Images of compatible systems of Galois representations of global function fields

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Set-up

- $\mathbb{F}_q$ the finite field of q elements, of characteristic p
- $X/\mathbb{F}_q$ a smooth geometrically connected curve
- $F = \mathbb{F}_q(X)$ the function field of X
- $|X|$ the set of closed points of X
- $\Gamma_{\mathrm{K}} = \mathrm{Gal}(\mathrm{K}^{\mathrm{sep}}/\mathrm{K})$ for any field K; e.g. $\Gamma_{\mathbb{F}_\mathrm{q}} \cong \widehat{\mathbb{Z}}$
- $\pi_1(X)$ the arithmetic fundamental group of X (have surjection $\Gamma_F \to \pi_1(X)$)
- $\pi_1^{geo}(X) := \pi_1(X_{\overline{\mathbb{F}}_q})$ the geometric fundamental group of X
- For $x \in |X|$ have $Frob_x \in \pi_1(X)$ (unique up to conjugacy)

One has the fundamental short exact sequence

$$1 \to \pi_1^{geo}(X) \to \pi_1(X) \to \Gamma_{\mathbb{F}_q} \to 1.$$

ℓ-adic étale cohomology

Let $\mathbb{L}$ be the set of all prime numbers $\ell \neq p$.
Let $Y \to X$ be a smooth proper morphism. Consider

$$V_\ell := H^i(Y_{F^{sep}}, \mathbb{Q}_\ell)$$

ℓ-adic étale cohomology (i fixed, $\ell \in \mathbb{L}$).

Theorem (Grothendieck, Deligne et al.)

- $d := \dim_{\mathbb{Q}_\ell} V_\ell$ *is finite and independent of* ℓ
- *have a representation* $\rho_\ell \colon \pi_1(X) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell) \cong GL_d(\mathbb{Q}_\ell)$.
- $\mathrm{charpol}_{Frob_x|V_\ell}(T) \in \mathbb{Z}[T]$ *is independent of* ℓ *for any* $x \in |X|$.
- *The action of* $\pi_1^{geo}(X)$ *is semisimple.*
  (i.e, every subrepresentation has a complement)
  $\Rightarrow G_\ell^{geo} := \overline{\rho_\ell(\pi_1^{geo}(X))}^{Zar} \subset \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell)$ *is a reductive group*

Elliptic curves example

Let $E \to X$ be an elliptic curve.
Let $W_\ell \cong \mathbb{Q}_\ell^2$ be the ℓ-adic Tate-module of E. Then

$$H^1(E_{F^{sep}}, \mathbb{Q}_\ell) \cong W_\ell^*$$

as a $\pi_1(X)$-module. For $x \in |X|$ one has

$$\mathrm{charpol}_{Frob_x|W_\ell}(T) = T^2 - a_x(E)T + q_x \in \mathbb{Z}[T]$$

with $q_x =$ cardinality of the residue field $\mathbb{F}_x$ at $x \in |X|$ and

$$\#E(\mathbb{F}_x) = 1 - a_x(E) + q_x.$$

If E has no complex multiplication, then

$$G_\ell^{geo} = SL_{2,\mathbb{Q}_\ell} \text{ and } G_\ell := \overline{\rho_\ell(\pi_1(X))}^{Zar} = GL_{2,\mathbb{Q}_\ell}$$

A result of Igusa

Theorem (Igusa, 1959)

Suppose E has no CM. Then the representation

$$\prod_{\ell\in\mathbb{L}} \rho_\ell \colon \pi_1^{geo}(X) \longrightarrow \prod_{\ell\in\mathbb{L}} SL_2(\mathbb{Z}_\ell)$$

arising from $H^1(E_{F^{sep}}, \mathbb{Q}_\ell)$ has open image.

Serre (1972) proved analog for $\Gamma_K \to \prod_\ell GL_2(\mathbb{Z}_\ell)$, K a number field.

Corollary ($Y = E$ and $V_\ell = H^1(\dots)$)

- *G_ℓ^{geo} comes from a reductive group over $\mathbb{Z}_\ell$.*
- *$\rho_\ell(\pi_1(X)^{geo}) \subset SL_2(\mathbb{Q}_\ell)$ is compact open for all $\ell \neq p$.*
- *$\rho_\ell(\pi_1(X)^{geo}) = SL_2(\mathbb{Z}_\ell)$ for almost all ℓ.*
- *$\pi_1(X)^{geo}$ act semisimply on $H^1(E_{F^{sep}}, \mathbb{F}_\ell)$ for almost all ℓ*
- *the fields $\overline{\mathbb{F}_p}(F^{sep})^{Ker\,\rho_\ell}$, $\ell \in \mathbb{L}$, are 'almost independent'.*

General result on ℓ-adic cohomologies

Let $Y \to X$ be smooth proper, fix i and let $V_\ell = H^i(Y_{F^{sep}}, \mathbb{Q}_\ell)$.

Proposition (Serre '67)

$\rho_\ell(\pi_1^{geo}(X)) \subset G_\ell^{geo}(\mathbb{Q}_\ell)$ is open for all $\ell \in \mathbb{L}$

Theorem (Cadoret-Hui-Tamagawa '16)

After replacing X by a finite étale cover one has:

1. *$\pi_1(X)^{geo}$ acts semisimply on $H^1(Y_{F^{sep}}, \mathbb{F}_\ell)$ for almost all ℓ*
2. *there is a group scheme $\mathfrak{G}_\ell^{geo}/\mathbb{Z}_\ell$, reductive for almost all ℓ, such that $G_\ell^{geo} = \mathfrak{G}_\ell^{geo} \times_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ and $\rho_\ell(\pi_1^{geo}(X))$"="$\mathfrak{G}_\ell^{geo}(\mathbb{Z}_\ell)$*
3. *the following image is special adelic (in the sense of Hui-Larsen)*
$$\Big(\prod_{\ell\in\mathbb{L}} \rho_\ell\Big)(\pi_1^{geo}(X)) \subset \prod_{\ell\in\mathbb{L}} \mathfrak{G}_\ell^{geo}(\mathbb{Z}_\ell)$$

Part 2 'for all ℓ in a set of density 1' is due to Larsen ('95).

E-rational compatible systems

Let E be a number field with $\mathcal{P}'_E$ its set of finite places not above p.

Definition

An E-rational n-dimensional compatible system $\rho_\bullet$ consists of

1. a cont. homomorphism $\rho_\lambda \colon \pi_1(X) \to GL_n(E_\lambda)$ for all $\lambda \in \mathcal{P}'_E$.
2. a polynomial $P_x \in E[T]$ monic of degree n for all $x \in |X|$

such that

$$\text{charpol}_{\rho_\lambda(Frob_x)} = P_x \text{ for } E \hookrightarrow E_\lambda, \forall x \in |X|, \lambda \in \mathcal{P}'_E$$

Call $\rho_\bullet$ *semisimple* if all ρ_λ are semisimple.
Call $\rho_\bullet$ *pure of weight w (in $\mathbb{Z}$)* if for each $x \in |X|$ the roots of P_x are Weil q_x numbers of weight w.

Example

The earlier system $(V_\ell)_{\ell \in \mathbb{L}}$ is $\mathbb{Q}$-rational and pure of weight i.

Background results I

Theorem (Cadoret-Tamagawa '13, Böckle-Gajda-Petersen '13)
If $E = \mathbb{Q}$, there exists $X' \to X$ finite such that

$$\Big(\prod_{\ell\in\mathbb{L}} \rho_\ell\Big)\big(\pi_1^{geo}(X')\big) = \prod_{\ell\in\mathbb{L}} \big(\rho_\ell(\pi_1^{geo}(X'))\big).$$

Analog of a similar result by Serre in the number field case.

For arbitrary E: can apply the result to the $\mathbb{Q}$-rational system defined by $\rho_\ell = \prod_{\lambda|\ell} \rho_\lambda$.

Background results II

Proposition (classical)

Given a sequence of Frobenius polynomials $(P_x)_{x\in|X|}$ there is up to conjugacy at most one semisimple compatible system with these P_x.

Theorem (Serre '81, Larsen-Pink '92)

There exists $X' \to X$ finite such that $G_\lambda^{(geo)} := \overline{\rho_\lambda^{(geo)}(\pi_1(X'))}^{Zar}$ is connected for all λ.

Theorem (Deligne '80)

G_λ^{geo} is semisimple and $G_\lambda^{geo,o} = ([G_\lambda, G_\lambda])^o$ for all λ.

From now on consider only $\rho_\bullet$ which are **semisimple and** E-**rational**, and such that all $G_\lambda^{(geo)}$ are **connected**. $[\rho_\bullet \rightsquigarrow \rho_\bullet^{ss}]$

Then all $G_\lambda^{(geo)}$ are reductive (semisimple).

Background results III

Conjecture (Larsen-Pink '95)

There exists G/E reductive such that $G \times_E E_\lambda = G_\lambda$ for all $\lambda \in \mathcal{P}'_E$.

Theorem (Cheewhye Chin '04)

Over some finite extension $E' \supset E$ the above conjecture holds. (Call the group M/E' the Chin group*).*

Fix for each $\lambda \in \mathcal{P}'_E$ an $\mathcal{O}_\lambda$-lattice $\Lambda_\lambda \subset E_\lambda^n$ stable under $\pi_1(X)$.

Theorem (Larsen-Pink '95)

Let $\mathfrak{G}_\lambda^{(geo)}/\mathcal{O}_\lambda$ be the Zariski closure of $G_\lambda^{(geo)}$ in $\mathrm{Aut}_{\mathcal{O}_\lambda}(\Lambda_\lambda)$. Then the group scheme $\mathfrak{G}_\lambda^{(geo)}$ is smooth over $\mathcal{O}_\lambda$ for almost all λ.

Wishlist for reduction of $\rho_\bullet$

Using the lattices Λ_λ, can assume

$$\rho_\lambda\colon \pi_1(X) \to GL_n(\mathcal{O}_\lambda)$$

Define k_λ as the residue field of E_λ, denote the reduction of ρ_λ to k_λ by $\bar\rho_\lambda\colon \pi_1(X) \to GL_n(k_\lambda)$.

Wishlist (for almost all λ)

1. *$\bar\rho_\lambda$ is semisimple.*
2. *the reduction $\mathfrak{G}_{k_\lambda}^{(geo)} = \mathfrak{G}_\lambda^{(geo)} \times_{\mathcal{O}_\lambda} k_\lambda$ is reductive.*
3. *recover $\mathfrak{G}_{k_\lambda}^{(geo)}$ from the finite group $\bar\rho_\lambda(\pi_1^{(geo)}(X))$.*

The Nori envelope/Serre saturation

A square matrix is unipotent (nilpotent) if all eigenvalues are 1 (0).

$\exists$ exp, log in $GL_n^{unip}(\overline{\mathbb{F}_\ell}) \underset{\exp}{\overset{\log}{\rightleftarrows}} M_{n\times n}^{nilp}(\overline{\mathbb{F}_\ell})$ as truncated power series,

if $\ell > n$. For $u \in GL_n^{unip}(\overline{\mathbb{F}_\ell})$, $t \in \overline{\mathbb{F}_\ell}$ set $u^t := \exp(t \cdot \log u)$.

Definition (Nori envelope/Serre saturation)

The Nori envelope of a subgroup $H \subset GL_n(\overline{\mathbb{F}_\ell})$ is

$$H^{sat} = \overline{\langle u^t \mid u \in H \cap GL_n^{unip}(\overline{\mathbb{F}_\ell}), t \in \overline{\mathbb{F}_\ell}\rangle \cdot H}^{Zar}$$

Lemma

If H lies in $GL_n(\mathbb{F}_{\ell^e})$, then H^{sat} is defined over $\mathbb{F}_{\ell^e}$.

Examples $SL_n(\mathbb{F}_{\ell^e})^{sat} = SL_{n,\mathbb{F}_{\ell^e}}$; $GL_1(\mathbb{F}_{\ell^e})^{sat} = GL_1(\mathbb{F}_{\ell^e})$ (discrete).

Theorem (Serre)

If H acts semisimply on $\overline{\mathbb{F}_\ell}^n$, then H^{sat} is reductive.

The main theorem

Let $\rho_\bullet$, G_λ, Λ_λ, $\mathfrak{G}_\lambda$ be as above.

Theorem 1 (Böckle-Gajda-Petersen)

After passing to a finite cover of X, for all but finitely many $\lambda \in \mathcal{P}'_E$ the following hold:

1. $\mathfrak{G}^{geo}_{k_\lambda} \subset GL_{n,k_\lambda}$ *is saturated, i.e,* $\mathfrak{G}^{geo}_{k_\lambda} = \left(\mathfrak{G}^{geo}_\lambda(k_\lambda)\right)^{sat}$.
2. $\bar{\rho}_\lambda$ *is semisimple as a representation of* $\pi_1^{geo}(X)$.
3. $\mathcal{H}^{geo}_{k_\lambda} := \bar{\rho}_\lambda(\pi_1^{geo}(X))^{sat}$ *is reductive and defined over* k_λ.
4. $\mathcal{H}^{geo}_{k_\lambda} \subseteq \mathfrak{G}^{geo}_{k_\lambda}$ *is an equality.*

Corollary

Suppose the Chin group M of $\rho_\bullet$ is absolutely simple, the Chin representation $M \hookrightarrow GL_n$ is the adjoint representation, and E is minimal (as defined by Pink). Then

$$\Big(\prod_{\ell\in\mathbb{L}} \rho_\lambda\Big)(\pi_1(X)) \subset \prod_{\ell\in\mathbb{L}} \mathfrak{G}_\lambda(\mathcal{O}_\lambda)$$

is special adelic in the sense of Hui-Larsen.

Note here M is semisimple $\Rightarrow$ have result not only for $\pi_1^{geo}(X)$.

Global Langlands over function fields I

Let $\overline{X}$ be the smooth compactification of X.
Denote by N an effective divisor of $\overline{X}$ with support in $\overline{X} \setminus X$.

Theorem (L. Lafforgue (any n), Drinfeld ($n = 2$))

Part I:
Let Π be a cuspidal automorphic representation for $GL_{n/\mathbb{A}_F}$ of level N, central character $\tau\colon \pi_1(X) \to GL_1(\overline{\mathbb{Q}})$ of finite order, Hecke field E_0, and Hecke polynomial $P_{\Pi,x} \in E_0[T]$ at all $x \in |X|$.

Then for some $E \supset E_0$ there exists an E-rational compatible system

$$\rho_\lambda\colon \pi_1(X) \to GL_n(E_\lambda), \text{ for } \lambda \in \mathcal{P}'_E,$$

with Frobenius polynomials $(P_{\Pi,x})_{x\in|X|}$, such that $\det \rho_\bullet = \tau$, and $\rho_\bullet$ is absolutely irreducible and pure of weight zero.

Global Langlands over function fields II

Theorem (L. Lafforgue (any n), Drinfeld ($n = 2$))

Part II: Let $\rho\colon \pi_1(X) \to GL_n(\overline{\mathbb{Q}_\ell})$ be continuous and absolutely irreducible with finite order determinant τ and conductor at most N.

Then there exists a cuspidal automorphic representation for $GL_{n/\mathbb{A}_F}$ of level N, and central character τ such that

$$\rho = \iota \circ \rho_{\Pi,\lambda}$$

for some continuous embedding $\iota\colon E_\lambda \hookrightarrow \overline{\mathbb{Q}_\ell}$.
In particular, ρ is a member of a compatible system.

Part III: The above correspondence is compatible with the local Langlands correspondence at all $x \in |\overline{X}| \smallsetminus |X|$.

Consequences I

Corollary (Passage to irreducibility and trivial determinant)

Let $\rho_\bullet$ be E-rational semisimple compatible.

1. *After possible enlarging E, one has*

$$\rho_\bullet = \rho_{\bullet,1} \oplus \ldots \oplus \rho_{\bullet,r}$$

   for absolutely irreducible compatible systems $\rho_{\bullet,i}$

2. *After possibly passing to a finite cover $X' \to X$, can write*

$$\rho_{\bullet,i} = \rho'_{\bullet,i} \otimes \tau_{\bullet,i}$$

   with $\tau_{\bullet,i}$ one-dimensional and $\rho'_{\bullet,i}$ pure of weight zero.

Further reduction steps reduce Theorem 1 to the case:
$\rho_\bullet$ is absolutely irreducible, $\det \rho_\bullet = 1$, all G_λ are connected.

Consequences II

Corollary (Conductor)

The conductor of the ρ_λ in a semisimple compatible system $\rho_\bullet$ is independent of λ.

Let $\tau\colon \pi_1(X) \to GL_1(\overline{\mathbb{Q}})$ be fixed and continuous. Let N be fixed as before.

Corollary (Finiteness)

For any N and n there are only finitely many absolutely irreducible n-dimensional compatible systems $\rho_\bullet$ with conductor bounded by N and $\det \rho_\bullet = \tau$.

Consequences of de Jong's and Gaitsgory's results

Using mainly results of de Jong and Gaitsgory one has

Theorem 2 (B.-Harris-Khare-Thorne)

Suppose $G \hookrightarrow GL_n$ *is reductive over* $W(\mathbb{F}_{\ell^e})$ *and* $\ell > 2 \dim G_\lambda$.
Let $\bar{\rho}\colon \pi_1(X) \to G(\mathbb{F}_{\ell^e}) \hookrightarrow GL_n(\mathbb{F}_{\ell^e})$ *be absolutely irreducible.*
Let $\chi\colon \pi_1(X) \to GL_1(W(\mathbb{F}_{\ell^e}))$ *be a continuous lift of* $\det \bar{\rho}$.

Then $\bar{\rho}$ *has a lift* $\rho\colon \pi_1(X) \to G(W(\mathbb{F}_{\ell^e}))$ *with* $\det \rho = \chi$.

Moreover if $\bar{N}$ *is the conductor of* $\bar{\rho}$, *and* $\bar{T}$ *that of* χ, *then the conductor of* ρ *can be bounded* $\bar{N} + n \cdot \bar{T} + n(\overline{X} \setminus X)$.

Residually compatible systems

Let $\mathcal{P} \subset \mathcal{P}'_E$ be infinite.
Write $\mathfrak{p}_\lambda \subset \mathcal{O}_E$ for the maximal ideal defined by $\lambda \in \mathcal{P}'_E$.

Definition

An E-rational n-dim. residually compatible system $\bar{\rho}_\bullet$ over $\mathcal{P}$ is

1. a cont. homomorphism $\bar{\rho}_\lambda\colon \pi_1(X) \to GL_n(k_\lambda)$ for all $\lambda \in \mathcal{P}$.
2. a Polynomial $\bar{P}_x \in \mathcal{O}_E[\frac{1}{p}][T]$ monic of degree n for all $x \in |X|$

such that

$$\mathrm{charpol}_{\bar{\rho}_\lambda(Frob_x)} \equiv \bar{P}_x \pmod{\mathfrak{p}_\lambda},\ \forall x \in |X|, \lambda \in \mathcal{P}$$

Lemma 1

For any residually compatible system $\bar{\rho}_\bullet$ of bounded conductor there exists a unique semisimple compatible system $\rho_\bullet$ over some $E' \supset E$ such that $\bar{P}_x = P_x \ \ \forall x \in |X|$. Moreover if all $\bar{\rho}_\lambda$ are reducible, then so is $\rho_\bullet$.

Proof of Lemma 1

Over some finite extension k'_λ of k_λ have

$$\bar\rho_\lambda^{ss} = \bar\rho_{\lambda,1} \oplus \ldots \oplus \bar\rho_{\lambda,n_\lambda}$$

with $\bar\rho_{\lambda,i}$ absolutely irreducible.

Use knowledge of eigenvalues of $Frob_x$ via $\bar P_x$ for one x to ensure: there is a finite set of lists $(\tau_1, \ldots, \tau_s)$ of finite order characters such that each $(\det \bar\rho_{\lambda,1}, \ldots, \det \bar\rho_{\lambda,n_\lambda})$ is one list mod $\mathfrak{p}_\lambda$

Use Theorem 2 to obtain a lift $\rho_{\lambda,1} \oplus \ldots \oplus \rho_{\lambda,n_\lambda}$ with $\det \rho_{\lambda,i} = \tau_i$.

Finiteness of lists and of partitions of n and conductor bound in Theorem 2 (for GL_{n_i}) shows:
there exist automorphic representations $\Pi_1, \ldots, \Pi_s$ (for $GL_{n_i,\mathbb{A}_F}$) such that $\oplus_j \rho_{\Pi_j,\lambda} \equiv \bar\rho_\lambda \;\, mod \;\, \mathfrak{p}_\lambda$ for infinitely many $\lambda \in \mathcal{P}$.

Absolute irreducibility

Corollary (Drinfeld)

Suppose $\rho_\bullet$ is absolutely irreducible. Then $\bar{\rho}_\lambda$ is absolutely irreducible for almost all $\lambda \in \mathcal{P}'_E$.

Proof.

Suppose infinitely many $\bar{\rho}_\lambda$ are reducible. They form a residually compatible reducible system. By Lemma 1 the latter arises from a reducible compatible system $\rho'_\bullet$.
Now $P'_x = P_x$ for all $x \in |X|$ gives a contradiction. □

Recall: Main Theorem in the absolutely irreducible case

Suppose $\rho_\bullet$ is absolutely irreducible and the G_λ are connected semisimple. Need to show:

Theorem 1' (Böckle-Gajda-Petersen)

After passing to a finite cover of X, for all but finitely many $\lambda \in \mathcal{P}'_E$ the following hold:

1. $\mathfrak{G}_{k_\lambda} \subset GL_{n,k_\lambda}$ *is saturated.*
2. $\bar{\rho}_\lambda$ *is absolutely irreducible. (Drinfeld).*
3. $\mathcal{H}_{k_\lambda} := \bar{\rho}_\lambda(\pi_1(X))^{sat}$ *is semisimple and defined over k_λ.*
4. $\mathcal{H}_{k_\lambda} \subseteq \mathfrak{G}_{k_\lambda}$ *is an equality.*

Part 2 was just shown. Part 1 I will not discuss.
Part 3 follows from part 2 and an earlier quoted result of Serre.
The inclusion in 4 follows from 1 and the definitions.

Saturated image and the Chin group

Recall $\mathcal{H}_{k_\lambda} = \bar{\rho}_\lambda(\pi_1(X))^{sat}$.

Lemma 2

Suppose $\rho_\bullet$ is E-rational absolutely irreducible with $\det \rho_\bullet = 1$. Assume $\bar{\rho}_\lambda(\pi_1(X))$ is ℓ_λ-generated ($\ell_\lambda = \mathrm{Char}\, k_\lambda$) for almost all λ.

Then for almost all $\lambda \in \mathcal{P}$ there exists a semisimple group $\mathfrak{H}_\lambda / W(k_\lambda)$ with generic fiber G_λ and special fiber $\mathcal{H}_{k_\lambda}$.

Proof of Lemma 2

For $\ell_\lambda \gg 0$ we have:

- $\mathcal{H}_{k_\lambda}$ is semisimple by Theorem 1'(iii).
- $\dim \mathcal{H}_{k_\lambda} \leq \dim \mathfrak{G}_{k_\lambda} = \dim G_\lambda$.
- $\mathcal{H}_{k_\lambda}$ is connected because $\bar{\rho}_\lambda(\pi_1(X))$ is ℓ-generated.
- The irreducible representation $\bar{r}\colon \mathcal{H}_{k_\lambda} \hookrightarrow GL_n$ over k_λ is of low weight (ℓ_λ-restricted) because $\mathcal{H}_{k_\lambda}$ is saturated.

Using results of Jantzen (and Serre):
There is a lift $r\colon \mathfrak{H}_\lambda \hookrightarrow GL_n$ of $\bar{r}$ to $W(k_\lambda)$ with $\mathfrak{H}_\lambda$ semisimple.
By Theorem 2 there is a lift
$\rho'_\lambda\colon \pi_1(X) \to \mathfrak{H}_\lambda(W(k_\lambda)) \hookrightarrow GL_n(W(k_\lambda))$ of $\bar{\rho}_\lambda$.
Have $\rho'_\lambda \cong \rho_{\Pi_\lambda,\lambda}$ for some Π_λ. The number of possible Π_λ is finite.
$\Rightarrow \rho_{\Pi,\lambda} = \rho'_\lambda$ for one Π and almost all λ.
Also have $\bar{\rho}_\bullet = \bar{\rho}_{\Pi,\bullet}$ (and thus $P_x = P'_x \ \forall x \in |X|$) $\Rightarrow \rho_\bullet \cong \rho_{\Pi,\bullet}$.

Conclusion

Corollary

$\mathcal{H}_{k_\lambda} = \mathfrak{G}_{k_\lambda}$ *for almost all* λ*.*

Proof.

We know already that $\mathcal{H}_{k_\lambda} \subseteq \mathfrak{G}_{k_\lambda}$.

By passing to a finite cover $X' \to X$ one can achieve that all groups $\bar{\rho}_\lambda(\pi_1(X))$ are ℓ_λ-generated. This does not change $\mathfrak{G}_{k_\lambda}$.

By Lemma 2 we have $\dim \mathcal{H}_{k_\lambda} = \dim G_\lambda = \dim \mathfrak{G}_{k_\lambda}$. □

An M-compatible system

To end, let me explain the idea of the reduction step:
Theorem 1' ($\rho_\bullet$ absolutely irreducible) implies Theorem 1.

Theorem (B.-Harris-Khare-Thorne, building on Chin)

Suppose $\rho_\bullet$ is semisimple, say with Chin group M over E. After enlarging E there is an M-compatible system

$$\rho_\lambda^M\colon \pi_1(X) \to M(E_\lambda),\ \lambda \in \mathcal{P}'_E,$$

and a representation $\alpha\colon M \to GL_n$, defined over E, such that

$$\alpha \circ \rho_\lambda^M = \rho_\lambda \text{ for all } \lambda \in \mathcal{P}'_E$$

The system $\rho_\bullet^M$ is unique up to conjugacy.

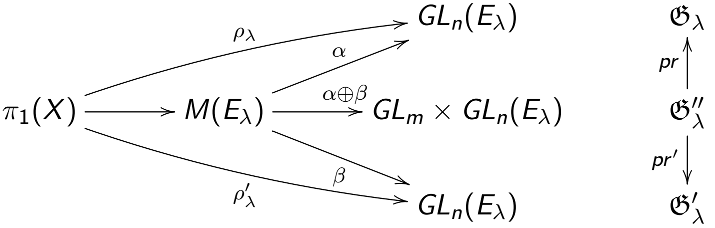
Note M-compatible means that for all λ and x the conjugacy class of $\rho_\lambda^M(Frob_x)$ lies in $M(\overline{\mathbb{Q}})$ and is independent of λ.

Changing the given representation

Say $\rho_\bullet = \bigoplus_i \rho_{i,\bullet}$ with $\rho_{i.\bullet}$ absolutely irreducible of weight zero.
Let $\rho_\bullet^M$ and α be as above, so that $\rho_\bullet = \alpha \circ \rho_\bullet^M$.

Let $\beta\colon M \to SL_m$ be almost faithful and irreducible (over E).
Then $\rho'_\bullet := \beta \circ \rho_\bullet^M$ is absolutely irreducible of weight zero.
Apply Theorem 1' to $\rho'_\bullet \Rightarrow$ almost all $\mathfrak{G}'_\lambda/\mathcal{O}_\lambda$ are semisimple

To finish off: compare $\mathfrak{G}_\lambda$ to $\mathfrak{G}'_\lambda$ via the three representations



Thank you!