Vortices and Cones

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Analysis of Gauge Theoretic Moduli Spaces
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I. Introduction to Vortices

- A vortex is a gauge theoretic soliton on a 2-d Riemann surface M. It couples a complex Higgs field ϕ (with no singularities) to a U(1) connection a. A zero of ϕ represents a vortex centre.
- We assume that M has a (local) complex coordinate $z = x_1 + ix_2$, and a conformally compatible metric

$$ds_0^2 = \Omega_0(z, \bar{z})\, dz d\bar{z}\,.$$

  The total area A_0 of M plays an important role in the theory, as does the Gaussian curvature K_0. We specialise later to surfaces with constant curvature.
- To have N vortices with positive multiplicity, the first Chern number needs to be N. Physically, there is a magnetic flux $2\pi N$.

II. Bogomolny Vortices

- The Bogomolny vortex equations are

$$\begin{aligned} D_{\bar{z}}\phi &= 0\,, \\ *f &= -C_0 + C|\phi|^2\,, \end{aligned}$$

  where C_0 and C are real constants. By choice of scale, we can fix C_0 and C to have values $-1, 0$ or $+1$.
- Here $*f = \frac{1}{\Omega_0} f_{12} = \frac{1}{\Omega_0}(\partial_1 a_2 - \partial_2 a_1)$ is the magnetic field.
- The first equation $\partial_{\bar{z}}\phi - ia_{\bar{z}}\phi = 0$ can be solved for a:

$$a_{\bar{z}} = -i\partial_{\bar{z}}(\log\phi)\,, \quad a_z = i\partial_z(\log\overline{\phi})\,.$$

- The second equation then reduces to the scalar, gauge invariant equation

$$-\frac{1}{2\Omega_0}\nabla^2 \log|\phi|^2 = -C_0 + C|\phi|^2\,.$$

- It is convenient to set $|\phi|^2 = \phi\overline{\phi} = e^{2u}$. Then

$$-\frac{1}{\Omega_0}\nabla^2 u = -C_0 + Ce^{2u} \,,$$

  with the Beltrami Laplacian of u on the left.
- u has logarithmic singularities ($u \to -\infty$) at the zeros of ϕ, so there are N additional delta functions at the vortex centres.
- A genuine vortex solution has u bounded above. The maximum principle (or positivity of N) then only allows those C_0 and C for which $-C_0 + Ce^{2u}$ is positive for some u.

- The *five* surviving vortex types are
  (i) Standard (Taubes) vortices ($C_0 = -1$, $C = -1$);
  (ii) "Bradlow" vortices ($C_0 = -1$, $C = 0$);
  (iii) Ambjørn–Olesen vortices ($C_0 = -1, C = 1$);
  (iv) Jackiw–Pi vortices ($C_0 = 0$, $C = 1$);
  (v) Popov vortices ($C_0 = 1$, $C = 1$).
- For Standard vortices, N-vortex solutions exist on M provided $2\pi N < A_0$ (Taubes, Bradlow, Garcia-Prada). The moduli space is $\mathcal{M}_N = M^N_{\mathrm{symm}}$, as there is a unique vortex given N unordered (possibly coincident) points on M.
- For Standard vortices the magnetic field is maximal at a vortex centre (Meissner effect); for the vortex types with $C = 1$ it is minimal (anti-Meissner effect). For Bradlow vortices the magnetic field is 1 uniformly over M, so $2\pi N = A_0$.

Vortices and Self-Dual Yang–Mills

- The vortex equations are a dimensional reduction of the self-dual Yang–Mills equation in 4-d. They arise by imposing invariance under a three-dimensional symmetry group S acting on a two-dimensional orbit (Mason and Woodhouse). A Higgs field is naturally generated this way.
- Vortices are therefore “S-invariant instantons”.
- For Standard vortices S is SO(3) acting on a 2-sphere; for Jackiw–Pi vortices S is E_2 acting on a plane; for Popov vortices S is SU(1,1) acting on a hyperbolic plane.
- The gauge group always reduces to U(1), but in 4-d it needs to be SU(2), E_2 or SU(1,1), depending on the value of C (Contatto and Dunajski).

III. Vortices as Conical Singularities

- Vortices have a geometric interpretation. Define the Baptista metric on M

$$ds^2 = |\phi|^2 ds_0^2 = e^{2u} ds_0^2 \,.$$

- This is conformal to the original metric, with conformal factor $\Omega = e^{2u}\Omega_0$, but has conical singularities with cone angle 4π – conical excess 2π – at the N vortex centres. (The conical excess is $2\pi m$ if a vortex has multiplicity m.)
- The gauge invariant vortex equation can be expressed as

$$(K - C)\Omega = (K_0 - C_0)\Omega_0 \,,$$

where K, K_0 are the Gaussian curvatures of Ω, Ω_0. (Recall $K = -\frac{1}{2\Omega}\nabla^2 \log \Omega$.)

IV. Integrable Vortices

- The vortex equations are integrable if $K_0 = C_0$, i.e. if the background metric on M has the appropriate constant curvature.
- Finding vortex solutions then reduces to finding a (Baptista) metric on M with unchanged topology and conformal structure, constant curvature $K = C$, and N conical singularities of cone angle 4π. This requires solving Liouville's equation.
- The Gauss–Bonnet theorem places constraints on N in terms of C_0 and C and the genus $\mathfrak{g}$ of M.
- All this generalizes Witten's observation that SO(3)-invariant SU(2) instantons on $\mathbb{R}^4$ are equivalent to U(1) vortices on $\mathbb{H}^2$. Vortex solutions are found by solving Liouville's equation using Blaschke functions – rational maps from $\mathbb{H}^2$ to $\mathbb{H}^2$.

- Jackiw–Pi vortices are constructed using rational functions on the plane (Horvathy and Zhang), or quasi-elliptic functions on the torus (Olesen; Akerblom et al.).
- Popov vortices on a sphere are constructed using rational functions (NSM, Q. Chen et al.).
- The solution is

$$|\phi|^2 = e^{2u} = \frac{(1 + C_0|z|^2)^2}{(1 + C|f(z)|^2)^2} \left| \frac{df}{dz} \right|^2 ,$$

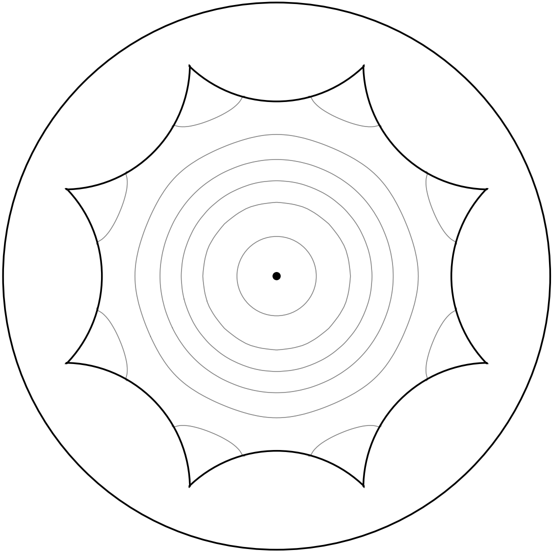
  and one may locally fix the gauge by choosing

$$\phi = \frac{1 + C_0|z|^2}{1 + C|f(z)|^2} \frac{df}{dz} .$$

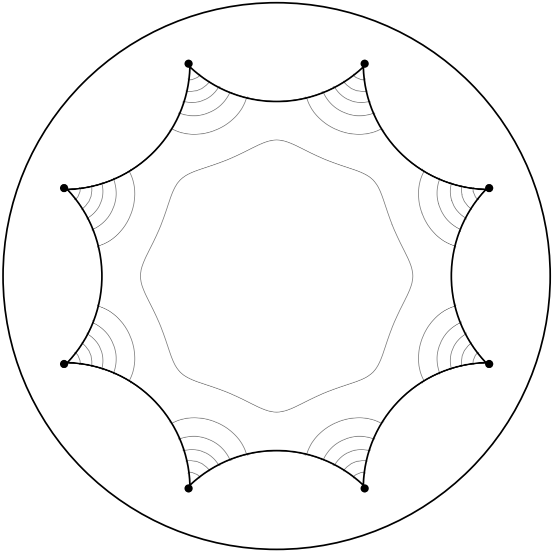
- Vortex centres are the ramification points, where $\frac{df}{dz} = 0$.
- Globally, f is a holomorphic map from M, with curvature C_0, to a smooth surface with curvature C. $|\phi|^2$ is the ratio between the target metric pulled back to M and the background metric of M, at corresponding points.
- The pulled-back metric is the Baptista metric and has conical singularities at the ramification points of f.

Vortices on the (compact) Bolza Surface

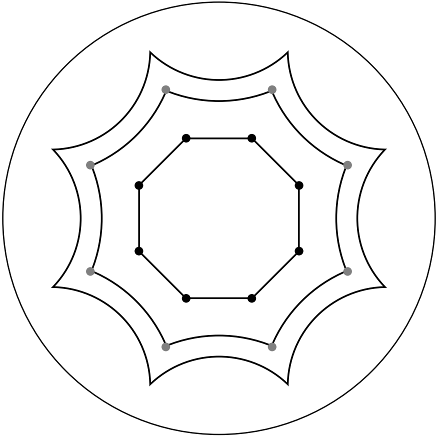
- The Gauss–Bonnet/Bradlow constraint for a Standard N-vortex on a hyperbolic, genus $\mathfrak{g}$ surface ($K_0 = -1$) is $N < 2\mathfrak{g} - 2$.
- The Bolza surface is the most symmetric genus 2 surface. Here, $N = 1$ is the only possibility. An explicit Standard vortex is known (Maldonado and NSM), with its centre at a symmetry point. The Baptista metric is hyperbolic, with one 4π cone angle.
- Bradlow vortices with $N = 2$ exist on the Bolza surface. Here, the Baptista metric is flat. (The Bolza surface with Baptista metric is a translation surface). The simplest solution has a single vortex of multiplicity 2, and the Baptista metric has one 6π cone angle.



Contours of $|\phi|^2 = e^{2u}$ for Standard $N = 1$ vortex on Bolza surface.

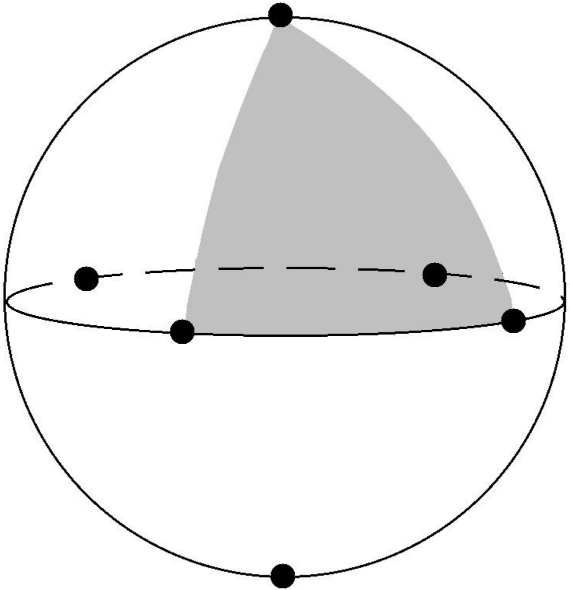


Contours of $|\phi|^2 = e^{2u}$ for vortex relocated to vertex.



Bolza octagon (outer) superimposed on the Poincaré disc; Baptista octagon (middle) of the $N = 1$ Standard vortex; flat Baptista octagon (inner) of the $N = 2$ Bradlow vortex. In all cases, opposite edges are identified.

- The $N = 1$ Standard vortex with centre in general position is not known. Can the Baptista geometry be sketched?
- $N = 2$ Bradlow vortices with separated vortex centres should arise from generic holomorphic 1-forms ω on the Bolza surface, and the Baptista metric is $|\omega|^2$.
- One can find an $N = 6$ Ambjørn–Olesen vortex on the Bolza surface. The vortices are at the branch points of the double covering of the sphere, and the Baptista metric is the pulled-back round metric on the double covered sphere.
- There are many more vortex solutions related to branched covering maps.



Bolza surface double covers the Riemann sphere.

Popov Vortices on a 2-sphere

- Popov vortices on the 2-sphere of unit radius (the integrable case) arise from rational maps $f : S^2 \to S^2$. The Baptista metric is the pulled-back (round) metric.
- For f of degree n, the vortex number is the ramification number $N = 2n - 2$, and is always even.
- Rational maps have $4n + 2$ real moduli, but the vortices are invariant under SO(3) rotations (isometries) of the target S^2, so Popov N-vortices have $4n - 1 = 2N + 3$ real moduli.
- The moduli space of Popov vortices in non-integrable cases is not known (I think).

V. Energy and Dynamics of Vortices

- The static energy function for all the vortex types we have considered is

$$E = \int_M \left\{ \frac{1}{\Omega_0^2} f_{12}^2 - \frac{2C}{\Omega_0} \left(\overline{D_1\phi} D_1\phi + \overline{D_2\phi} D_2\phi \right) + \left(-C_0 + C|\phi|^2 \right)^2 \right\} \Omega_0 \, d^2x \, .$$

  E is not positive definite for $C > 0$, so not all vortex types are stable.
- Manipulation of E (completing the square) shows that Bogomolny vortices are always stationary points of E, but not always minima. Standard vortices are energy minima.

- The static energy can be extended to a Lagrangian for fields on $\mathbb{R} \times M$, with metric $dt^2 - \Omega_0\, dz d\bar{z}$,

$$L = \int_M \Big\{ -\frac{1}{2} f_{\mu\nu} f^{\mu\nu} - 2C\overline{D_\mu \phi} D^\mu \phi - \big(-C_0 + C|\phi|^2\big)^2 \Big\} \Omega_0\, d^2x\,.$$

- The kinetic energy ($\mu = 0$ terms) is

$$T = \int_M \Big\{ \frac{1}{\Omega_0} f_{0i} f_{0i} - 2C\overline{D_0\phi} D_0\phi \Big\} \Omega_0\, d^2x\,.$$

  The electric field f_{0i} contributes positively, but $D_0\phi$ contributes negatively if $C = 1$.

- L is a dimensionally reduced pure Yang–Mills Lagrangian in $4+1$ dimensions (F. Contatto and M. Dunajski). For $C = 1$ and $C = 0$ the gauge group in 4-d is non-compact (SU(1,1) and E_2, resp.), leading to an exotic kinetic energy.

Dynamics on Moduli Space

- For dynamics tangent to the moduli space, we can set $a_0 = 0$ provided Gauss's law is satisfied (field dynamics orthogonal to gauge orbits). We then write $\partial_0 \phi = \phi \eta$. The quantity η determines the time derivatives of all fields, and obeys the linearized (scalar) vortex equation

$$-\frac{1}{\Omega_0}\nabla^2 \eta = 2Ce^{2u}\eta \,.$$

- The kinetic energy simplifies to

$$T = \int_M \left\{ \frac{1}{\Omega_0} \partial_i \bar{\eta} \partial_i \eta - 2Ce^{2u} \bar{\eta}\eta \right\} \Omega_0 \, d^2 x \,.$$

- Integrating by parts, and using the equation for η, this reduces to line integrals around the vortex centres – Strachan–Samols localization.

Dynamics of Standard Vortices

- Standard vortices have positive kinetic energy. The moduli are the vortex centres, and the kinetic energy on $\mathcal{M}_N = M^N_{\mathrm{symm}}$ is a quadratic form in vortex velocities. This defines a metric on $\mathcal{M}_N$. Slowly moving vortices follow geodesics in $\mathcal{M}_N$.
- The metric on $\mathcal{M}_N$ is Kähler. This is shown using the local formula of Strachan–Samols, or by a more general Kähler quotient argument.
- The cohomology class of the Kähler form on $\mathcal{M}_N$ is explicitly known. From this one can calculate the volume of moduli space (NSM and Nasir, Perutz), and deduce the (classical) statistical mechanics of vortices for large N.
- Quantization of vortex motion also depends on the topology of the moduli space and its Kähler structure.

Dynamics of Exotic Vortices

- For integrable Popov vortices there are $2N + 3$ moduli, and the kinetic energy seems to vanish identically (Contatto and Dunajski, NSM). Positive and negative contributions cancel. We do not understand this phenomenon well.
- E.g., An $N = 2$ Popov vortex with centres at $Z = 0$ and $Z = \infty$ is described by the rational function $f(z; t) = c(t)z^2$. The fields, and vortex sizes, vary as c varies, but the kinetic energy is zero. This follows from the localization formula, and has been checked by direct integration.
- A Jackiw–Pi vortex moving linearly on a torus also has zero kinetic energy.
- NSM and E. Walton are investigating non-integrable cases of the exotic vortices, and are trying to understand the dimension of the moduli space, and whether it has a non-trivial Kähler metric.

VI. Summary

- The Standard Bogomolny/Taubes U(1) vortex equation can be extended to five distinct vortex equations, with parameters C_0 and C. All vortex types give a Baptista metric $|\phi|^2 ds_0^2$ with conical singularities with cone angle 4π.
- Each vortex equation is integrable on a surface of constant curvature $K_0 = C_0$, and reduces to Liouville's equation. Vortex solutions can be found using holomorphic maps $f(z)$, or branched coverings. The Baptista metric then has constant curvature $K = C$ away from the conical singularities.
- A gauge theoretic metric on the moduli space of vortices is derived using the vortex kinetic energy. This metric is quite well understood using the Strachan/Samols localization formula, but vanishes for certain vortex types.
- For integrable vortices, is there a relation to the Weil–Petersson metric on the moduli space of constant curvature surfaces with conical singularities?

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