Indefinite theta series and theta liftings
a survey

Stephen Kudla
(Toronto)

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Introduction

- The theory of theta series for positive definite quadratic forms was highly advanced in the 19th century.
- Its extension to the case of indefinite quadratic forms was developed by Hecke, Siegel, Maass and others in the first half of the 20th century.
- Weil recast and extended much of this theory in a new language. His formulation emphasized the role of representation theory and the theory of distributions.
- In this language, Howe introduced the concept of reductive dual pairs and, motivated by Weyl's classical invariant theory, proposed his local and global duality conjectures.
- By now, based on the subsequent work of a very large group of people, we have arrived at a very elaborate and fairly complete theory of the theta correspondence of automorphic representations.
- I will not attempt to describe this general theory in this lecture!

  Rather, I want to describe an interesting class of examples which lie somewhat outside of the usual scope of this theory.

Introduction

- Recently, a new type of indefinite theta series was found by Alexandrov, Banerjee, Manschot and Pioline [ABMP], arising in their work on string theory.

  Their series provide a generalization to the case of signature $(n-2,2)$ of the Zwegers theta functions for lattices of signature $(n-1,1)$.

  A generalization to the case of any signature (p,q) was proposed in [ABMP] and was subsequently worked out by Nazaroglu [Naz].

  Other examples were described by Martin Raum and by Sanders Zwegers in his Dublin lectures.
- In this talk I will explain how these theta series can be obtained by using my old work with John Millson.

  This provides some additional 'geometric' insight.
- This is joint work with Jens Funke.

Classical vs indefinite theta series

Let V, $(\ ,\ )$ be a rational inner product space and $L \subset V$ is a lattice with $L \subset L^\vee$ and with quadratic form $Q(x) = \frac{1}{2}(x, x)$.

If Q is positive definite, the classical theta series

$$\theta(\tau, h, L) = \sum_{x \in h+L} \mathbf{q}^{Q(x)}, \qquad \tau = u + iv, v > 0, \quad \mathbf{q} = e^{2\pi i \tau}, \quad h \in L^\vee / L,$$

is a holomorphic modular form of weight $\frac{1}{2} \dim V$.

In the indefinite case, where $\mathrm{sig}(V) = (p, q)$, $pq > 0$, the series fails to converge.

One approach to handling this is due to Siegel, etc. as mentioned above.

Another was advocated by Zagier.

Zagier's approach

An approach to a theory of indefinite theta series, advocated by Zagier, is to cut down the summation to

$$\sum_{\substack{x\in h+L \\ x\in\Xi}} \mathbf{q}^{Q(x)},$$

where

$$\begin{aligned} \Xi &= \text{ a suitable subset of } V^+, \qquad \text{e.g.} \\ \Xi &= \text{ a positive cone in } V^+, \end{aligned}$$

This a holomorphic 'indefinite theta series'.

The problem is that such series are seldom modular forms, since they do not behave well with respect to Poisson summation!

Zwegers' Example

Suppose $\mathrm{sig}(V) = (n-1, 1)$.
Take two negative vectors C and C' in $V_{\mathbb{R}}$,

$$Q(C) < 0, \quad Q(C') < 0, \quad (C, C') < 0,$$

and let

$$\Phi_1(x; C, C') = \frac{1}{2}(\mathrm{sgn}(x, C) - \mathrm{sgn}(x, C')).$$

Then

$$\vartheta(\tau, L) = \sum_{x \in h+L} \Phi_1(x; C, C')\, \mathbf{q}^{Q(x)}$$

is a holomorphic function of τ, *but is not modular.*

Zwegers' Example

Zwegers' striking result is that this series can be 'completed' by the addition of a non-holomorphic series to yield a non-holomorphic modular form.

Let

$$E^c(u) = 2\,\mathrm{sgn}(u) \int_{|u|}^{\infty} e^{-\pi t^2}\, dt$$

be the 'complementary error function'.

Let

$$\vartheta^c(\tau, L, C, C') = \sum_{x \in h+L} \frac{1}{2} \left(E^c((x, \underline{C}')\sqrt{2v}) - E^c((x, \underline{C})\sqrt{2v}) \right) \mathbf{q}^{Q(x)}.$$

Then Zwegers showed that the sum

$$\hat{\vartheta}(\tau, L, C, C') := \vartheta(\tau, L, C, C') + \vartheta^c(\tau, L, C, C')$$

is a (non-holomorphic) modular form of weight $n/2$.

The relation is connected with the phenomena of Mock modular forms, as explained in Zwegers thesis.

Zwegers' Example

Note that if we set

$$E(u) = 2\,\mathrm{sgn}(u)\int_0^{|u|} e^{-\pi t^2}\,dt,$$

then

$$\mathrm{sgn}(u) = E(u) + E^c(u).$$

Hence the completion can be written as

$$\hat{\vartheta}(\tau, L, C, C') = \sum_{x \in h+L} \frac{1}{2}\left(E((x, \underline{C})\sqrt{2v}) - E((x, \underline{C'})\sqrt{2v}) \right) \mathbf{q}^{Q(x)}.$$

The 'holomorphic part' can be recovered by looking at the limit as $v \longrightarrow \infty$ in the coefficients of $\mathbf{q}^{Q(x)}$.

The ABMP and Nazaroglu examples, which I will describe shortly, generalize this picture to signature (p, q).

Basic setup

It turns out that the 'exotic' indefinite theta series of Zwegers, ABMP and Nazaroglu, and others, all arise very naturally from my old joint work with John Millson.

To fix some notation, let

$$
\begin{aligned}
V &= \text{rational inner product space, } Q(x) = \frac{1}{2}(x,x) \\
\mathrm{sig}(V) &= (p,q), \quad m = p+q = \dim V, \\
D &= \text{space of oriented negative } q\text{-planes in } V_{\mathbb{R}} \subset \mathrm{Gr}_q(V_{\mathbb{R}}) \\
\dim(D) &= pq \\
L &= \text{lattice in } V \\
L &\subset L^{\vee}. \\
\Gamma_L &= \{\gamma \in O(V) \mid \gamma L = L, \gamma\,|_{L^{\vee}/L} = 1\}, \\
A^r(D) &= \text{smooth } r\text{-forms on } D.
\end{aligned}
$$

Theta forms

Millson and I constructed a Schwartz form:

$$\varphi_{KM} \in [\,\mathcal{S}(V_{\mathbb{R}}) \otimes A^q(D)\,]^G, \qquad G = \mathrm{SO}(V),$$

i.e., for $x \in V_{\mathbb{R}}$ and $g \in G$,

$$\varphi_{KM}(g^{-1}x) = g^*\varphi_{KM}(x) \in A^q(D).$$

Thus, $\varphi_{KM}(x)$ is a G_x-invariant q-form on D.
For example, $\varphi_{KM}(0)$ is an invariant form.

Moreover, φ_{KM} is **closed**

$$d\varphi_{KM}(x) = 0.$$

Write

$$\varphi_{KM}(x) = \varphi^0_{KM}(x)\,e^{-2\pi Q(x)}.$$

Theta forms

The **theta form**

$$\theta(\tau, L, \varphi_{KM}) = v^{-\frac{m}{4}} \sum_{x \in h+L} \varphi^0_{KM}(x\sqrt{v})\, \mathbf{q}^{Q(x)},$$

defined using φ_{KM} has the following properties:

(1) In the variable τ,

$$\theta(\tau, L, \varphi_{KM}) = \text{a (non-holomorphic) modular form of weight } \frac{p+q}{2}.$$

(by the standard Poisson summation argument)

(2)

$$\theta(\tau, L, \varphi_{KM}) = \text{a closed } \Gamma_L\text{-invariant } q\text{-form on } D.$$

hence defines a closed q-form on $M_L = \Gamma_L \backslash D$.

Relation to geodesic cycles

In particular, the theta forms define cohomology classes for the locally symmetric space $M_L = \Gamma_L \backslash D$ related to totally geodesic cycles.

This is why we were constructing them.

Here is the basic construction:

$$
\begin{aligned}
x \in V, &\qquad x \neq 0 \\
D_x &= \{z \in D \mid z \perp x\} \\
V_x &= x^{\perp} \\
D_x &\simeq D(V_x) \\
\operatorname{codim}(D_x) = q, &\qquad \text{if } Q(x) > 0. \\
D_x &= \text{empty if } Q(x) \leq 0
\end{aligned}
$$

So we have subsymmetric spaces $D_x \hookrightarrow D$, equivariant for $SO(p-1, q) \simeq SO(V_x) \hookrightarrow SO(V) \simeq SO(p, q)$.

Relation to geodesic cycles

Let

$$\mathrm{pr}_{\Gamma_L} : D \longrightarrow \Gamma_L \backslash D = M_L.$$

For $Q(x) > 0$, let

$$Z(x) = \mathrm{pr}_{\Gamma_L}(D_x) = \text{totally geodesic codimension } q\text{-cycle in } M_L$$

$$\Gamma_x \backslash D_x \longrightarrow Z(x) \subset \Gamma \backslash D, \qquad \text{immersion.}$$

Notice that this depends only on the Γ_L-orbit of x.
Millson and I proved the following:

Theorem (KM). Suppose that η is a closed and compactly supported $(p-1)q$-form on M_L. Then

$$\int_{M_L} \eta \wedge \theta(\tau, L, \varphi_{KM}) = \int_{M_L} \eta \wedge \varphi_{KM}(0) + \sum_{\substack{x \in h+L \\ Q(x)>0 \\ \bmod \Gamma_L}} \left(\int_{Z(x)} \eta \right) \mathbf{q}^{Q(x)}.$$

Relation to geodesic cycles

An alternative formulation in terms of homology is the following:

Theorem (KM). Suppose that S is a compact closed (i.e., $\partial S = 0$) oriented q-cycle on M_L. Then

$$\int_S \theta(\tau, L, \varphi_{KM}) = \int_S \varphi_{KM}(0) + \sum_{\substack{x \in h+L \\ Q(x)>0 \\ \bmod \Gamma_L}} I(S, Z(x))\, \mathbf{q}^{Q(x)},$$

where $I(S, Z(x))$ is the **intersection number** of the cycles S and $Z(x)$.

In particular, the series here and the one in the previous theorem are termwise absolutely convergent and define holomorphic modular forms of weight $\frac{p+q}{2}$.

Other theta integrals

Main Idea: Since the theta form $\theta(\tau, L, \varphi_{KM})$ is a closed q-form, we can consider integrals

$$I(\tau, L, S) = \int_S \theta(\tau, L, \varphi_{KM})$$

over other singular q-chains S in D or M_L.

If S is compact, $I(\tau, L, S)$ is always a (non-holomorphic) modular form of weight $m/2$.

Key Point: For suitable choices of S, the 'completed' modular forms of Zwegers, ABMP, Nazaroglu and others arise in this way!

I will next describe how the relevant q-chains are defined.

Singular q-cubes

Cubical Data ([ABMP] and [Naz]): Consider a collection

$$\mathcal{C}^{\square} = \{\{C_1, C_{1'}\}, \{C_2, C_{2'}\}, \dots, \{C_q, C_{q'}\}\}$$

of pairs of negative vectors in $V_{\mathbb{R}}$.

Under a suitable condition, the data $\mathcal{C}^{\square}$ defines a singular q-cube in D.

Definition. For $s = [s_1, \dots, s_q] \in [0,1]^q$, let

$$B_j(s_j) = (1 - s_j)C_j + s_j C_{j'}.$$

We say that $\mathcal{C}^{\square}$ is in **good position**, if

$$\mathrm{span}\{B_1(s_1), \dots, B_q(s_q)\}_{\text{p.o.}} \in D$$

for all $s \in [0,1]^q$.

Singular q-cubes

In particular, this implies that

(0) We can define a **singular q-cube** $S(\mathcal{C}^\square)$

$$\phi_{\mathcal{C}^\square} : [0,1]^q \longrightarrow D, \qquad s \longmapsto \mathrm{span}\{B_1(s_1), \ldots, B_q(s_q)\}_{\text{p.o.}}.$$

(1) For all $I \subset \{1, \ldots, q\}$, let $C_j^I = C_j$ if $j \notin I$ and $C_j^I = C_j'$ if $j \in I$.
Then $z^I = \mathrm{span}\{C_1^I, C_2^I, \ldots, C_q^I\}_{\text{p.o.}} \in D$, so we obtain 2^q (not necessarily distinct) points in D, the vertices of the q-cube.

(2) All the z^I lie in the same component of D, since they are connected by paths in $\phi_{\mathcal{C}^\square}([0,1]^q)$.

Tetrahedral data

Another interesting case is that of singular q-simplices (tetrahedra).

These were discussed by Zwegers in his Dublin lectures.

Given a collection

$$\mathcal{C}^{\triangle} = \{C_0, C_1, \dots, C_q\}$$

of negative vectors in $V_{\mathbb{R}}$.

We impose the condition that for any j, $0 \leq j \leq q$,

$$z_j = \mathrm{span}\{C_0, \dots, \widehat{C_j}, \dots, C_q\}_{\text{p.o.}} \in D.$$

Moreover, if the matrix

$$\mathrm{Adj}((C_i, C_j))$$

has non-negative off diagonal entries, we obtain a singular q-simplex

$$\phi_{\mathcal{C}^{\triangle}} : \Delta_q \longrightarrow D.$$

Generalized error functions

Definition (ABMP): For $z = \mathrm{span}\{c_1, c_2, \ldots, c_r\} = \mathrm{span}\{\boldsymbol{c}\}$ a negative r-plane in $V_{\mathbb{R}}$ and $x \in V_{\mathbb{R}}$, the **generalized error function** is given by

$$E_r(x, \boldsymbol{c}) = \int_z e^{\pi(y-\mathrm{pr}_z(x), y-\mathrm{pr}_z(x))} \,\mathrm{sgn}(y, \boldsymbol{c})\, dy,$$

where

$$\mathrm{sgn}(y, \boldsymbol{c}) = \mathrm{sgn}(y, c_1)\,\mathrm{sgn}(y, c_2)\, \ldots\, \mathrm{sgn}(y, c_r).$$

Note that for $q = 1$,

$$E_1(x, C) = E((x, \underline{C})) = 2\,\mathrm{sgn}((x, \underline{C})) \int_0^{|(x,\underline{C})|} e^{-\pi t^2}\, dt.$$

Also note that, if x is ‘regular’ with respect to $\boldsymbol{c}$,

$$\lim_{t \longrightarrow \infty} E_r(tx, \boldsymbol{c}) = \prod_j \mathrm{sgn}(x, \boldsymbol{c}_j) = \mathrm{sgn}(x, \boldsymbol{c}).$$

We can now give the formulas for the integrals

$$I(\tau, L, \mathcal{C}^\star) = \int_{S(\mathcal{C}^\star)} \theta(\tau, L, \varphi_{KM}), \qquad \star = \square \text{ or } \triangle.$$

Theorem (FK).

(1.1) In the cubical case, assume that $\mathcal{C}^\square$ is in good position. Then

$$I(\tau, L, \mathcal{C}^\square) = \sum_{x \in h+L} 2^{-q} \sum_{I} (-1)^{|I|-1}\, E_q(x\sqrt{2v}, C^I)\, \mathbf{q}^{Q(x)}.$$

(1.2) The 'holomorphic part' is given by

$$\vartheta(\tau, L, \mathcal{C}^\square) = \sum_{x \in h+L} \Phi_q(x, \mathcal{C}^\square)\, \mathbf{q}^{Q(x)}.$$

where

$$\Phi_q(x, \mathcal{C}^\square) = 2^{-q}(\mathrm{sgn}(x, C_1) - \mathrm{sgn}(x, C_{1'}))\dots(\mathrm{sgn}(x, C_q) - \mathrm{sgn}(x, C_{q'})).$$

The indefinite theta series

(2.1) In the simplicial case, assume that $\mathcal{C}^{\triangle}$ is in good position. Then

$$I(\tau, L, \mathcal{C}^{\triangle}) = \sum_{x \in h+L} (-1)^q 2^{-q} \sum_{r=0}^{[q/2]} \sum_{\substack{I \\ |I|=2r+1}} E_{q-2r}(\mathcal{C}^{(I)}; x\sqrt{2v})\, \mathbf{q}^{Q(x)},$$

where, for a subset $I \subset \{0, 1, \dots, q\}$, let $\mathcal{C}^{(I)}$ be the collection of $q+1-|I|$ elements where the C_i with $i \in I$ have been omitted.

(2.2) The 'holomorphic part' is given by

$$\vartheta(\tau, L, \mathcal{C}^{\triangle}) = \sum_{x \in h+L} \Phi_q(x, \mathcal{C}^{\triangle})\, \mathbf{q}^{Q(x)}.$$

where

$$\Phi_q(x, \mathcal{C}^{\triangle}) = 2^{-q-1} \left(\prod_{j=0}^{q} (1 - \operatorname{sgn}(x, C_j)) + (-1)^q \prod_{j=0}^{q} (1 + \operatorname{sgn}(x, C_j)) \right).$$

The indefinite theta series

In both cases, for $\mathcal{C}^\star$ in good position, the series

$$\vartheta(\tau, L, \mathcal{C}^\star) = \sum_{x \in h+L} \Phi_q(x, \mathcal{C}^\star)\, \mathbf{q}^{Q(x)}$$

is termwise absolutely convergent.

Thus $I(\tau, L, \mathcal{C}^\star)$ is its modular completion.

Indeed, in the cubical case it coincides with that in [ABMP] and [Naz].

(3) If $x \in V$ with $\Phi_q(x, \mathcal{C}^\star) \neq 0$, then
   (i) $S(\mathcal{C}^\star) \cap D_x = \phi_{\mathcal{C}}(s(x))$ for a unique parameter $s(x) \in [0,1]^q$ (resp. Δ_q).
   (ii) The map $\phi_{\mathcal{C}}$ is immersive at $s(x)$.
   (iii)
$$\Phi_q(x, \mathcal{C}^\star) = I(S(\mathcal{C}), D_x).$$

   (If $s(x)$ lies on the boundary, the intersection number is weighted accordingly.)

Open Problems

There are a number of interesting open problems arising here.

1. **Convexity Problem:** Give precise conditions on a collection $\mathcal{C}^{\square}$, expressed in terms of the Gram matrix, which insure that $\mathcal{C}^{\square}$ is in good position.
2. Compute the integral of $\varphi_{KM}(x)$ over more general singular simplices or singular cubes.

   For example, the generic simplex has $q+1$ vertices, each spanned by q negative vectors, so that these could span a space of dimension $q(q+1)$ and hence of signature (q^2, q). What is $\varphi(\tau, L, \mathcal{C}^{\triangle})$ in that case?
3. What happens when vertices are allowed to go to the boundary?
4. Do the resulting non-holomorphic modular forms have any meaning/use? In physics? For automorphic representation theory?