting 1000 extra ratios for both series, which we expect to be accurate to more than 10 significant digits.

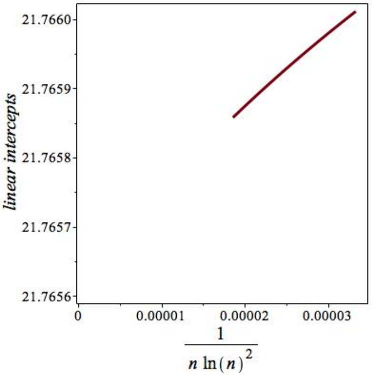
Using these additional 1000 terms, we reconstruct the above plots. The locus passes through a maximum, as expected, and the linear intercepts are now decreasing with increasing n, as predicted. We also show the same plot, but with the abscissa restricted to ratios corresponding to $700 \leq n \leq 1100$. The value of the ordinate at the origin is precisely $4\sqrt{3}\pi$, and the extrapolated locus is convincingly going through the origin.

THE METHOD OF SERIES EXTENSION BY DIFFERENTIAL APPROXIMANTS

The number of terms we can predict varies from problem to problem. In this case we are extremely fortunate, in that the standard deviation of the coefficient estimates increases extremely slowly, and so we are confident in predicting 1000 extra ratios for both series, which we expect to be accurate to more than 10 significant digits.
Using these additional 1000 terms, we reconstruct the above plots. The locus passes through a maximum, as expected, and the linear intercepts are now decreasing with increasing n, as predicted. We also show the same plot, but with the abscissa restricted to ratios corresponding to $700 \leq n \leq 1100$. The value of the ordinate at the origin is precisely $4\sqrt{3}\pi$, and the extrapolated locus is convincingly going through the origin.

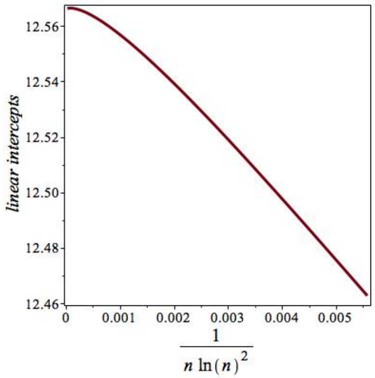


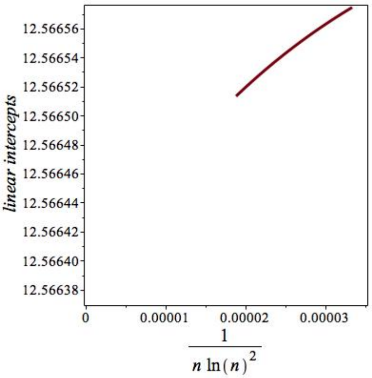
Plot of linear intercepts of ratios of $A(x)$ vs. $1/n\log^2 n$, using an extra 1000 ratios.



Plot of linear intercepts of ratios of $A(x)$ vs. $1/n\log^2 n$, using ratios 700 to 1100.

The corresponding plots for planar orientations where the value of the ordinate at the origin is precisely 4π is:





Plot of linear intercepts of ratios of $U(x)$ vs. $1/n \log^2 n$, using an extra 1000 ratios.

Plot of linear intercepts of ratios of $U(x)$ vs. $1/n \log^2 n$, using ratios 700 to 1100.

Estimating the exponent I

Now that we have good grounds to conjecture the exact value of the critical points, we are in a better position to estimate the exponent. If

$$f(x) = (1 - \mu \cdot x)^{-\alpha} \left(\frac{1}{\mu \cdot x} \log \frac{1}{1 - \mu \cdot x} \right)^{\beta},$$

then

$$[x^n]f(x) = \frac{\mu^n \cdot n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^{\beta} \left(1 + \frac{c_1}{\log n} + \frac{c_2}{\log^2 n} + \frac{c_3}{\log^3 n} + \frac{c_4}{\log^4 n} + \cdots \right)$$

where

$$c_k = \binom{\beta}{k} \Gamma(\alpha) \frac{d^k}{ds^k} \frac{1}{\Gamma(s)} \bigg|_{s=\alpha}.$$

When α is a negative integer the Γ function diverges, so that certain constants vanish. In particular, provided that α is a negative integer and β is not a positive integer, one has

$$[x^n]f(x) = \frac{\mu^n \cdot n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^\beta \left(\frac{c_1}{\log n} + \frac{c_2}{\log^2 n} + \frac{c_3}{\log^3 n} + \frac{c_4}{\log^4 n} + \cdots \right),$$

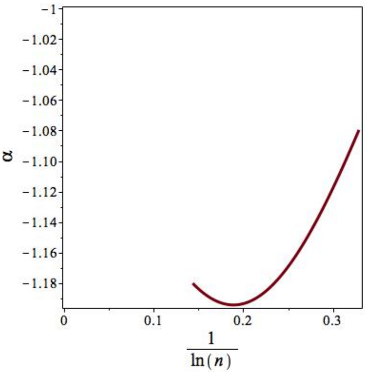
Then the ratio of successive coefficients is

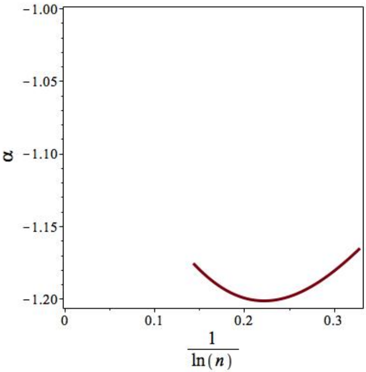
$$r_n = \frac{[x^n]f(x)}{[x^{n-1}]f(x)} = \mu \left(1 + \frac{\alpha - 1}{n} + \frac{\beta - 1}{n \log n} - \frac{c_1}{n \log^2 n} + o\left(\frac{1}{n \log^2 n} \right) \right),$$

So one can estimate α from the sequence

$$\alpha_n = \left(\frac{r_n}{\mu} - 1 \right) \cdot n + 1 = \alpha + \frac{\beta - 1}{\log n} - \frac{c_1}{\log^2 n} + o\left(\frac{1}{\log^2 n} \right).$$

Plots of α_n against $1/\log n$ for both $U(x)$ and $A(x)$ respectively are shown below, and it can be seen that having many more than 100 terms is essential. In fact the minimum in both plots occurs at around $n = 100$, and it is only with our extended data that the limit $\alpha = -1$ becomes plausible.





Plot of exponent α estimates from $U(x)$ vs. $1/\log n$, using an extra 1000 ratios.

Plot of exponent α estimates from $A(x)$ vs. $1/\log n$, using an extra 1000 ratios.

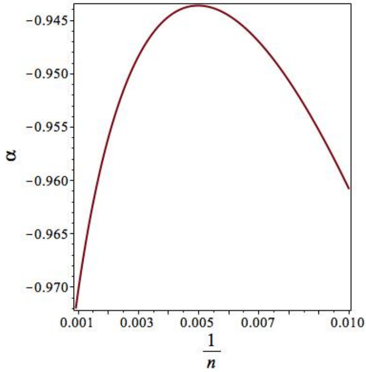
To take into account higher-order terms in the asymptotics, we attempted a linear fit to the assumed form (also assuming α is a negative integer, otherwise β replaces $\beta - 1$),

$$\left(\frac{r_n}{\mu} - 1\right) \cdot n + 1 = \alpha + \frac{\beta - 1}{\log n} - \frac{c_1}{\log^2 n} + o\left(\frac{1}{\log^2 n}\right). \qquad (2)$$

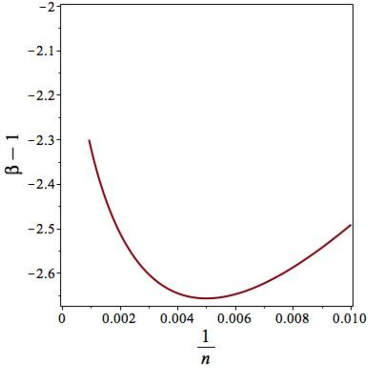
We did this by solving the linear system given by setting $n = m - 1,\ n = m,\ n = m + 1$ in the preceding equation, and solving for $\alpha,\ \beta,\ c_1$, with m ranging from 20 to the maximum possible value 1100. We obtain an m-dependent sequence of estimates of the terms $\alpha,\ \beta,\ c_1$, which we show plotted against appropriate powers of $1/m$. These are shown below for planar orientations. (The corresponding plots for 4-valent orientations are similar in appearance, so are not shown).

In this way we see that both α and β are plausibly going to -1, as appropriate for a singularity of the form

$$\frac{c \cdot \mu \cdot x \cdot (1 - \mu \cdot x)}{\log(1 - \mu \cdot x)}.$$



Plot of exponent α estimates from eqn (2).

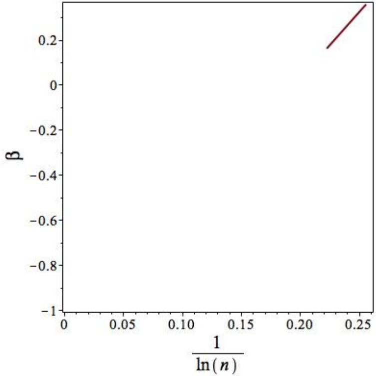


Plot of exponent $\beta - 1$ estimates from eqn (2).

Finally, if we accept that $\alpha = -1$, we can refine the estimate of β, since in that case

$$\left(\frac{r_n}{\mu} - 1 + \frac{2}{n}\right) n \log n = \beta - 1 + O\left(\frac{1}{\log n}\right). \tag{3}$$

This curve is plausibly tending to $\beta = -1$, though the fact that the abscissa is $1/\log n$ means that one would really need many more terms, around 22,000, even to get to 0.1 on the abscissa.



Conclusion

We conjecture that both general Eulerian orientations enumerated by edges, $U(x)$ and 4-valent Eulerian orientations enumerated by vertices $A(x)$ (generalised 6-vertex model) have a singularity structure of the form

$$\frac{c \cdot \mu \cdot x \cdot (1 - \mu \cdot x)}{\log(1 - \mu \cdot x)}.$$

For $U(x)$, $\mu = 4\pi$, and for $A(x)$, $\mu = 4\sqrt{3}\pi$.

POSTSCRIPT

It's all true!

As for the method of series extension

The ratio c_{998}/c_{997} is exactly
21.718170986407648634371728755726..... Our prediction was
21.718170986407648634371728755826.....

That is to say, it differs only in the 30th digit!

POSTSCRIPT

It's all true!
As for the method of series extension
The ratio c_{998}/c_{997} is exactly
21.7181709864076486343717287557\26..... Our prediction was
21.718170986407648634371728755826.....
That is to say, it differs only in the 30th digit!

SOLUTIONS ARE DIFFERENTIALLY ALGEBRAIC

General:

$$432*z^2-864*z+192+(972*z^5-648*z^4+120*z^3-4*z^2)*diff(diff(F(z),z),z)+(3240*z^4-3024*z^3+648*z^2-24*z)*diff(F(z),z)+(-648*z^6+108*z^5)*diff(F(z),z)^2-54*z^8*diff(F(z),z)^3+(1944*z^3-2592*z^2+528*z-24)*F(z)+(972*z^6-324*z^5+6*z^4)*F(z)*diff(diff(F(z),z),z)+(1944*z^5-1296*z^4+36*z^3)*F(z)*diff(F(z),z)-486*z^7*F(z)*diff(F(z),z)^2+(972*z^4-972*z^3+36*z^2)*F(z)^2+243*z^7*F(z)^2*diff(diff(F(z),z),z)$$

4-valent:

$$2+(16*z^3-z^2)*diff(diff(F(z),z),z)+(52*z^2-4*z)*diff(F(z),z)+24*z^4*diff(F(z),z)^2-16*z^6*diff(F(z),z)^3+(-2+8*z)*F(z)+(-64*z^4+2*z^3)*F(z)*diff(diff(F(z),z),z)+(-160*z^3+8*z^2)*F(z)*diff(F(z),z)-96*z^5*F(z)*diff(F(z),z)^2+(-32*z^2+4*z)*F(z)^2+64*z^5*F(z)^2*diff(diff(F(z),z),z)+64*z^4*F(z)^2*diff(F(z),z)$$