New Crossing Lemmas

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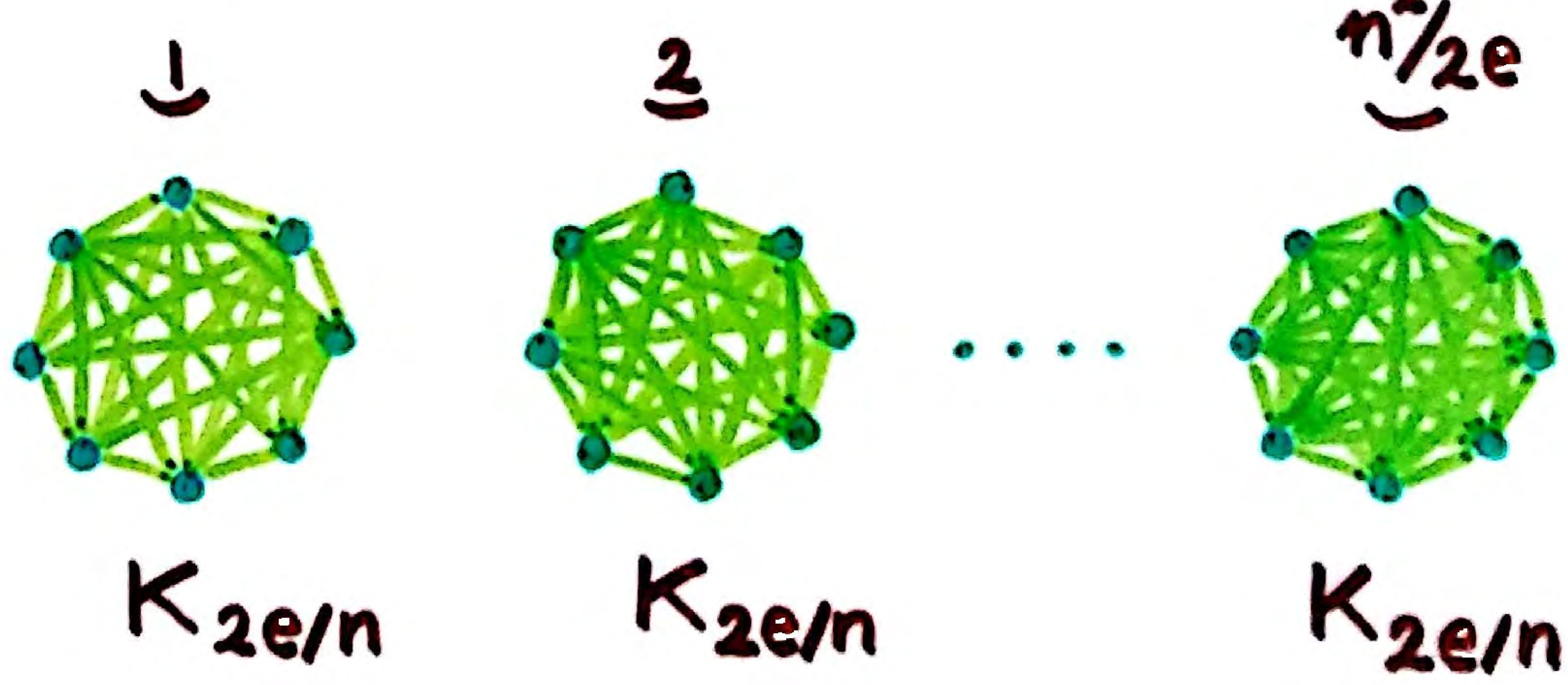
Theorem (Ajtai-Chvátal-Newborn-Szemerédi 1982, Leighton 1983)

For every simple graph with n vertices and $e \geq 4n$ edges,

$$cr(G) \geq \frac{1}{64} \frac{e^3}{n^2}$$

Construction



Lemma

$$cr(G) \geq e - (3n-6) > e - 3n$$

Proof of Theorem

Pick each $v \in V(G)$ with probability p, and let $G' \subseteq G$ denote the subgraph induced by these points.

$$E[cr(G')] > E[e'] - 3E[n']$$

#edges of G' #vertices of G'

$$p^4 cr(G) > p^2 e - 3pn$$

Set $p = 4n/e$.

MULTIGRAPHS

Theorem (Székely 1997)

Let G be a multigraph with n vertices, $e \geq 4mn$ edges and maximum edge multiplicity m. Then

$$cr(G) \geq \frac{1}{64} \frac{e^3}{mn^2}$$

This bound cannot be improved.

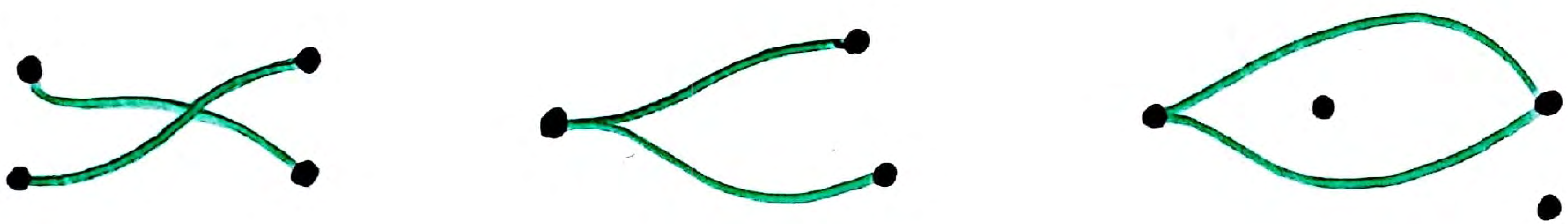
Proof

Keep at most one edge between any pair of vertices, each with probability $\frac{1}{m}$.

SIMPLE TOPOLOGICAL MULTIGRAPH

G has a drawing in the plane such that graph

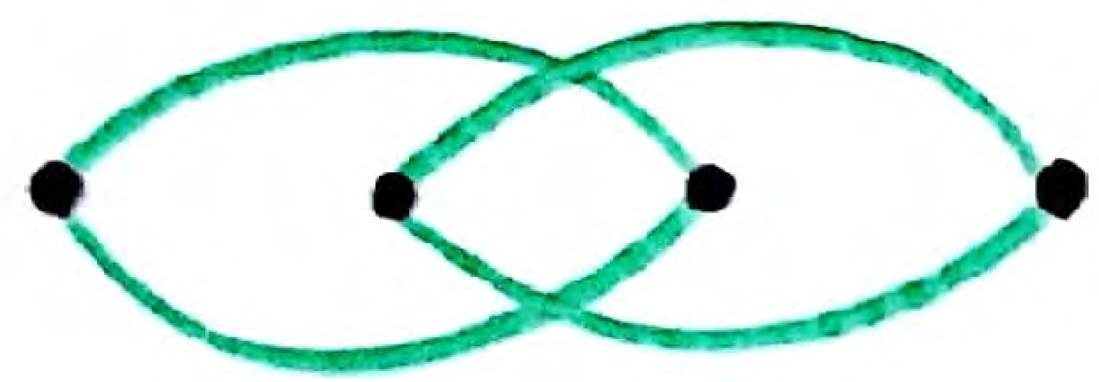
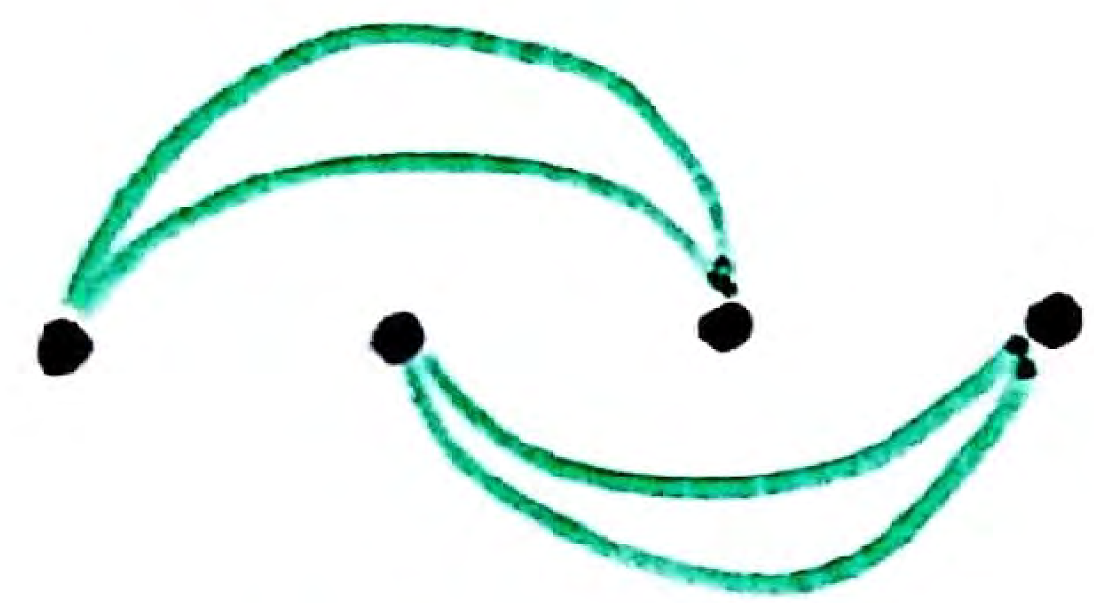
- any two independent edges cross $\leq$ once
- no two edges that share 1 or 2 endpoints cross
- there is ≥ 1 vertex inside and outside every region enclosed by two parallel edges



Lemma. Let G be a planar simple multigraph with n vertices and e edges. Then

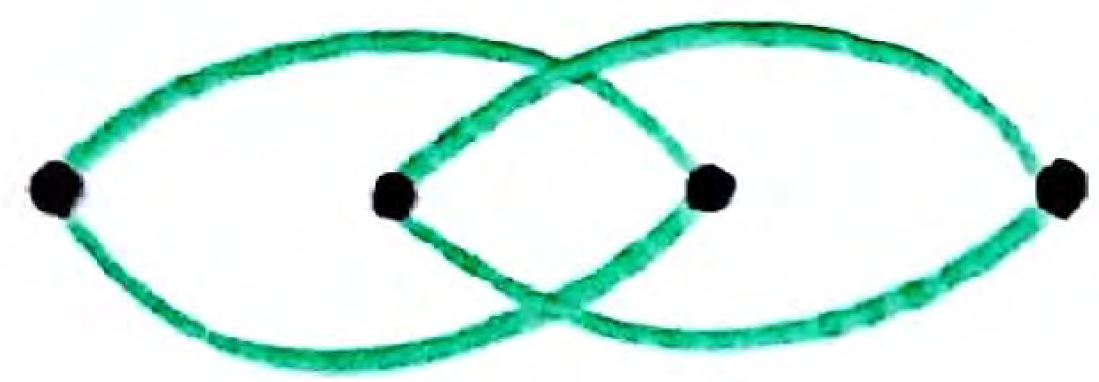
$$e \leq 3n - 6$$

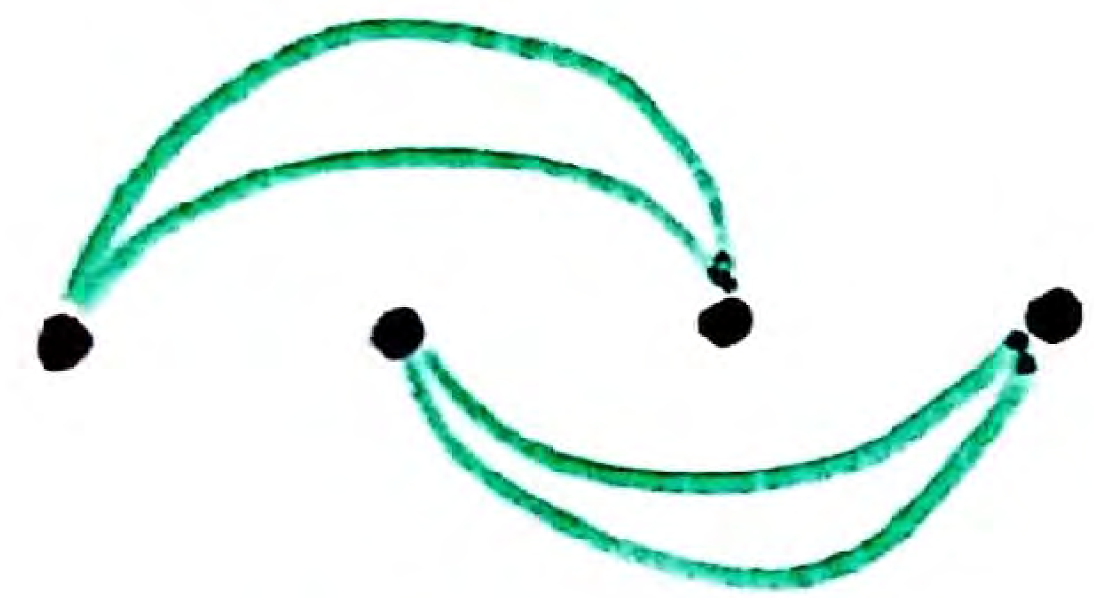
$cr_s(G)$ crossing number of the simple topological multigraph G

$$cr_s(G) = 2 \geqslant cr(\bar{G}) = 0$$

$cr_s(G)$ crossing number of the simple topological multigraph G





$cr_s(G) = 2 \quad \geq \quad cr(\bar{G}) = 0$

Theorem (P. - Tóth 2017)

Let G be a simple topological multigraph with n vertices and $e \geq 4n$ edges. Then

$$cr_s(G) \geq \frac{1}{10^3} \frac{e^3}{n^2}$$

This bound cannot be improved.

Lemma. Let G be a simple topological multigraph with n vertices and e edges. Then

$$e < n^2$$

CROSSING NUMBER AND BISECTION WIDTH

$b(G)$ bisection width : minimum number of edges whose removal cuts G into two subgraphs, each having $\geq \frac{1}{3}|V(G)|$ vertices.

Theorem (P.-Shahrokhi-Szegedy 1996)
Let G be a graph with n vertices of degree $d_1, \dots, d_n$. Then

$$b(G) \leq 10\sqrt{cr(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}$$

Crossing Number and Bisection Width for Simple Topological Multigraphs

$b_s(G)$ bisection width of the simple topological multigraph G: minimum number of edges whose removal splits G into disjoint simple topological multigraphs G_1, G_2 with $|V(G_1)|, |V(G_2)| \geq \frac{1}{3}|V(G)|$

Theorem (P.-Tóth 2017)
Let G be a simple topological multigraph with n vertices of degree $d_1, \dots, d_n$. Then

$$b_s(G) \leq 10^3 \sqrt{cr_s(G)} + 10^3 \sqrt{\sum_{i=1}^{n} d_i^2}$$

Crossing Lemma for Curves

n = # curves ⟵ n = # vertices

T = # touching points  ⟵ e = # edges

X = # crossing points  ⟵ cr = # crossings

for graphs

Crossing Lemma for Curves

n = # curves $\longleftarrow$ n = # vertices

T = # touching points  $\longleftarrow$ e = # edges

X = # crossing points  $\longleftarrow$ $\underbrace{cr = \#\text{ crossings}}_{\text{for graphs}}$

Theorem (P. - Rubin - Tardos 2017)

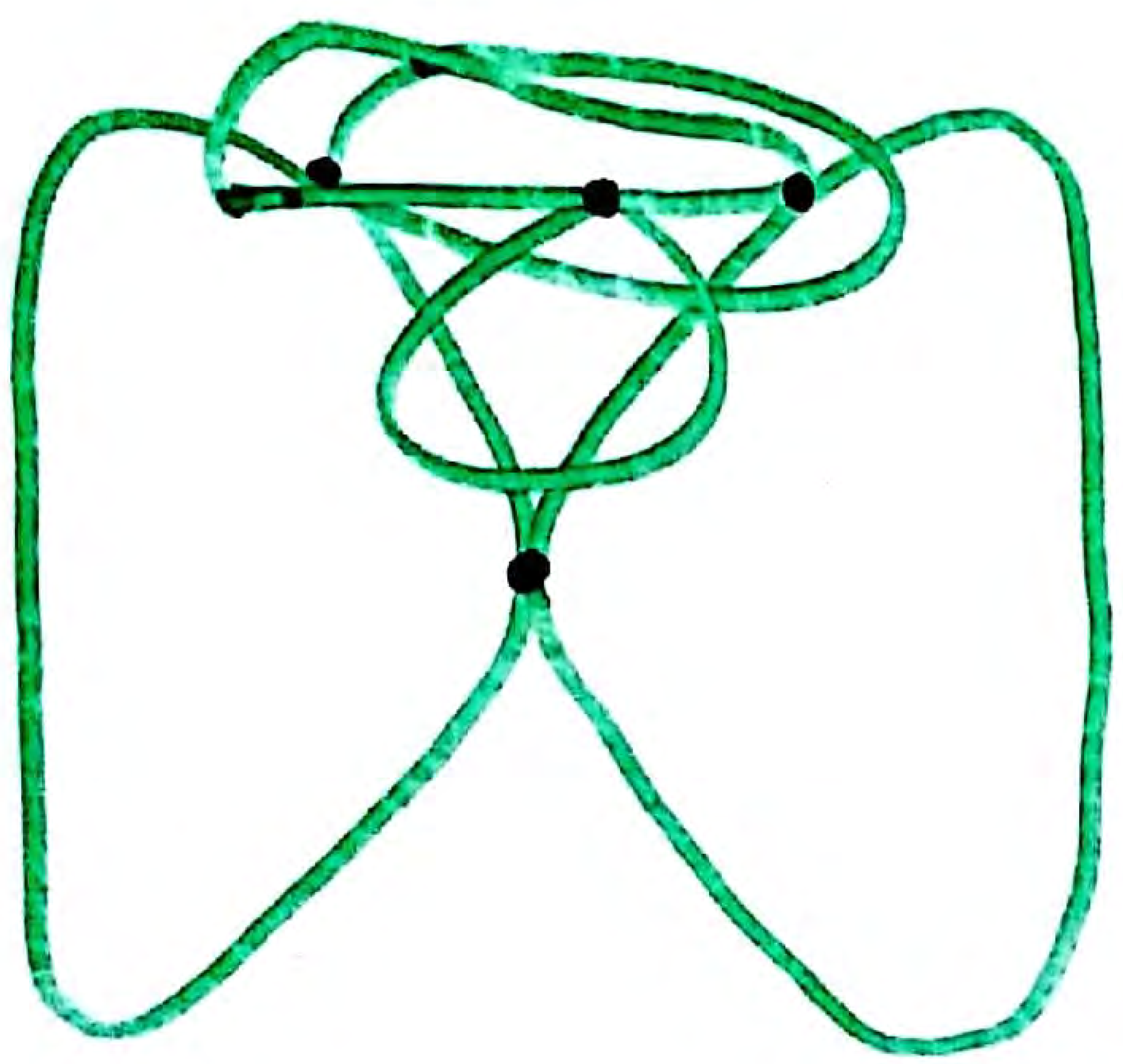
For any system of n curves in the plane, no 3 of which pass through the same point, the number of crossing points X satisfies

$$X \geq c\,T\left(\log\log\frac{T}{n}\right)^{1/100},$$

provided that $T \geq c'n$. Here $c, c' > 0$ are constants.

Corollary (Richter-Thomassen conjecture 1995, PRT 2017)

The total number of intersection points (touchings + crossings) between n pairwise intersecting simple closed curves in the plane, no 3 of which pass through the same point, is $\geq (1-o(1))\,n^2$.



$$(1-o(1))\binom{n}{2}2$$

X_{pair} = # CROSSING PAIRS OF CURVES

$\wedge\!\backslash$

X = # crossing points

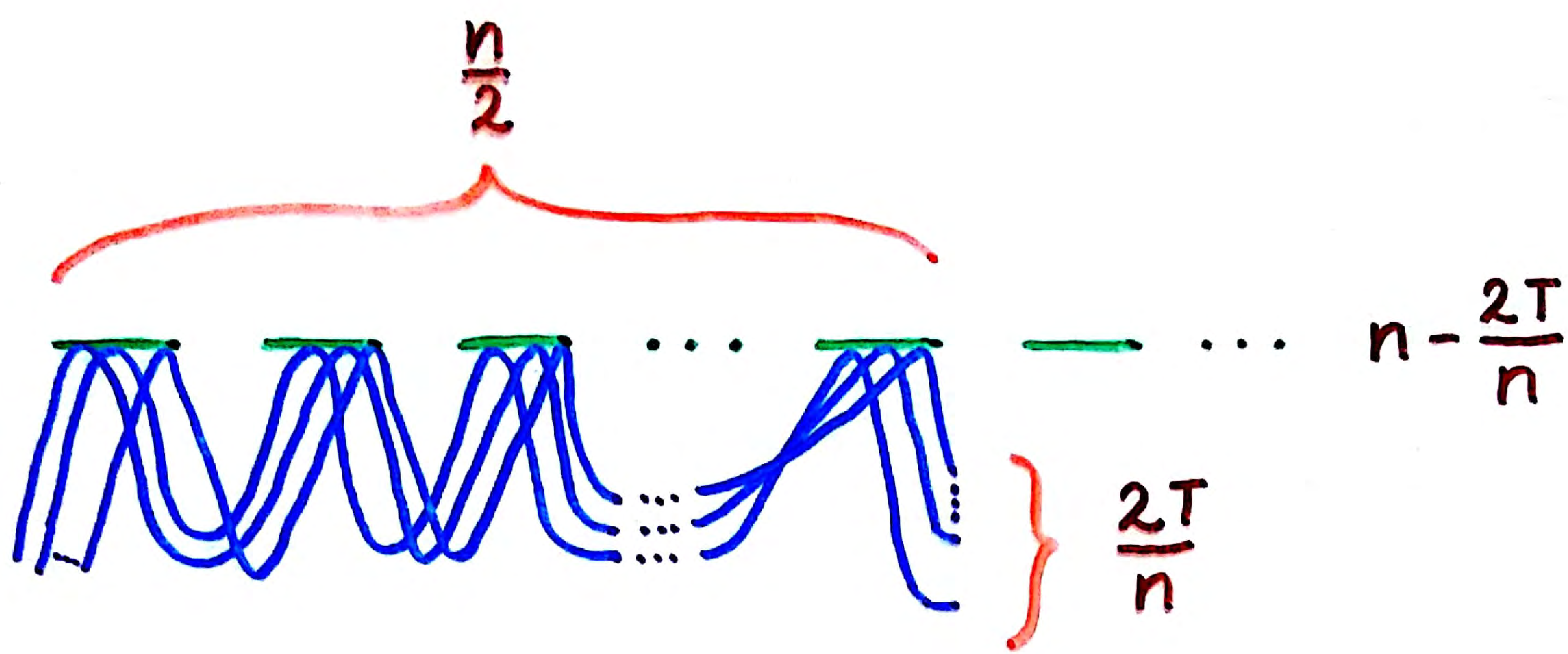
T = # touching points (pairs of curves)

Theorem (P. - Tóth 2017)

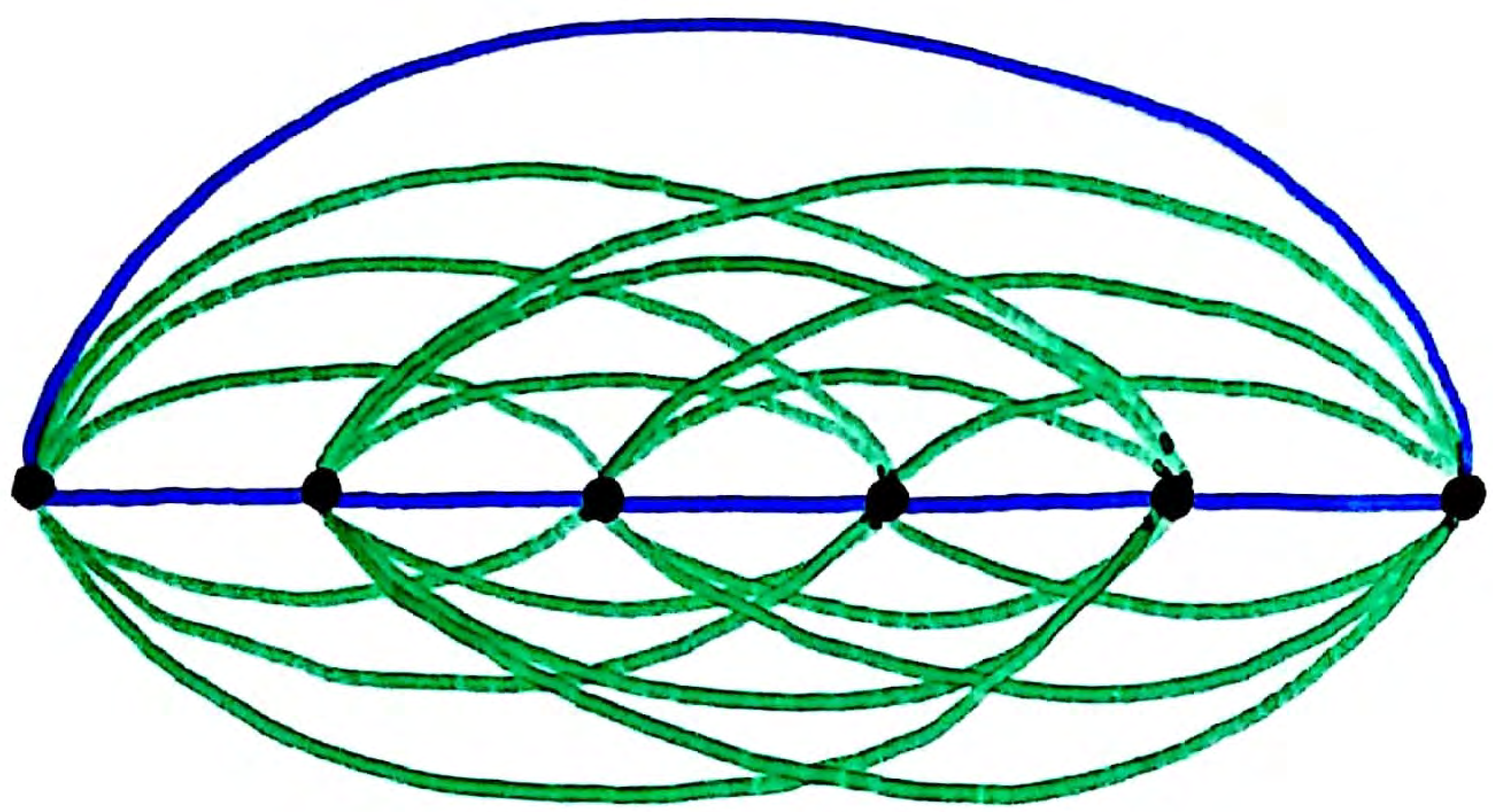
For any family of n curves, no 3 of which pass through the same point, we have

$$X_{pair} \geq \frac{1}{10^5} \frac{T^2}{n^2},$$

provided that $T \geq 10n$.



$$X_{pair} \leq \binom{2T/n}{2} \leq 2\frac{T^2}{n^2}$$



$$e = 2\binom{n}{2} - n = n(n-2)$$