Singular perturbations for port-Hamiltonian systems, normal hyperbolicity and non-hyperbolicity

Hildeberto Jardon Kojakhmetov, **Jacquelien M.A. Scherpen**

Jan C. Willems Center for Systems and Control
Faculty of Science and Engineering
University of Groningen

BIRS Women in Control: New Trends in Infinite Dimensions, 20 July 2017

university of groningen

My PDE background

- Port-Hamiltonian modeling and control of nonlinear piezoelectric material, with Thomas Voß, results published from 2009-2014.
- Structure preserving discretization, then design passivity based (energy shaping and damping injection) controller.
- Currently, ongoing work with Kirsten Morris to understand various issues related to the modeling and discretization (controllability, stabilizability, voltage versus current control, etc.).

My PDE background

- Port-Hamiltonian modeling and control of nonlinear piezoelectric material, with Thomas Voß, results published from 2009-2014.
- Structure preserving discretization, then design passivity based (energy shaping and damping injection) controller.
- Currently, ongoing work with Kirsten Morris to understand various issues related to the modeling and discretization (controllability, stabilizability, voltage versus current control, etc.).
- Also, discrete geometry approach for the structure preserving discretization of port-Hamiltonian systems, with Marko Seslija and Arjan van der Schaft, results published from 2010-2014.
- Results for various systems, such as reaction diffusion systems, very useful for stability analysis and control!

The ODE case: Port-Hamiltonian systems *Maschke, van der Schaft, 1992*

General description in x coordinates on some n dimensional manifold:

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

where
$J(x) = -J^T(x)$: interconnection structure (related to Dirac structures)
$R(x) = R^T(x) \geq 0$): damping
$H(x) > 0$: is the Hamiltonian (total energy).

The ODE case: Port-Hamiltonian systems *Maschke, van der Schaft, 1992*

General description in x coordinates on some n dimensional manifold:

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

where
$J(x) = -J^T(x)$: interconnection structure (related to Dirac structures)
$R(x) = R^T(x) \geq 0)$: damping
$H(x) > 0$: is the Hamiltonian (total energy).

Nice property: $\boxed{\dot{H} = -\frac{\partial^T H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + y^T u \leq y^T u}$

The ODE case: Port-Hamiltonian systems *Maschke, van der Schaft, 1992*

General description in x coordinates on some n dimensional manifold:

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

where
$J(x) = -J^T(x)$: interconnection structure (related to Dirac structures)
$R(x) = R^T(x) \geq 0)$: damping
$H(x) > 0$: is the Hamiltonian (total energy).

Nice property: $\boxed{\dot{H} = -\frac{\partial^T H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + y^T u \leq y^T u}$

Passivity! Very useful for Passivity Based Control, Control based on the port-Hamiltonian structure!

Pde version of PH systems *Maschke, van der Schaft*

- Hamiltonian

$$H(x(z)) = \int_V \mathcal{H}(x(z))dV$$

where $\mathcal{H}(x(z))$ is the energy density depending on the state $\mathbf{x}$ at a specific point $z \in V$ in the n dimensional volume $V \subseteq \mathcal{Z}$.

- From Stokes-Dirac structure, interconnection structure represented by skew-adjoint differential operator $J(x) = -J(x)^*$.

- Ports are either boundary or distributed ports.

Pde version of PH systems

- No losses: distributed port model

$$\begin{aligned}\dot{x} &= J(x)\frac{\delta H}{\delta x} + B(x)u \\ y &= B^*(x)\frac{\delta H}{\delta x}\end{aligned}$$

  plus boundary conditions.

- Passivity property

$$\frac{dH}{dt} = \int_V \frac{\delta^\top H}{\delta x}\left(J\frac{\delta H}{\delta x} + Bu\right) dV = \int_V y^\top u dV.$$

- Passive interconnection preserves PH structure.

university of groningen

Reaction-diffusion systems

E.g., *Smoller 1994, Teman 1997*.

- *Seslija, van der Schaft, Scherpen, Automatica 2014*, obtained a PH model to study the effect of diffusion on balanced reaction networks governed by mass action kinetics.
- Diffusion may lead to instability.
- Geometric perspective, PH structure makes immediate study of passivity possible.

Reaction-diffusion systems

E.g., *Smoller 1994, Teman 1997*.

- *Seslija, van der Schaft, Scherpen, Automatica 2014*, obtained a PH model to study the effect of diffusion on balanced reaction networks governed by mass action kinetics.
- Diffusion may lead to instability.
- Geometric perspective, PH structure makes immediate study of passivity possible.
- Stability with help of Krasowskii-LaSalle principle in general difficult, as it needs compactness and global boundedness of solutions.
- For constant diffusion systems some asymptotic stability is proved, and with Neumann boundary conditions conjecture about more general stability is possible.
- Application to compartmental stability of a glycolisis pathway reaction is possible.

PH reaction diffusion model

$$
\begin{aligned}
\frac{\partial x}{\partial t} &= div\left(R_d(x)gradLn(\frac{x}{x^*})\right) - ZBK(x^*)B^T e^{Z^T Ln(\frac{x}{x^*})}, \\
e_b &= \left(Ln(\frac{x}{x^*})\right)|_{\partial M} \\
f_b &= \left(-R_d(x)gradLn(\frac{x}{x^*}) \cdot K(x^*)B^T e^{Z^T Ln(\frac{x}{x^*})}\right)|_{\partial M}
\end{aligned}
$$

with Z complex stoichiometric matrix, x concentrations, x^* equilibrium concentrations, $K(x) > 0$, diagonal matrix with reaction constants, R_d energy diffusion operator.

with the Gibbs free energy $G(x) = x^T Ln(\frac{x}{x^*}) + (x - x^*)^T \mathbf{1}$ defining total energy (Hamiltonian)

$$\mathcal{G} = \int_{\mathcal{M}} \mathcal{G}$$

university of groningen

Slow-Fast reaction-diffusion systems, an open problem

- Studies for fast reaction, slow diffusion scenarios mainly done for analysis purposes, see e.g., *Bykov, Cherkinsky, Goldshtein, Krapivnik, Maas, arXiv, Jan. 2017*.

Slow-Fast reaction-diffusion systems, an open problem

- Studies for fast reaction, slow diffusion scenarios mainly done for analysis purposes, see e.g., *Bykov, Cherkinsky, Goldshtein, Krapivnik, Maas, arXiv, Jan. 2017*.
- Fast reaction layer often causes unexpected effects, such as in flame propagation in combustion theory, segragation and aggregation of biological individuals, chemical attack on porous materials (e.g., concrete or natural stone), etc.

Slow-Fast reaction-diffusion systems, an open problem

- Studies for fast reaction, slow diffusion scenarios mainly done for analysis purposes, see e.g., *Bykov, Cherkinsky, Goldshtein, Krapivnik, Maas, arXiv, Jan. 2017*.
- Fast reaction layer often causes unexpected effects, such as in flame propagation in combustion theory, segragation and aggregation of biological individuals, chemical attack on porous materials (e.g., concrete or natural stone), etc.
- Control problems are not treated so far, as composite control as in ode case is not yet having a counter part in pde case (to the best of my knowledge).

Slow-Fast reaction-diffusion systems, an open problem

- Studies for fast reaction, slow diffusion scenarios mainly done for analysis purposes, see e.g., *Bykov, Cherkinsky, Goldshtein, Krapivnik, Maas, arXiv, Jan. 2017*.
- Fast reaction layer often causes unexpected effects, such as in flame propagation in combustion theory, segragation and aggregation of biological individuals, chemical attack on porous materials (e.g., concrete or natural stone), etc.
- Control problems are not treated so far, as composite control as in ode case is not yet having a counter part in pde case (to the best of my knowledge).

Next: ode case.

Regular perturbation

Consider the algebraic problem

$$x^2 + \varepsilon x - 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

It has solutions

$$x_{1,2} = \frac{-\varepsilon \pm \sqrt{4 + \varepsilon^2}}{2} = \pm 1 + O(\varepsilon)$$

Regular perturbation

Consider the algebraic problem

$$x^2 + \varepsilon x - 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

It has solutions

$$x_{1,2} = \frac{-\varepsilon \pm \sqrt{4 + \varepsilon^2}}{2} = \pm 1 + O(\varepsilon)$$

The limit equation is

$$x^2 - 1 = 0$$

which has solutions $x_{1,2} = \pm 1$.

Regular perturbation

Consider the algebraic problem

$$x^2 + \varepsilon x - 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

It has solutions

$$x_{1,2} = \frac{-\varepsilon \pm \sqrt{4 + \varepsilon^2}}{2} = \pm 1 + O(\varepsilon)$$

The limit equation is

$$x^2 - 1 = 0$$

which has solutions $x_{1,2} = \pm 1$.

"The solutions of the limit equation are
ε-close to those of the original problem"

Singular perturbation

Consider the algebraic problem

$$\varepsilon x^2 + x - 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

It has solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{4\varepsilon + 1}}{2\varepsilon} \in O(1/\varepsilon)$$

Singular perturbation

Consider the algebraic problem

$$\varepsilon x^2 + x - 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

It has solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{4\varepsilon + 1}}{2\varepsilon} \in O(1/\varepsilon)$$

The limit equation is

$$x - 1 = 0$$

which has (one) solution $x = 1$.

Singular perturbation

Consider the algebraic problem

$$\varepsilon x^2 + x - 1 = 0, \qquad 0 < \varepsilon \ll 1.$$

It has solutions

$$x_{1,2} = \frac{-1 \pm \sqrt{4\varepsilon + 1}}{2\varepsilon} \in O(1/\varepsilon)$$

The limit equation is

$$x - 1 = 0$$

which has (one) solution $x = 1$.

"The solutions of the limit equation are
not close to those of the original problem"

Singularly perturbed ODEs (a.k.a. Slow-Fast Systems)

slow

$$\dot{x} = f(x, z, \varepsilon)$$
$$\varepsilon \dot{z} = g(x, z, \varepsilon)$$

$$\tau = \frac{t}{\varepsilon}$$
$$0 < \varepsilon \ll 1$$

fast

$$x' = \varepsilon f(x, z, \varepsilon)$$
$$z' = g(x, z, \varepsilon)$$

$\varepsilon \to 0$ $\varepsilon \to 0$

DAE

$$\dot{x} = f(x, z, 0)$$
$$0 = g(x, z, 0)$$

layer

$$x' = 0$$
$$z' = g(x, z, 0)$$

$\mathcal{S}$

critical manifold

university of groningen

Singularly perturbed ODEs (a.k.a. Slow-Fast Systems)

slow

$$\dot{x} = f(x, z, \varepsilon)$$
$$\varepsilon \dot{z} = g(x, z, \varepsilon)$$

$$\tau = \frac{t}{\varepsilon} \quad \longleftrightarrow \quad 0 < \varepsilon \ll 1$$

fast

$$x' = \varepsilon f(x, z, \varepsilon)$$
$$z' = g(x, z, \varepsilon)$$

$\varepsilon \to 0$ (slow system) $\qquad\qquad$ $\varepsilon \to 0$ (fast system)

DAE

$$\dot{x} = f(x, z, 0)$$
$$0 = g(x, z, 0)$$

layer

$$x' = 0$$
$$z' = g(x, z, 0)$$

$\mathcal{S}$

critical manifold

Singularly perturbed ODEs (a.k.a. Slow-Fast Systems)

slow

$$\dot{x} = f(x, z, \varepsilon)$$
$$\varepsilon \dot{z} = g(x, z, \varepsilon)$$

$$\tau = \frac{t}{\varepsilon} \qquad 0 < \varepsilon \ll 1$$

fast

$$x' = \varepsilon f(x, z, \varepsilon)$$
$$z' = g(x, z, \varepsilon)$$

$\varepsilon \to 0$ (slow system) $\qquad$ $\varepsilon \to 0$ (fast system)

DAE

$$\dot{x} = f(x, z, 0)$$
$$0 = g(x, z, 0)$$

layer

$$x' = 0$$
$$z' = g(x, z, 0)$$

$\mathcal{S}$

critical manifold

Singularly perturbed ODEs (a.k.a. Slow-Fast Systems)

slow:

$$\dot{x} = f(x, z, \varepsilon)$$
$$\varepsilon \dot{z} = g(x, z, \varepsilon)$$

$$\tau = \frac{t}{\varepsilon} \quad \longleftrightarrow \quad 0 < \varepsilon \ll 1$$

fast:

$$x' = \varepsilon f(x, z, \varepsilon)$$
$$z' = g(x, z, \varepsilon)$$

$\varepsilon \to 0$ (slow $\to$ DAE) $\qquad$ $\varepsilon \to 0$ (fast $\to$ layer)

DAE:

$$\dot{x} = f(x, z, 0)$$
$$0 = g(x, z, 0)$$

layer:

$$x' = 0$$
$$z' = g(x, z, 0)$$

$\mathcal{S}$

critical manifold

Normal hyperbolicity

Definition (Critical manifold)

$$\mathcal{S} = \{(x,z) \in \mathcal{X} \times \mathcal{Z} \mid g(x,z,0) = 0\}$$

$\mathcal{S}$ is said to be *Normally Hyperbolic* if spec $\left\{ \frac{\partial g}{\partial z}(x,z,0) \right\}$ has nonzero real part.

Normal hyperbolicity

Definition (Critical manifold)

$$\mathcal{S} = \{(x,z) \in \mathcal{X} \times \mathcal{Z} \,|\, g(x,z,0) = 0\}$$

$\mathcal{S}$ is said to be *Normally Hyperbolic* if spec $\left\{ \frac{\partial g}{\partial z}(x,z,0) \right\}$ has nonzero real part.

Recall the reduced systems

$$\dot{x} = f(x,z,0) \qquad\qquad x' = 0$$
$$0 = g(x,z,0) \qquad\qquad z' = g(x,z,0)$$

$\rightarrow$ The manifold $\mathcal{S}$ is the phase-space of the DAE and the set of equilibrium points of the layer equation.

$\mathcal{S}$ is NH if each point $(x,z) \in \mathcal{S}$ is a hyperbolic equilibrium point of the layer equation.

Normal hyperbolicity

Definition (Critical manifold)

$$\mathcal{S} = \{(x, z) \in \mathcal{X} \times \mathcal{Z} \,|\, g(x, z, 0) = 0\}$$

$\mathcal{S}$ is said to be *Normally Hyperbolic* if spec $\left\{ \frac{\partial g}{\partial z}(x, z, 0) \right\}$ has nonzero real part.

If $\mathcal{S}$ is NH, then $\exists\, h_0(x)$ such that locally[1]

$$\mathcal{S} = \{(x, z) \in \mathcal{X} \times \mathcal{Z} \,|\, z = h_0(x)\}$$

[1] Implicit Function Theorem

Normal hyperbolicity

Definition (Critical manifold)

$$\mathcal{S} = \{(x, z) \in \mathcal{X} \times \mathcal{Z} \,|\, g(x, z, 0) = 0\}$$

$\mathcal{S}$ is said to be *Normally Hyperbolic* if spec $\left\{ \frac{\partial g}{\partial z}(x, z, 0) \right\}$ has nonzero real part.

If $\mathcal{S}$ is NH, then $\exists\, h_0(x)$ such that locally[1]

$$\mathcal{S} = \{(x, z) \in \mathcal{X} \times \mathcal{Z} \,|\, z = h_0(x)\}$$

Then, the flow along $\mathcal{S}$ is given by the reduced *slow* system

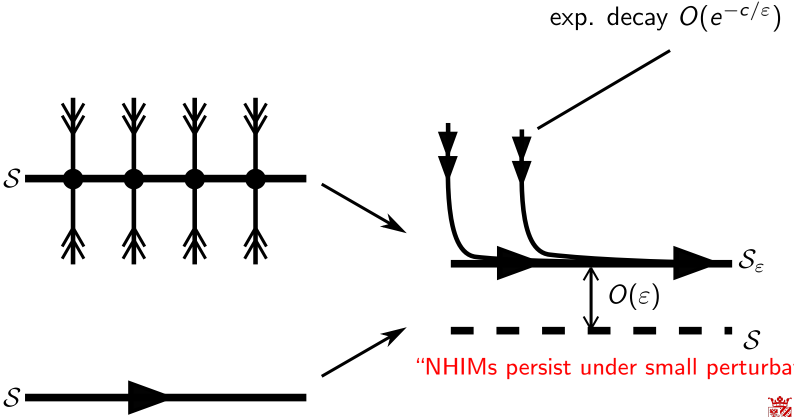
$$\dot{x} = f(x, h_0(x), 0)$$

[1]Implicit Function Theorem

Geometric Singular Perturbation Theory *N. Fenichel, 1979*

Let $\bar{\mathcal{S}}$ be NH and $\mathcal{S} \subseteq \bar{\mathcal{S}}$ be compact. Then, for $\varepsilon > 0$ sufficiently small

- $\exists$ an invariant manifold $\mathcal{S}_\varepsilon$ diffeomorphic to $\mathcal{S}$
- The flow along $\mathcal{S}_\varepsilon$ is ε-close to the flow along $\mathcal{S}$



"NHIMs persist under small perturbations"

Application to control - Composite control *Kokotović et al.*

Relies on Tikhonov's theorem (1935).

$$\dot{x} = f(x, z, \varepsilon, u) \qquad x' = \varepsilon f(x, z, \varepsilon, u)$$
$$\varepsilon \dot{z} = g(x, z, \varepsilon, u) \qquad z' = g(x, z, \varepsilon, u)$$

$$\dot{x} = f(x, z, 0, u) \qquad x' = 0$$
$$0 = g(x, z, 0, u) \qquad z' = g(x, z, 0, u)$$

Let $\mathcal{S}$ be NH and $u = u_s(x) + u_f(x, z), \quad u_f(x, z)|_{\mathcal{S}} = 0$

$$\dot{x} = f_r(x, z, u_s) \qquad z' = \bar{g}_x(z, u_f)$$

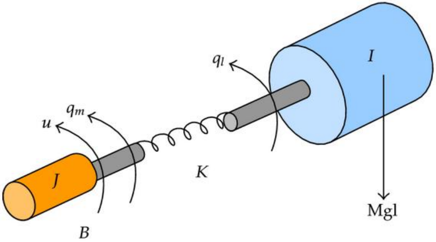
Stabilize reduced subsystems and combine for overall control

Main message of Normal Hyperbolicity

"Model order reduction and composite control only hold around hyperbolic points."

Motivation

Flexible-joint robots are a standard example of two time scale mechanical systems



Goal: to follow a desired trajectory with only position measurements
Assumption: $|K|$ is large

Flexible-joint manipulator

Joint flexibility can be attributed to:

- Harmonic drives
- Transmission belts
- Long shafts
- Robotic hands
- Variable stiffness drives for safety/interaction purposes
- ⋮

Some preliminary remarks:

- Flexible-joint robots have been studied for many years
- Port-Hamiltonian systems + singular perturbations have a wide range of applicability

Standard mechanical systems in the PH framework

Generalized coordinates q, generalized momenta p. Hamiltonian:

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + V(q)$$

$V(q) > 0$ potential energy, $M(q) = M^T(q) > 0$ mass inertia matrix.

Standard mechanical systems in the PH framework

Generalized coordinates q, generalized momenta p. Hamiltonian:

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + V(q)$$

$V(q) > 0$ potential energy, $M(q) = M^T(q) > 0$ mass inertia matrix.

Model without damping:

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + B(x)u \\ y &= B^T(x) \frac{\partial H}{\partial p}(q,p) \end{aligned}$$

Input is a generalized force, output is a generalized velocity, $u^T y$ is the supplied power.

Port-Hamiltonian model of a flexible-joint robot 1/4

$q_1 \in \mathbb{R}^n$ links' coordinate,

- Link's kinetic energy:

$$K_l(q_1, \dot{q}_1) = \frac{1}{2}\dot{q}_1^T M_l(q_1)\dot{q}_1$$

- Motor's kinetic energy:

$$K_m(\dot{q}_2) = \frac{1}{2}\dot{q}_2^T I \dot{q}_2$$

$q_2 \in \mathbb{R}^n$ motors' coordinate

- Potential energy due to gravity

$$P_g(q_1) = \sum_{i=1}^{n} (P_{g,l_i}(q_1) + P_{g,m_i}(q_1))$$

- Potential energy due to joint stiffness

$$P_s(q_1, q_2) = \frac{1}{2}(q_1 - q_2)^T K(q_1 - q_2),$$

where $K \in O(1/\varepsilon)$.

university of groningen

Port-Hamiltonian model of a flexible-joint robot 2/4

Total energy

$$H = \frac{1}{2}\dot{q}_1^T M_l(q_1)\dot{q}_1 + \frac{1}{2}\dot{q}_2^T I \dot{q}_2 + P_g(q_1) + \frac{1}{2\varepsilon}(q_1 - q_2)^T(q_1 - q_2)$$

Let

$$\varepsilon z = q_1 - q_2.$$

Then

$$\bar{H} = \boxed{\frac{1}{2}\dot{q}_1^T (M_l(q_1) + I)\dot{q}_1 + P_g(q_1)} + \varepsilon \left(-\dot{q}_1^T I \dot{z} + \frac{1}{2}\varepsilon \dot{z}^T I \dot{z} + \frac{1}{2} z^T z\right)$$

Rigid robot

Let $q = (q_1, z)$, $\bar{H}$ can be written as

$$\bar{H} = \frac{1}{2} p^T M_\varepsilon{}^{-1}(q) p + V_\varepsilon(q),$$

where

$$M_\varepsilon = \begin{bmatrix} M_l(q_1) + I & -\varepsilon I \\ -\varepsilon I & \varepsilon^2 I \end{bmatrix}, \qquad p = M_\varepsilon \dot{q}, \qquad V_\varepsilon(q) = P_g(q_1) + \frac{1}{2} \varepsilon z^T z$$

Major obstruction for *good* model.

Let $q = (q_1, z)$, $\bar{H}$ can be written as

$$\bar{H} = \frac{1}{2} p^T M_\varepsilon^{-1}(q) p + V_\varepsilon(q),$$

where

$$M_\varepsilon = \begin{bmatrix} M_l(q_1) + I & -\varepsilon I \\ -\varepsilon I & \varepsilon^2 I \end{bmatrix}, \qquad p = M_\varepsilon \dot{q}, \qquad V_\varepsilon(q) = P_g(q_1) + \frac{1}{2}\varepsilon z^T z$$

Port-Hamiltonian model of a flexible-joint robot 3/4

Let $q = (q_1, z)$, $\bar{H}$ can be written as

Major obstruction for *good* model.

$$\bar{H} = \frac{1}{2} p^T M_\varepsilon{}^{-1}(q) p + V_\varepsilon(q),$$

where

$$M_\varepsilon = \begin{bmatrix} M_l(q_1) + I & -\varepsilon I \\ -\varepsilon I & \varepsilon^2 I \end{bmatrix}, \qquad p = M_\varepsilon \dot{q}, \qquad V_\varepsilon(q) = P_g(q_1) + \frac{1}{2}\varepsilon z^T z$$

What is good model? Consider

$$\begin{bmatrix} \dot{x}_1 \\ \varepsilon \dot{x}_2 \end{bmatrix} = J(x, \varepsilon)\frac{\partial H}{\partial x} + G(x, \varepsilon) u \qquad \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{J}_{11} & \bar{J}_{12} \\ \bar{J}_{21} & \bar{J}_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}}{\partial x_1} \\ \frac{\partial \bar{H}}{\partial x_2} \end{bmatrix} + \begin{bmatrix} \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} u$$

NH implies $x_2 = h_0(x_1, u)$. Then, reduced system is not necessarily in port-Hamiltonian format.

Let $q = (q_1, z)$, $\bar{H}$ can be written as

Major obstruction for *good* model.

$$\bar{H} = \frac{1}{2} p^T M_\varepsilon{}^{-1}(q) p + V_\varepsilon(q),$$

where

$$M_\varepsilon = \begin{bmatrix} M_l(q_1) + I & -\varepsilon I \\ -\varepsilon I & \varepsilon^2 I \end{bmatrix}, \qquad p = M_\varepsilon \dot{q}, \qquad V_\varepsilon(q) = P_g(q_1) + \frac{1}{2}\varepsilon z^T z$$

Solution: use canonical change of coordinates[2][3] to obtain

$$\bar{H}_\varepsilon(\bar{q}, \bar{p}) = \frac{1}{2}\bar{p}^\top \bar{p} + \bar{V}_\varepsilon(\bar{q})$$

[3] *Fujimoto, K. and Sugie, T. (2001).*

[3] *Viola, G., Ortega, R., Banavar, R., Acosta, J.A., and Astolfi, A. (2007).*

Port-Hamiltonian model of a flexible-joint robot 4/4

By using a canonical transformation we can write the port-Hamiltonian model of a flexible-joint robot as

$$\begin{bmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{q}}_2 \\ \dot{\bar{p}}_1 \\ \dot{\bar{p}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & t_1^{-T} & \alpha^T \\ 0 & 0 & 0 & \frac{t_4^{-T}}{\varepsilon} \\ -t_1^{-1} & 0 & j_1 & j_{21} - \frac{j_{22}}{\varepsilon} \\ -\alpha^T & -\frac{t_4^{-1}}{\varepsilon} & -j_{21} + \frac{j_{22}}{\varepsilon} & j_{31} - \frac{j_{32}}{\varepsilon} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}_\varepsilon}{\partial \bar{q}_1} \\ \frac{\partial \bar{H}_\varepsilon}{\partial \bar{q}_2} \\ \frac{\partial \bar{H}_\varepsilon}{\partial \bar{p}_1} \\ \frac{\partial \bar{H}_\varepsilon}{\partial \bar{p}_2} \end{bmatrix} + \begin{bmatrix} 0_{n\times n} \\ 0_{n\times n} \\ \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} \bar{v}$$

where

$$t_i = t_i(\bar{q}), \quad \alpha = \alpha(\bar{q}), \quad j_\bullet = j_\bullet(\bar{q}, \bar{p})$$

are all invertible.

Reduced models

Reduced slow (rigid):

$$\begin{bmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{p}}_1 \end{bmatrix} = \begin{bmatrix} 0 & t_1^{-T} \\ -t_1^{-1} & j_1 \end{bmatrix} \begin{bmatrix} \frac{\partial H_0}{\partial \bar{q}_1} \\ \frac{\partial H_0}{\partial \bar{p}_1} \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ g_1(\bar{q}_1, \bar{p}_1) \end{bmatrix} u_s$$

Reduced fast:

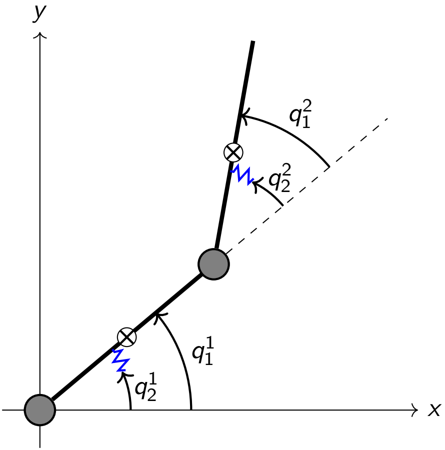
$$\begin{bmatrix} \bar{q}_2' \\ \bar{p}_2' \end{bmatrix} = \begin{bmatrix} 0 & t_4^{-T} \\ -t_4^{-1} & j_{32} \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}}{\partial \bar{q}_2} \\ \frac{\partial \bar{H}}{\partial \bar{p}_2} \end{bmatrix} + \begin{bmatrix} 0_{n \times n} \\ g_2(\bar{q}_1, \bar{p}_1, \bar{q}_2, \bar{p}_2) \end{bmatrix} u_f$$

where $(\bar{q}_1, \bar{p}_1)$ are fixed parameters

Both reduced systems are port-Hamiltonian

Simulation

Control of a 2DOF flexible joint robot with only position measurements



Goal: To make both links follow the desired trajectory

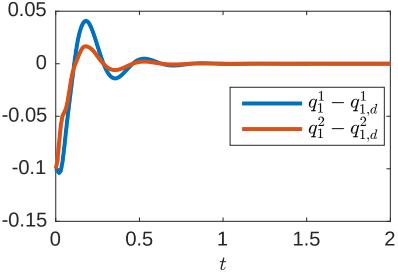
$$q_d = 0.1 + 0.05\sin(t)$$

with *only position measurements*.

Control of the rigid model[4]

$$u_s = M_l \ddot{q}_{1,d} + \frac{\partial}{\partial q_1}(M_l \dot{q}_{1,d})\dot{q}_{1,d} - \frac{1}{2}\frac{\partial}{\partial q_1}(\dot{q}_{1,d}^T M_l \dot{q}_{1,d}) - K_p(q_1 - q_{1,d})$$
$$- K_c(q_1 - q_{1,d} - q_{1,c})$$

$$\dot{q}_{1,c} = K_d^{-1} K_c(q_1 - q_{1,d} - q_{1,c})$$



[4] *Dirksz and Scherpen (2013).*

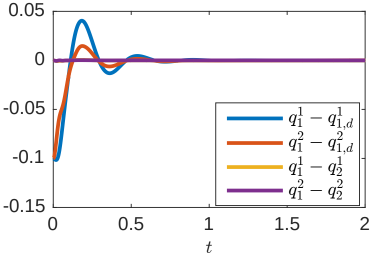
Composite control of the flexible model[5]

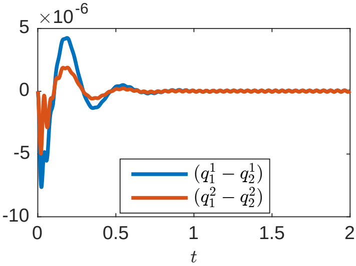
$$u = u_s + u_f\,,$$

where u_f stabilizes the fast subsystem with reference

$$z_d = \frac{1}{\varepsilon}(q_{1,d} - q_{2,d}) = (0, 0).$$

$$u_f = -L_p z - L_c(z - z_c)$$
$$\dot{z}_c = L_d^{-1} L_c(z - z_c)$$

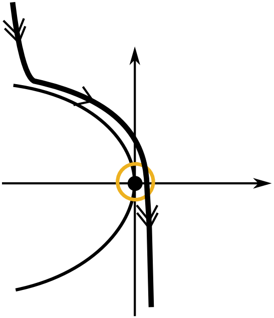




5 *Jardón-Kojakhmetov, Munoz-Arias, Scherpen, 2016.*

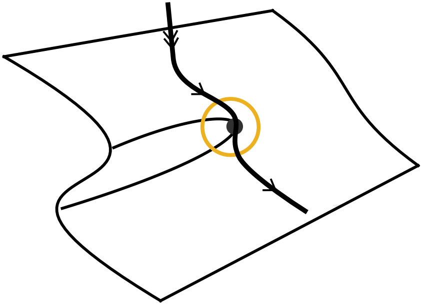
The fold

$$\dot{x} = 1$$
$$\varepsilon \dot{z} = -(z^2 + x)$$



The cusp

$$\dot{x}_1 = 1$$
$$\dot{x}_2 = 0$$
$$\varepsilon \dot{z} = -(z^3 + x_2 z + x_1)$$

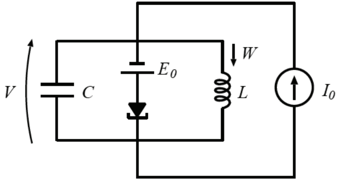


Why are non-hyperbolic points interesting?

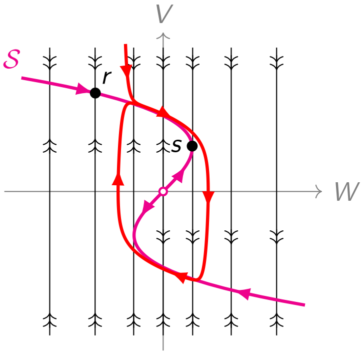
- They are difficult to study
- They are responsible for relaxation oscillations
- They are responsible for hidden effects (canards)
- They model complicated phenomena (mixed-mode oscillations, canards explosion)
- They appear in many mathematical models of
  - Electric circuits (van der Pol oscillator)
  - Biology (cell division, heartbeat)
  - Chemistry (biochemical reactions)
  - Neuroscience (nerve impulse)
  - Classical mechanics
  - Mathematics (16th Hilbert problem)

  ⋮

van der Pol oscillator



Source: `http://www.scholarpedia.org/article/Van_der_Pol_oscillator`



$r \equiv$ hyperbolic point $\rightsquigarrow$ "well understood"

$s \equiv$ non-hyperbolic point $\rightsquigarrow$ "?"

Goal: to stabilize a non-hyperbolic point

Geometric Desingularization

- Has its origins in algebraic geometry.

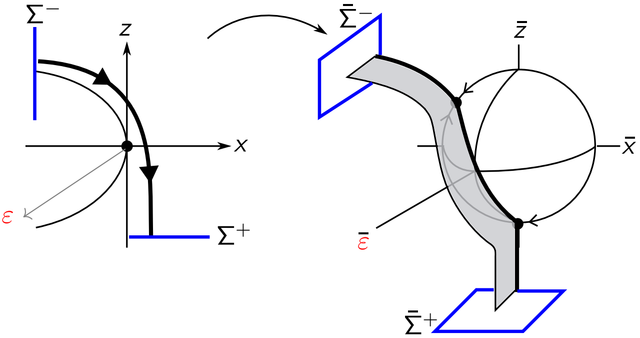


Figure: Schematic picture of a blow up of a fold point

- The blown up vector field is regular, hyperbolic
- The blown up vector field is equivalent to the original one

Stabilization of a folded point *Jardon-Kojakhmetov, Scherpen, 2017*

$$
\begin{aligned}
x' &= \varepsilon(Ax + Bz + u) \\
z' &= -(z^2 + x) \\
\varepsilon' &= 0
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
x &= r^2 \bar{x} \\
z &= r \bar{z} \\
\varepsilon &= r^3
\end{aligned}
$$

Stabilization of a folded point *Jardon-Kojakhmetov, Scherpen, 2017*

$$\begin{aligned} x' &= \varepsilon(Ax + Bz + u) \\ z' &= -(z^2 + x) \\ \varepsilon' &= 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x &= r^2\bar{x} \\ z &= r\bar{z} \\ \varepsilon &= r^3 \end{aligned} \quad \longrightarrow$$

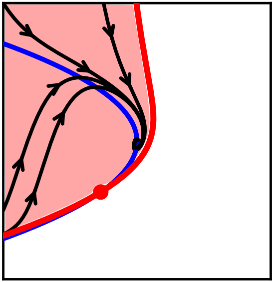
Design controller here!

$$\begin{aligned} \bar{x}' &= Ar^2\bar{x} + Br\bar{z} + \bar{u} \\ \bar{z}' &= -(\bar{z}^2 + \bar{x}) \\ r' &= 0 \end{aligned}$$

Stabilization of a folded point *Jardon-Kojakhmetov, Scherpen, 2017*

Design controller here!

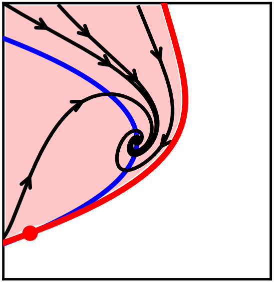
$$x' = \varepsilon(Ax + Bz + u) \qquad x = r^2\bar{x} \qquad \bar{x}' = Ar^2\bar{x} + Br\bar{z} + \bar{u}$$
$$z' = -(z^2 + x) \quad \rightarrow \quad z = r\bar{z} \quad \rightarrow \quad \bar{z}' = -(\bar{z}^2 + \bar{x})$$
$$\varepsilon' = 0 \qquad \varepsilon = r^3 \qquad r' = 0$$



closed-loop slow-fast system

$u = -Ax - Bz + \alpha\varepsilon^{-2/3}x + \beta\varepsilon^{-1/3}z$

$\longleftarrow$



closed-loop blown up v.f.

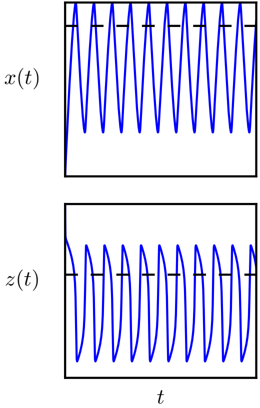
$\bar{u} = -Ar^2\bar{x} - Br\bar{z} + \alpha\bar{x} + \beta\bar{z}$

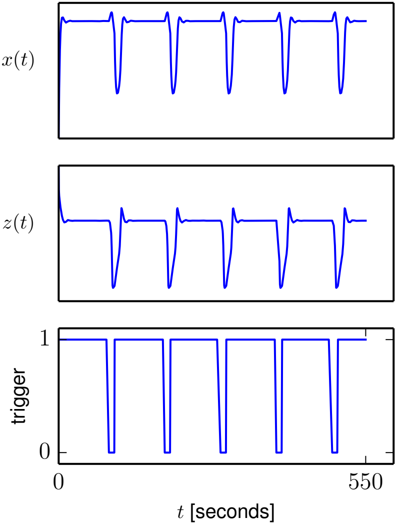
university of groningen

Application: Trigger control of the van der Pol oscillator

Jardón-Kojakhmetov, Scherpen 2016.

$$x' = \varepsilon(z + u), \qquad u = -z + O(\varepsilon^{-1/3})$$
$$z' = -z^3 + z - x$$





Adaptive stabilization of a non-hyperbolic point

Blow up + backstepping → injection of hyperbolicity, *Jardon-Kojakhmetov, del Puerto Flores, Scherpen, 2017*

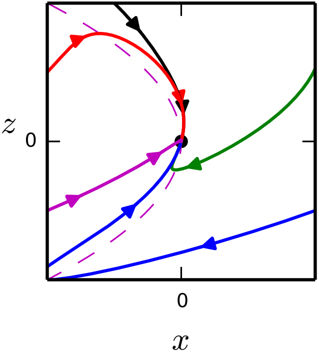
Consider the SFS

$$x' = \varepsilon(A_0 + Ax + Bz + u(x, z, \varepsilon))$$
$$z' = -(z^2 + x),$$

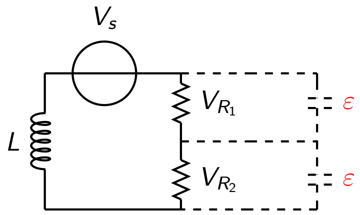
where A_0, A, B is *unknown*, together with some well designed control

$$u = w(x, \hat{a}, \varepsilon)$$
$$\hat{a}' = h(x, \hat{a}, \varepsilon)$$

Then the origin is a locally a.s. equilibrium point.

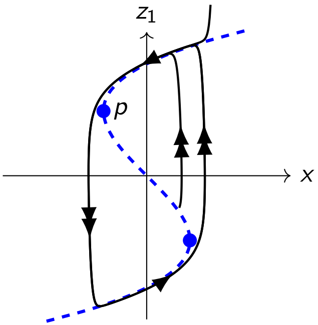


Adaptive control of an electric circuit with jumps



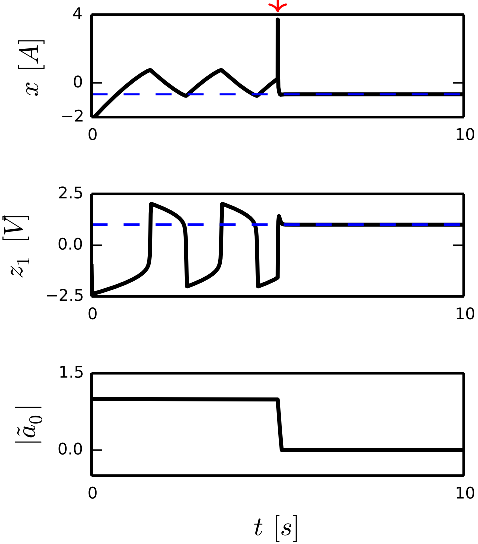
$$
\begin{aligned}
\dot{x} &= -\alpha_1 z_1 - \alpha_2 z_2 + u \\
\varepsilon \dot{z}_1 &= -f_1(z_1) + x \\
\varepsilon \dot{z}_2 &= -f_2(z_2) + x,
\end{aligned}
$$

R_1	–	non linear
R_2	–	linear
x	–	current through L
z_i	–	voltage at R_i



Adaptive control of an electric circuit with jumps

Open-loop $(0 < t < 5)$ — Controller is activated $(t = 5)$



Conclusions

- Starting from a slow fast PH system, we can rewrite it such that the slow and fast subsystems are both port-Hamiltonian.
- Model order reduction can be used to design a controller for a flexible-joint robot from a rigid one.
- We have presented a novel approach to stabilize non-hyperbolic points of slow-fast systems.
- The blow up technique allows us to desingularize a fold point and study the dynamics nearby.
- The “geometric desingularization” technique has been introduced into the control systems context.
- Geometric desingularization + well-known control strategies can be used to stabilize non-hyperbolic points of slow-fast systems.

Future research

For ODE systems

- Consideration of general “slow-fast PHSs”.
- Influence of ε on the transient performance.
- Regularization of Differential Algebraic port-Hamiltonian systems.
- Path following and trajectory tracking along non-hyperbolic sets.
- “Control” of canards and mixed-mode oscillations.
- Etc.

Future research

For PDE systems

- Extension of Tikhonov's theorem.
- Normally hyperbolic and non-hyperbolic extensions?
- Well-posedness issues
- ....
- Etc.

Questions?