Tight Binding Models for Longitudinally Driven Linear/Nonlinear Lattices

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Outline

- Introduction
- Study light propagation in longitudinal (z) direction with an optical lattice in transverse (x-y) plane
- Systematic method to obtain tight binding approximations for linear/NL lattices with detailed longitudinal structure
- Prototypes: honeycomb (HC) and staggered square (SS) lattices
- Examples of longitudinal structure: helical periodic variation with same rotation/different radii on sublattices, out of phase rotation between sublattices, different frequencies between sublattices,...

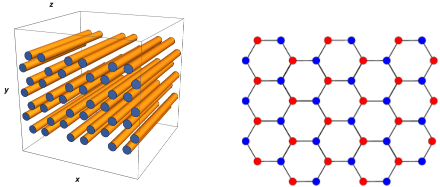
Outline – con't

- Obtain Floquet bands – indicates edge waves; typically two types
    - Unidirectional traveling edge waves: topological: no backscatter, stable w/r defects
    - Non-unidirectional/non-topological waves
- Further asymptotic models can be constructed; yields analytical insight: rapid and slow variation
- In nonlinear problem can find envelope edge solitons satisfying classical 1d-NLS eq; the solitons are stable, they persist over long distances; remain intact around defects, corners
- NL modes–solitons inherit underlying topological properties
- Conclusion

Refs: MJA, C. Curtis, YP Ma (2014-15); MJA, J. Cole (2017)

Introduction

- Investigations of optical lattices extensive
- Paradigm – HC lattice: 'Photonic Graphene' (PG)



Left: Uniform HC lattice: *z* direc'n; Right: x-y plane : HC lattice

- Segev group 2007–conical diffrct'n; MJA, Y. Zhu, C. Curtis constructed/studied TB models; found conical diffrct'n & various interesting new NL nonlocal eqn's in certain limits (2009-13)

Introduction–con't

- Topological edge waves were theoretically proposed/observed in magneto-optics, Wang et al 2008-09
- Such waves were found in photonics: HC lattice with longitudinal helical variation, Rechtsman et al 2013
- MJA, YP Ma, C. Curtis (2014), studied TB model, developed asymptotic description linear/NL under assumption of rapid helical variation
- Leykam et al (2016) studied staggered square lattice with helical variation and phase sh'fts between sublattices
- MJA & J. Cole (2017): systematic method to find TB models in lattices with longitudinal structure
- Topological edge/interface/surface waves in physics – very active field of research

Lattice NLS Equation

Maxwell's eq with paraxial approx. => NLS eq with ext pot'l

$$i\frac{\partial\psi}{\partial z} = -\frac{1}{2k_0}\nabla^2\psi + k_0\frac{\Delta n(x,y,z)}{n_0}\psi - \gamma|\psi|^2\psi$$

where: k_0 is input wavenumber

n_0 is the bulk refractive index

$\Delta n/n_0$ is the change of index change relative to n_0

γ is NL index

Non-dimensional NLS Equation

Rescale to non-dimensional form

$$x = \ell x' \ , \ y = \ell y' \ , \ z = z_* z' \ , \ \psi = \sqrt{P_*}\psi'$$

where: ℓ is the lattice scale; P_*: peak input power

Find non-dim NLS eq, $'$: dimensionless:

$$i\frac{\partial \psi'}{\partial z'} + (\nabla')^2\psi' - V(x', y', z')\psi' + \sigma|\psi'|^2\psi' = 0$$

where $z_* = 2k_0\ell^2, \quad V = 2k_0^2\ell^2(\Delta n/n_0), \quad \sigma = 2\gamma k_0 \ell^2 P_*$

Drop $'$ $=>$ normalized lattice NLS eq

$$i\frac{\partial \psi}{\partial z} + \nabla^2\psi - V(x, y, z)\psi + \sigma|\psi|^2\psi = 0$$

TB Limit

When $|V| \gg 1$, the tight binding (TB) limit, approx the potential by

$$V(\mathbf{r}) \approx \sum_{\mathbf{v}} V_j(\mathbf{r} - \mathbf{v}), \quad j = 1, 2 \;\; \text{(nonsimple HC or SS lattice)}$$

where $V_j(\mathbf{r})$, denotes the approximating potential with minima at site $S_{\mathbf{v}}$; $V_j(\mathbf{r})$ typically gaussian

Associated localized functions near the potential minima, termed orbitals, are used to approx ψ

TB approx used widely in physics to study lattice systems: uniform linear HC lattices ('Graphene'): Wallace 1947

Longitudinal Variation in Potentials

Typical case nonsimple lattice with two sublattices

$$V_1 = V_1(\mathbf{r} - \mathbf{h}_1(z)), \quad V_2 = V_2(\mathbf{r} - \mathbf{h}_2(z))$$

in nb'hd of sublattices $1, 2$ and $\mathbf{h}_j(z),\ j = 1, 2$ are prescribed (smooth) functions

Simple case, helical variation

$$\mathbf{h}_j(z) = \eta_j \left(\cos\left(\frac{z}{\varepsilon_j} + \chi_j\right),\ \sin\left(\frac{z}{\varepsilon_j} + \tilde{\chi}_j\right)\right), \quad j = 1, 2$$

Rotating frame

Move to coordinate frame co-moving with with the $V_1(\mathbf{r}, z)$ sublattice,

$$\mathbf{r}' = \mathbf{r} - \mathbf{h}_1(z) \quad , \quad z' = z$$

which after the phase transformation

$$\psi = \psi' \exp\left[-i \int_0^z |\mathbf{A}(\xi)|^2 d\xi\right] \text{ with } \mathbf{A}(z) = -\mathbf{h_1}'(z)$$

find lattice NLS with a pseudo-field $\mathbf{A}(z)$ -dropping $'$:

$$i\partial_z\psi + (\nabla + i\mathbf{A}(z))^2\psi - V(\mathbf{r}, z)\psi + \sigma |\psi|^2 \psi = 0$$

$V_1(r, z) = V_1(\mathbf{r})$, $V_2(r, z) = V_2(\mathbf{r} - \Delta\mathbf{h}_{21}(z))$, near sites $1, 2$ with $\Delta\mathbf{h}_{21}(z) = \mathbf{h}_2(z) - \mathbf{h}_1(z)$

NL HC Representation

In non-dim NLS eq using HC lattice with $|V| >> 1$ substitute

$$\psi(\mathbf{r}, z) \sim \sum_{v} [a_v(z)\phi_{1,v}(\mathbf{r}, z) + b_v(z)\phi_{2,v}(\mathbf{r}, z)]$$

where

$$\left(\nabla^2 - \tilde{V}_j(\mathbf{r}, z)\right)\phi_{j,v}(\mathbf{r}, z) = -E_j\phi_{j,v}(\mathbf{r}, z); \quad j = 1, 2$$

$\phi_{j,v}$ are termed orbitals

Substitute ψ into NLS eq. with pseudo-field, multiply $\phi_j(\mathbf{r} - \mathbf{p})e^{-i\mathbf{k}\cdot\mathbf{p}}; \quad j = 1, 2$ and integrate

Discrete HC System

$$
\begin{aligned}
i\frac{da_{mn}}{dz} + e^{i\varphi(z)}\left(\mathcal{L}_-(z)b\right)_{mn} + \sigma|a_{mn}|^2 a_{mn} = 0 \\
i\frac{db_{mn}}{dz} + e^{-i\varphi(z)}\left(\mathcal{L}_+(z)a\right)_{mn} + \sigma|b_{mn}|^2 b_{mn} = 0
\end{aligned}
$$

$$
(\mathcal{L}_-(z)b)_{mn} = L_0(z)b_{mn} + L_1(z)b_{m-1,n-1}e^{-i\theta_1(z)} + L_2(z)b_{m+1,n-1}e^{-i\theta_2(z)}
$$

$$
(\mathcal{L}_+(z)a)_{mn} = \tilde{L}_0(z)a_{mn} + \tilde{L}_1(z)a_{m+1,n+1}e^{i\theta_1(z)} + \tilde{L}_2(z)a_{m-1,n+1}e^{i\theta_2(z)},
$$

where $\varphi(z), \theta_j(z), L_j(z), \tilde{L}_j(z) \in \mathbb{R},\ j = 1, 2, 3$ known

Typical Rotation Patterns for Sublattices

- Same rotation, same or different radii:

$$\mathbf{h}_2(z) = R_a \mathbf{h}_1(z) = R_a \eta \left(\cos\left(\frac{z}{\varepsilon}\right),\ \sin\left(\frac{z}{\varepsilon}\right) \right) ,$$

- π-Phase offset rotation

$$\mathbf{h}_2(z) = \mathbf{h}_1(z + \varepsilon\pi) = -\eta \left(\cos\left(\frac{z}{\varepsilon}\right),\ \sin\left(\frac{z}{\varepsilon}\right) \right) ,$$

- Different frequencies

$$\mathbf{h}_j(z) = \eta \left(\cos\left(\frac{z}{\varepsilon_j}\right),\ \sin\left(\frac{z}{\varepsilon_j}\right) \right),\ \ j = 1, 2 ,$$

BCs – Linear Floquet Bands

Consider e.g. Zig-Zag; left end BCs

$$a_{mn} = 0 \quad \text{for} \quad n < 1 , \qquad b_{mn} = 0 \quad \text{for} \quad n < 0 ,$$
$$a_{mn} \to 0 \quad \text{as} \quad n \to \infty , \quad b_{mn} \to 0 \quad \text{as} \quad n \to \infty .$$

Look for solutions of the form

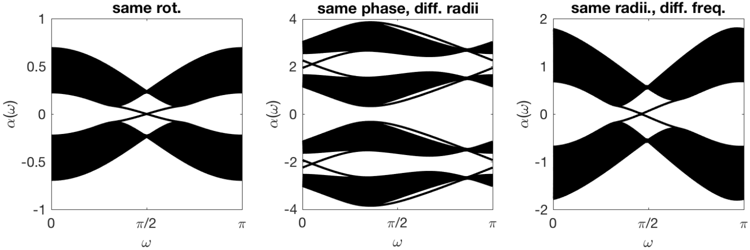
$$a_{mn}(z) = a_n(z;\omega)e^{im\omega} , \quad b_{mn}(z) = b_n(z;\omega)e^{im\omega} ,$$

Find linear difference eq with periodic coef; use Floquet thy:

$$f(z+L) = e^{-i\alpha(\omega)z} f(z)$$

HC Floquet Bands

HC lattice linear band structure–typical parameters



A B C

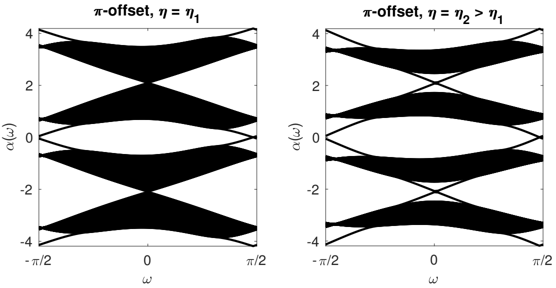
Fig A: same freq, same radii

Fig B: same freq, different radii ($R_2 = R_1/2$)

Fig C: diff freq ($1/\varepsilon_2 = \omega_2 = 2\omega_1 = 1/\varepsilon_1$), same radii

HC Floquet Bands –con't

HC lattice linear band structure–typical parameters



A B

Figs A & B: π offset, same rotation

Fig B vs Fig A: radius $\eta_2 > \eta_1$

Linear HC Edge Mode Dynamics

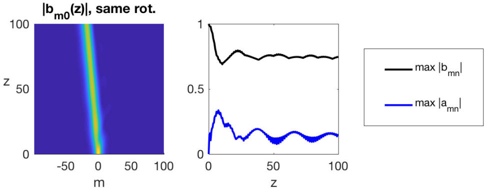


Fig Above: Same rotation, same radii

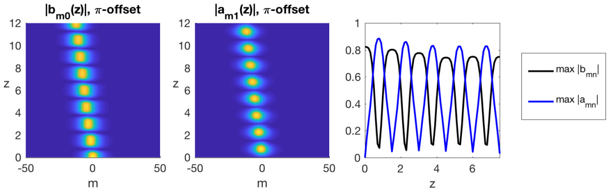
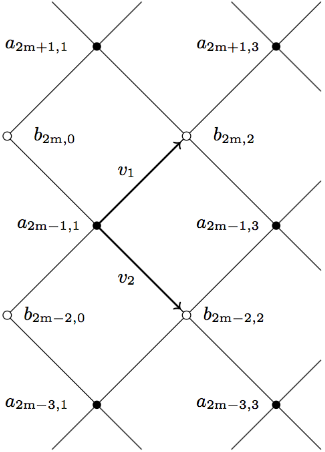


Fig Above: π offset, different radii

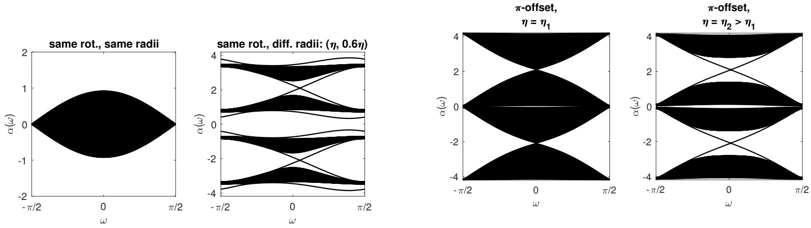
Staggered Square lattice



Analysis to find TB system and Floquet system similar to HC

Staggered Square (SS) Floquet bands

Staggered Square (SS) lattice linear band structure–typical parameters



A B C D

A & B: Same rotation

C & D: π offset, different radii

A: same radii (simple lattice)

B: different radii ($R_a = 0.6$)

Linear SS Edge Mode Dynamics

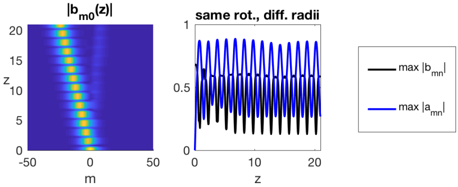


Fig Above: Same rotation, different radii ($R_a = 0.6$)

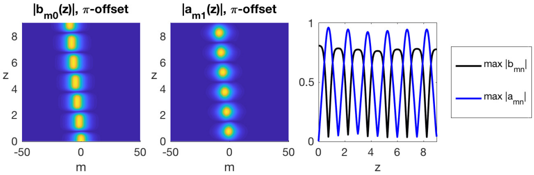


Fig Above: π offset, different radii:

Asymptotics: Rapidly Varying Helical HC Lattice

HC lattice, when each sublattice has same rotation/radius, eqs simplify

Let: $a_{mn} = a_n e^{im\omega}, b_{mn} = b_n e^{im\omega}$ find

$$i\partial_z a_n + e^{i\mathbf{d}\cdot\mathbf{A}}\left(b_n + \rho\gamma^* b_{n-1}\right) + \sigma|a_n|^2 a_n = 0$$
$$i\partial_z b_n + e^{-i\mathbf{d}\cdot\mathbf{A}}\left(a_n + \rho\gamma a_{n+1}\right) + \sigma|b_n|^2 b_n = 0$$

where $\gamma = \gamma(\omega, \mathbf{A}(z))$, ρ: geometric deformation parameter

$\mathbf{A}(z)$ periodic & rapidly varying in z:

$\mathbf{A} = \mathbf{A}(\zeta)$, $\zeta = \frac{z}{\varepsilon}, |\varepsilon| \ll 1$; expt's: $\varepsilon = 0.24$

e.g. $\mathbf{A} = \kappa(\sin\zeta, -\cos\zeta)$: 'helical waveguides'

Expt's: Rechtsman et al (2013); Theory: MJA, Curtis, Ma, Cole (2014, 2017)

Edge Modes: Zig-Zag (ZZ)

Multiple scales:
$a_n = a_n(z,\zeta)$; $b_n = b_n(z,\zeta)$; $\zeta = \frac{z}{\varepsilon}$
Expand a_n, b_n in powers of ε

Find at leading order: **Edge Modes (ZZ)**, exp decay, left end:

$$a_n \sim 0, \quad b_n \sim C(Z,\omega) r^n, \quad |r| = |r(\omega,\rho;\mathbf{A})| < 1, \; n \geq 0$$

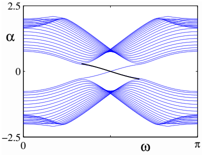
Linear problem (first order):

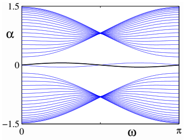
$$C(Z,\omega) = C_0 \exp\left(-i\alpha(\omega)Z\right), Z = \varepsilon z$$

C_0 const. $\quad \alpha(\omega) \equiv \alpha(\omega,\rho;\mathbf{A}) \in \mathbb{R}$: ‘edge dispersion relation’
Obtain explicit formulae (Floquet coef)

Linear Problem–Edge Dispersion Relation

Dispersion relations (helical waveguides): $\alpha(\omega)$: thin curves are 'bulk' modes; lines in the gap are edge modes ($\rho = 1$):





$\rho = 1$ $\rho = 0.4$

Left: : 'Topolgical Floquet Insulator' ($\mathcal{I} = 1$)
Right $\rho = 0.4$: Nontopological mode ($\mathcal{I} = 2$)

Nonlinear Edge Wave Envelope Evolution Eq

Discrete Zig-Zag edge mode:

$$a_{mn} \sim 0$$

$$b_{mn} \sim C(Z,y)e^{i\omega_0 m} r^n, \; |r| < 1$$

where slowly varying ($|\nu| \ll 1$) edge mode envelope C satisfies:

$$i\partial_Z C = \alpha_0 C - i\alpha_0' \nu C_y - \frac{\alpha_0''}{2}\nu^2 C_{yy} + \frac{i\alpha_0'''}{6}\nu^3 C_{yyy} - \alpha_{nl,0}|C|^2 C + \cdots$$

where $\alpha_0 = \alpha(\omega_0)$ etc

If $\mathbf{A} = \mathbf{0}$ then $\alpha = 0$: stationary mode

Linear/NL mode evolution discrete eq agrees with LS/NLS eq

NL topological mode: unidirectional, stable

Typical Linear Edge Wave Evolution–Defects

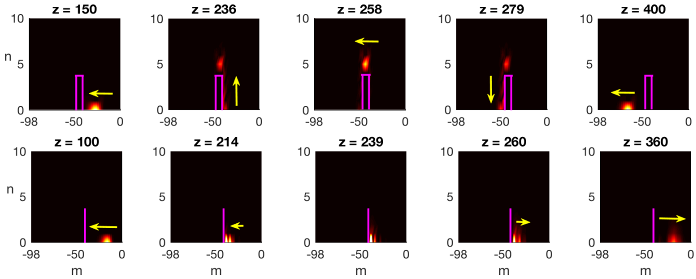


Fig: propagation across defect: left to right

Top fig: Topological mode – wave propagates unidirectionally without losing significant power ($\rho = 1, \omega = \pi/2, \alpha_{nl} = 0$)

Bottom fig: Nontopological mode – wave reflects, broadens/loses significant power ($\rho = 0.4, \omega = \pi/2, \alpha_{nl} = 0$)

NL Edge Wave Propagation Around Defects

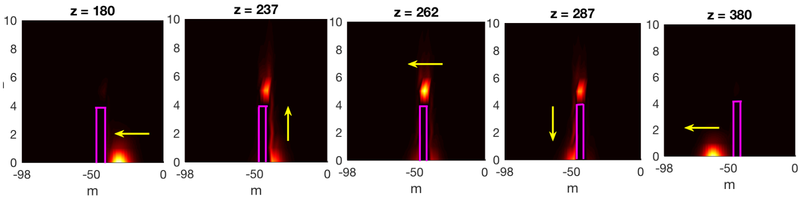
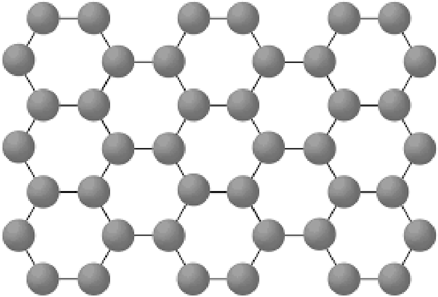


Fig: NL propagation across defect: left to right

NL topological edge wave $(\rho = 1, \alpha_0'' > 0, \alpha_{nl} \neq 0))$ propagates without losing significant power

NL edge solitons: unidirectional, propagates across defects

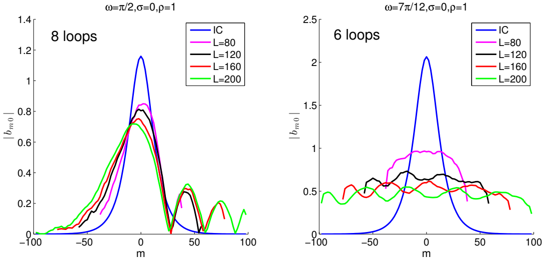
Bounded Graphene: Zig-Zag, Arm Chair Edges



Zig-Zag (ZZ): Left Right; Armchair: Top, Bottom

Mode Propagation–Linear

Linear propagation $\rho = 1$: topological case; different points on the dispersion curve

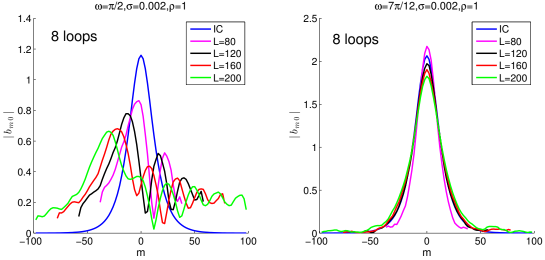


Left: Linear $\omega = \pi/2$
$\alpha'' = 0, \alpha''' \neq 0$

Right: Linear $\omega = 7\pi/12$
$\alpha'' \neq 0$

Mode Propagation–NL

NL propagation $\rho = 1$: topological case:; different points on the dispersion curve



Left: NL $\omega = \pi/2$
$\alpha'' = 0, \alpha''' \neq 0$

Right: NL $\omega = 7\pi/12$: NLS eq
$\alpha'' \neq 0$

Adiabatic HC Lattice

Take HC lattice, uniform rotation, and $\mathbf{A} = \mathbf{A}(Z)$, where $Z = \varepsilon z$

In lattice system: $a_n = a_n(z, Z), \; b_n = b_n(z, Z)$

Multiple scales asymptotics:

$$a_n \sim 0; \quad b_n \sim C(Z, \omega) b_n^S(Z)$$

where $\qquad b_n^S(Z) = \{ r^n(Z); \quad |r| < 1; \quad r = r(\omega, \rho; \mathbf{A}(Z)), \; n \geq 0 \}$

In general edge mode existence ($|r| < 1$) depends on ω, ρ, Z

Modes can 'disintegrate' under evolution

Adiabatic HC Lattice–con't

For $b_n \sim C(Z,\omega) b_n^S(Z)$, find find at leading order:

$$\partial_Z C + i\alpha_p(Z;\omega)C = 0,$$

where $\alpha_p(Z;\omega)$ may be calculated explicitly

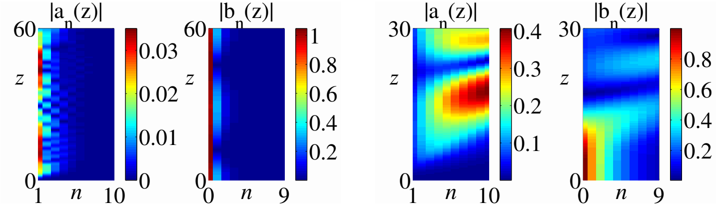
Typical case take: $\mathbf{A}(Z) = \kappa(\sin Z, -\cos Z)$

$\alpha_p(Z;\omega)$ is a periodic fcn in Z

$\alpha_p(Z;\omega)$ is termed the 'edge dispersion relation'

Linear Edge Modes–Under Evolution

Typical linear edge mode evolution–via discrete eq:
$\varepsilon = 0.1, \kappa = 0.3, \rho = 1$



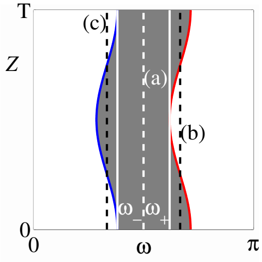
(a) (b)

$\omega_0 = \pi/2$

$\omega_0 = 2\pi/3$
Mode 'disintegrates'

Edge Mode Region–Under Evolution

Typical Edge Mode Region Under Evolution:
$\varepsilon = 0.1, \kappa = 0.3, \rho = 1$



Mode exists in region in gray; If begin inside $\omega : \omega_- < \omega < \omega_+$, then remain. Mode (a) exists for entire period; mode (b) disintegrates at some Z_*

Adiabatic NL Edge Wave Envelope Evolution Eq

Discrete edge mode: $a_{mn} \sim 0$

$$b_{mn} \sim C(Z, y) e^{i\omega_0 m} b_n^S(Z), \ |r| < 1$$

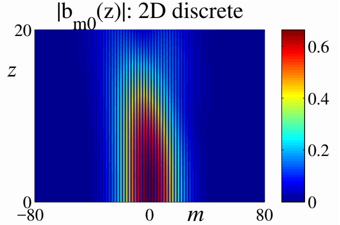
slowly varying ($|\nu| \ll 1$) edge mode envelope C NLS-type eq:

$$i\partial_Z C = \alpha_0(Z)C - i\alpha_0'(Z)\nu C_y - \frac{\alpha_0''(Z)}{2}\nu^2 C_{yy} + \frac{i\alpha_0'''(Z)}{6}\nu^3 C_{yyy}$$
$$- \alpha_{nl,0}(Z)|C|^2 C + \cdots$$

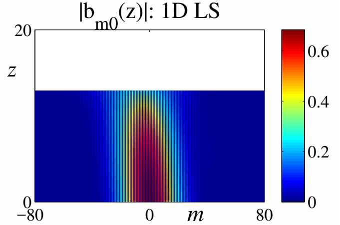
$\alpha_0(Z) = \alpha_p(\omega_0, Z), \alpha_0'(Z) = (\partial_\omega \alpha_p)(\omega_0, Z)$ etc.

Adiabatic: Linear Discrete vs. Envelope Eq

Linear discrete system vs. linear envelope eq – at edge: $n = 0$



(a)



(b)

Left: Discrete eq. evolution; Right: Envelope evolution:
Stopped at $z \approx 12$ edge state 'disintegrates'

$\varepsilon = 0.1, \kappa = 0.3, \rho = 1, \nu = 0.1, \omega_0 = 2\pi/3$

Conclusion

Photonic lattices with longitudinal variation

- Systematic method to find tight binding equations for complex longitudinally driven lattices
- Special case periodically driven –helical–lattices: Honeycomb and staggered square lattices
- In tight binding (TB) limit find/study discrete linear and NL systems governing wave propagation
- Find Floquet bands; study evolution of edge waves for different sublattice structures: different radii/different frequency/phase offsets...
- Find topological/nontopological edge waves

Conclusion: HC Edge States

Helical Variation I:

Rapid (fast) helical variation:

- Construct asymptotic theory
- Envelope of edge modes satisfy standard NLS eq
- NLS solitons topological case: unidirectional, propagate stably around defects
- Bounded Domain:
    - Linear modes affected by dispersion; integrity of pulses deteriorate
    - NL solitons propagate long distances with little degradation

Conclusion: HC Edge States

Helical Variation II:

Adiabatic helical variation:

- Construct asymptotic theory
- Edge states depend on slow longitudinal coordinate: Z
- Find envelope satisfies NLS-type eq: coefficients depend on Z
- Modes can propagate over entire period or 'disintegrate' over partial period
- Solitons can propagate intact over long distances

Ref.: MJA, C. Curtis, Y-P Ma, J. Cole: 2014 - 2017