Nerves Can Only Kill, and Also Serially!

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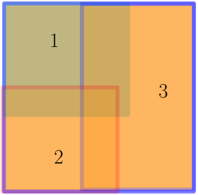

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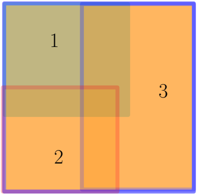

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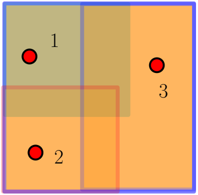

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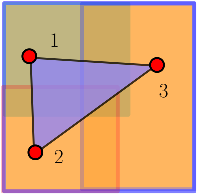

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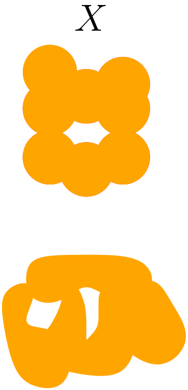

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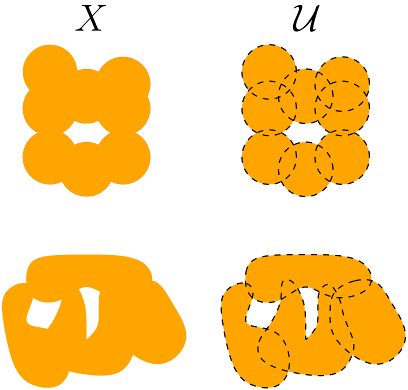

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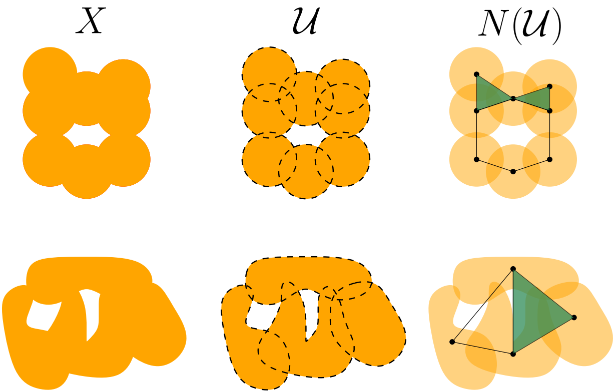
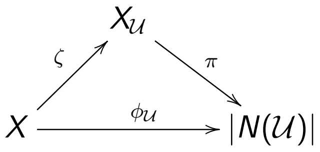


From space to nerve and H_1-classes

- X a path connected, paracompact space
- $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, a path connected cover, $X_\mathcal{U}$: blowup space
- $\phi_\mathcal{U} : X \to |N(\mathcal{U})|$ is a map where $\phi_\mathcal{U} = \pi \circ \zeta$



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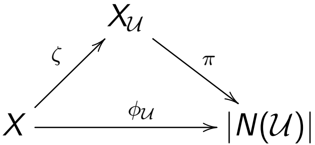

Theorem (Space-Nerve)

$\phi_{\mathcal{U}*} : H_1(X) \to H_1(|N(\mathcal{U})|)$ *is a surjection.*

Maps between covers

- Consider covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ and a map of sets $\xi : A \to B$ satisfying $U_\alpha \subseteq V_{\xi(\alpha)}$ for all $\alpha \in A$

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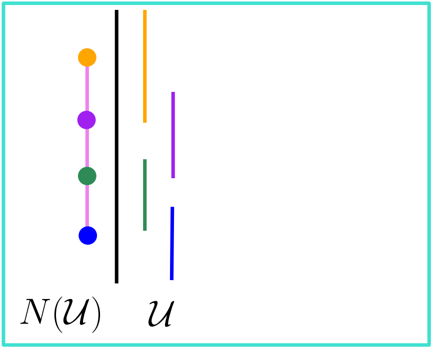

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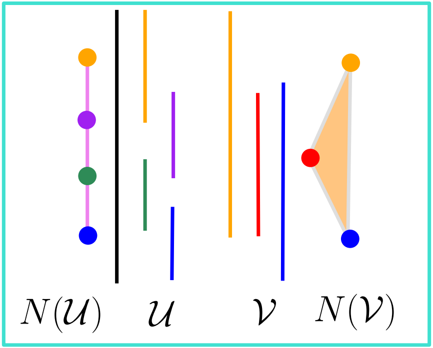

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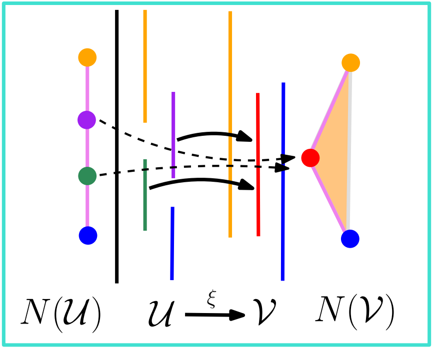

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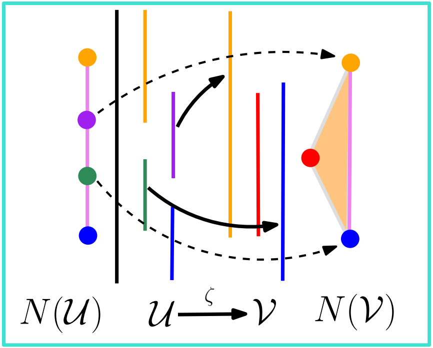

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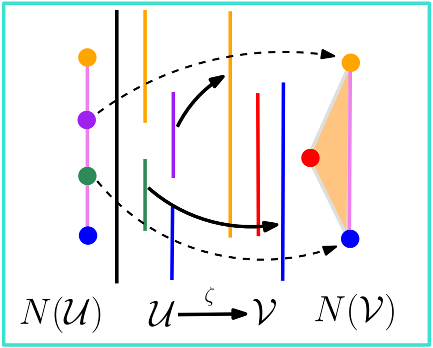
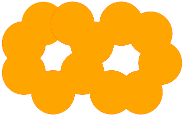
- ξ induces a simplicial map $N(\xi) : N(\mathcal{U}) \to N(\mathcal{V})$
- if $\mathcal{U} \xrightarrow{\xi_1} \mathcal{V} \xrightarrow{\xi_2} \mathcal{W}$, then $N(\xi_2 \circ \xi_1) = N(\xi_2) \circ N(\xi_1)$

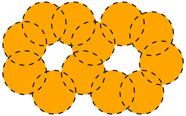


Nerve to nerve and H_1-classes

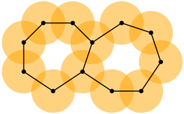
X



$\mathcal{U}_1$



$N(\mathcal{U}_1)$



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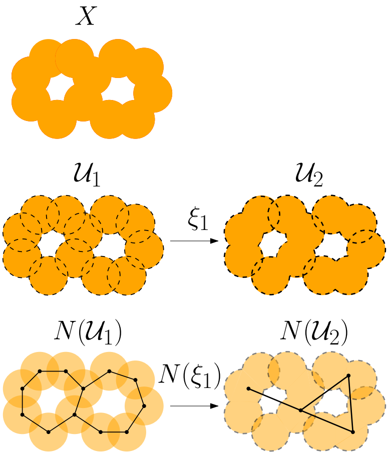

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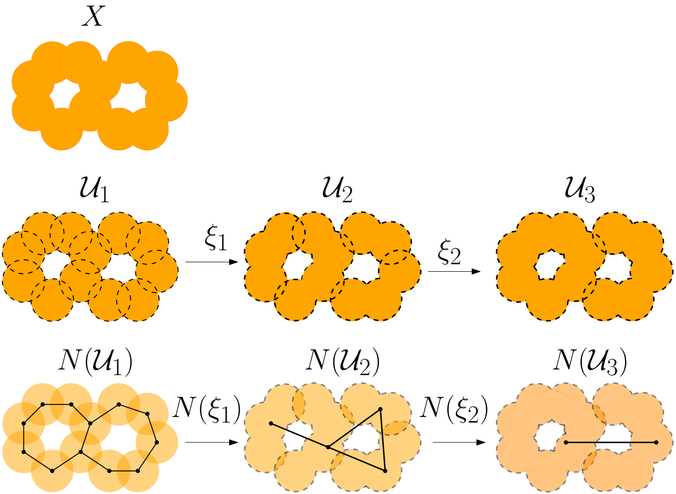

Nerve to nerve and H_1-classes

Proposition

$\mathcal{U}$ and $\mathcal{V}$ be two covers of X with a cover map $\mathcal{U} \xrightarrow{\theta} \mathcal{V}$. Then, $\phi_{\mathcal{V}} = \hat{\tau} \circ \phi_{\mathcal{U}}$ where $\tau : N(\mathcal{U}) \to N(\mathcal{V})$ is induced by θ.

Corollary

The maps $\phi_{\mathcal{U}*} : H_k(X) \to H_k(|N(\mathcal{U})|)$, $\phi_{\mathcal{V}*} : H_k(X) \to H_k(|N(\mathcal{V})|)$, and $\hat{\tau}_* : H_k(|N(\mathcal{U})|) \to H_k(|N(\mathcal{V})|)$ commute, that is, $\phi_{\mathcal{V}*} = \hat{\tau}_* \circ \phi_{\mathcal{U}*}$.

Theorem (Nerve-Nerve)

Let $\tau : N(\mathcal{U}) \to N(\mathcal{V})$ be induced by a cover map $\mathcal{U} \to \mathcal{V}$. Then, $\tau_* : H_1(N(\mathcal{U})) \to H_1(N(\mathcal{V}))$ is a surjection.

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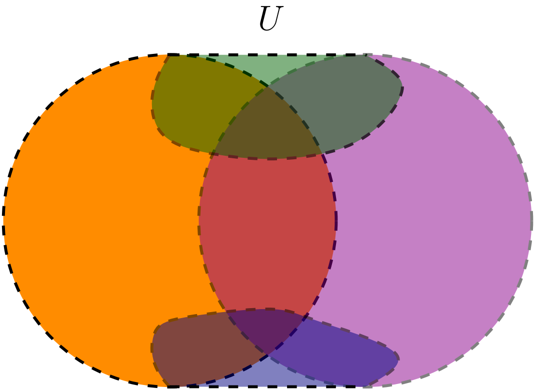
$$\lambda(\mathcal{U}) = \sup\{\delta \,|\, \forall X' \subseteq X \text{ with } s(X') \leq \delta, \exists U_\alpha \in \mathcal{U} \text{ where } U_\alpha \supseteq X'\}$$



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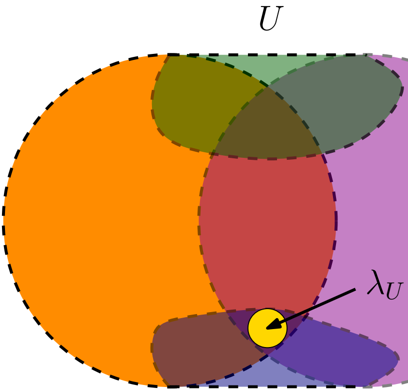

Persistent H_1-classes

Theorem (Persistent H_1-classes)

Let $z_1, z_2, \ldots, z_g$ be a minimal generator basis of $H_1(X)$ ordered with increasing sizes.

i. *Let $\ell \in [1, g]$ be the smallest integer so that $s(z_\ell) > \lambda(\mathcal{U})$. If $\ell \neq 1$, the class $\bar{\phi}_{\mathcal{U}*}[z_j] = 0$ for $j = 1, \ldots, \ell - 1$. Moreover, the classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell,\ldots,g}$ generate $H_1(N(\mathcal{U}))$.*

ii. *The classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell',\ldots,g}$ are linearly independent where $s(z_{\ell'}) > 4s_{max}(\mathcal{U})$.*

Maps and pseudometric

- $f : X \to Z$ where (Z, d_Z) a metric space
- $d_f(x, x') := \inf_{\gamma \in \Gamma_X(x,x')} \mathrm{diam}_Z(f \circ \gamma)$.

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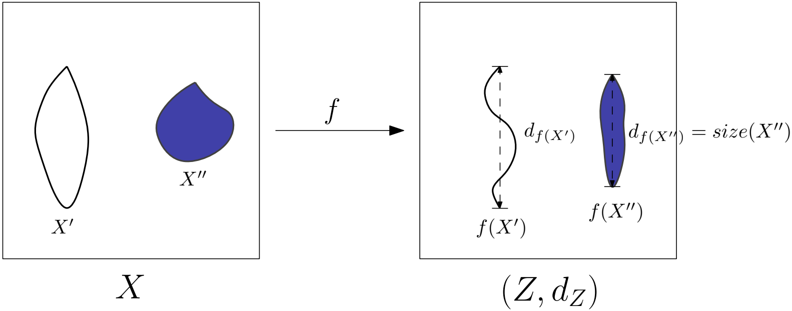
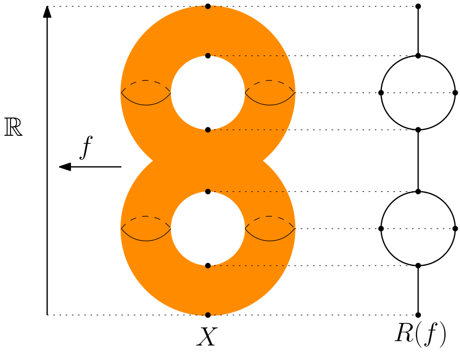


Reeb graph/space

- $H_1(X) = H_1^v \oplus H_1^h$
- $c \in H_1^h$ iff $c = [z]$ where $z \in f^{-1}(a)$
- Reeb graphs capture only vertical homology classes [D.-Wang 14]



Surviving H_1-classes in Reeb space

Theorem (Persistent H_1-classes)

Let $z_1, z_2, \ldots, z_g$ be a minimal generator basis of $H_1(X)$ ordered with increasing sizes (defined by d_f); $q : X \to R_f$ quotient map.

- *Let $\ell \in [1, g]$ be the smallest s.t. $s(z_\ell) \neq 0$. If no ℓ exists, $H_1(R_f)$ is trivial, otherwise $\{[q(z_i)]\}_{i=\ell \ldots g}$ is a basis of $H_1(R_f)$.*

Implication: Just like in Reeb graphs, only vertical homology classes survive in Reeb spaces (extension of a result of [D.-Wang 14])

Surviving H_1-classes in intrinsic Čech complex

- $C^\delta(Y)$: Čech complex of (Y, d_Y)
- $z_1, \ldots, z_g$: a minimal generator basis for $H_1(Y)$

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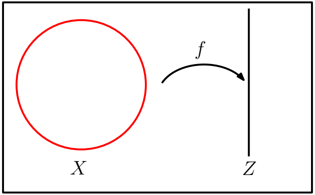
Theorem (Persistent H_1-classes)

- $\{\Phi_{\mathcal{U}*}(z_i)\}_{i=\ell \ldots g}$ *generate* $H_1(C^\delta(Y)))$ *where* ℓ *is the smallest s.t.* $s(z_\ell) > \delta$.
- $\{\Phi_{\mathcal{U}*}(z_i)\}_{i=\ell' \ldots g}$ *are linearly independent if* $s(z'_\ell) > 8\delta$

Maps and covers



- Let $f : X \to Z$ continuous, well-behaved and $\mathcal{U}$ a finite cover of Z.

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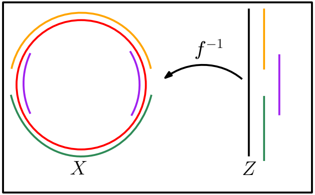

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- Connected components of $f^{-1}(U_\alpha) = \bigcup_{i=1}^{j_\alpha} V_{\alpha,i}$ form a cover $f^*(\mathcal{U})$ of X.

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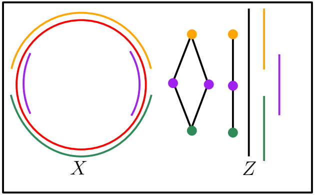

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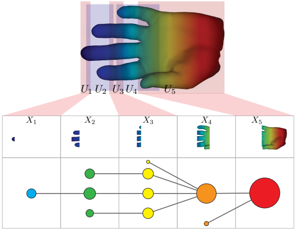
Mapper



Definition (Mapper)

[Singh-Carlsson-Mémoli] Let $f : X \to Z$ be continuous and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a finite open covering of Z. The Mapper is

$$\mathrm{M}(\mathcal{U}, f) := N(f^*(\mathcal{U}))$$

Tower of covers and complexes

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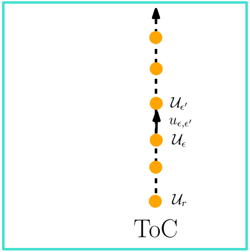

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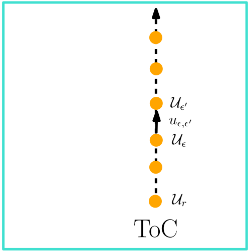

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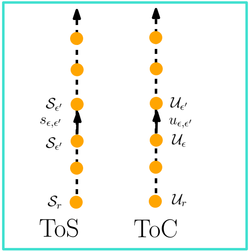

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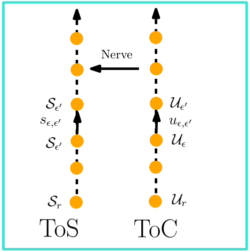

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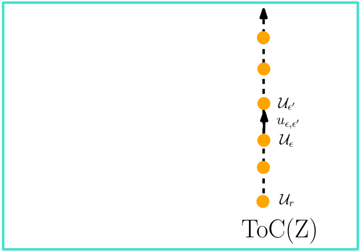

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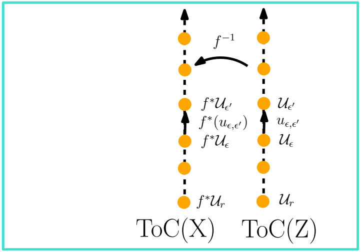
- Then, $f^*(\mathfrak{U})$ is ToC of X and $N(f^*(\mathfrak{U}))$is ToS



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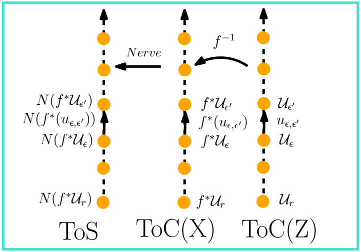

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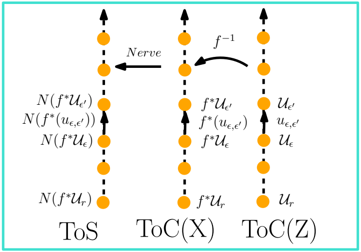


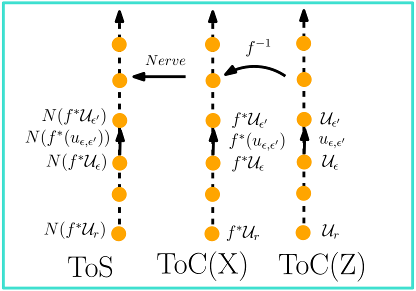
Multiscale Mapper:

$$\mathrm{MM}(\mathfrak{U}, f) := N(f^*(\mathfrak{U}))$$

Persistence diagram of MM

- $\mathrm{D_k MM}(\mathfrak{U}, f)$= persistence diagram of:

$$\mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_1}))\big) \to \mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_2}))\big) \to \cdots \to \mathrm{H}_k\big(N(f^*(\mathcal{U}_{\varepsilon_n}))\big)$$



Implication for multiscale mapper

Theorem

Consider the following multiscale mapper:

$$N(f^*\mathcal{U}_0) \to N(f^*\mathcal{U}_1) \to \cdots \to N(f^*\mathcal{U}_n)$$

- *surjection from $H_1(X)$ to $H_1(N(f^*\mathcal{U}_i))$ for each $i \in [0, n]$.*
- *For H_1-persistence module:*

$$\mathrm{H}_1\big(N(f^*\mathcal{U}_0)\big) \to \mathrm{H}_1\big(N(f^*\mathcal{U}_1)\big) \to \cdots \to \mathrm{H}_1\big(N(f^*\mathcal{U}_n)\big)$$

all connecting maps are surjections.

Persistent H_1-classes in MM

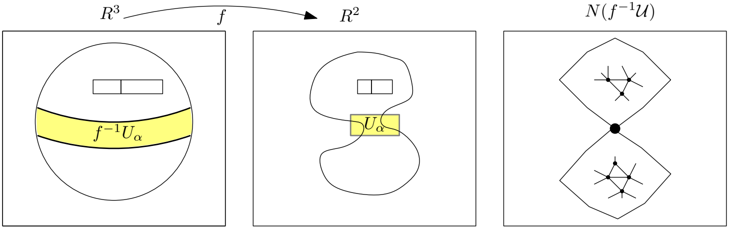
Theorem

Consider a H_1-persistence module of a multiscale mapper induced by a tower of path connected covers:

$$\mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_0})\big) \xrightarrow{s_{1*}} \mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_1})\big) \xrightarrow{s_{2*}} \cdots \xrightarrow{s_{n*}} \mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_n})\big)$$

Let $\hat{s}_{i*} = s_{i*} \circ s_{(i-1)*} \circ \cdots \circ \bar{\phi}_{\mathcal{U}_{\varepsilon_0}*}$. Then, $\hat{s}_{i*}$ renders the small classes of $H_1(X)$ trivial in $H_1(N(f^*\mathcal{U}_{\varepsilon_i}))$ as detailed in previous theorem.

Open Question



Conjecture: If t-wise intersections in $\mathcal{U}$ for all $t > 0$ have $\tilde{H}_{\leq k-t} = 0$, then $\phi_{\mathcal{U}*}$ is surjective for H_k

Thank You