**r-graph** = \( r \)-uniform hypergraph

**Definition**

An **F-decomposition** of an \( r \)-graph \( G \) is a set of edge-disjoint copies of \( F \) covering all edges of \( G \) (also called an \((n, q, r)\)-Steiner system if \( G = K_n^{(r)} \) and \( F = K_q^{(r)} \)).

\((7, 3, 2)\)-Steiner system = triangle decomposition of \( K_7^{(2)} \)
Designs and hypergraph decompositions

\( r \)-graph = \( r \)-uniform hypergraph

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\((7, 3, 2)\)-Steiner system = triangle decomposition of \( K_7^{(2)} \)

A set of distinct copies of \( K_q^{(r)} \) in \( G \) such that every edge of \( G \) is covered exactly \( \lambda \) times is a \((q, r, \lambda)\)-design of \( G \)
(also called an \((n, q, r, \lambda)\)-design if \( G = K_n^{(r)} \)).
It’s the year 1853...

For which \( n \) does a triple system of order \( n \) exist?

Jakob Steiner
It’s the year 1853...

For which \( n \) does a triple system of order \( n \) exist?

Jakob Steiner

6 years earlier...

**Theorem (Kirkman, 1847)**

A *triple system of order* \( n \) *exists if and only if* \( n \equiv 1, 3 \mod 6 \).
Arrh! It should read Kirkman system.

Thomas Kirkman
Thomas Kirkman

Wesley Woolhouse

Julius Plücker

Arrh! It should read Kirkman system.

EXCUSE ME!
Divisibility conditions

Question
When does $G$ have an $F$-decomposition?

If $G$ has a triangle decomposition, then
(a) the number of edges of $G$ is divisible by 3,
(b) every vertex has even degree.

Call $G$ triangle divisible if (a) and (b) are satisfied.
Divisibility conditions

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Theorem (Kirkman 1847)

A Steiner triple system of order \( n \) (i.e. a triangle decomposition of \( K_n \)) exists if and only if \( n \equiv 1, 3 \mod 6 \), i.e. if and only if \( K_n \) is triangle-divisible.
Divisibility conditions

Question
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Theorem (Kirkman 1847)
A Steiner triple system of order $n$ (i.e. a triangle decomposition of $K_n$) exists if and only if $n \equiv 1, 3 \mod 6$, i.e. if and only if $K_n$ is triangle-divisible.

Divisibility conditions can be generalised for arbitrary $q, r, \lambda$, in which case we say that $G$ is $(q, r, \lambda)$-divisible (or $K_q^{(r)}$-divisible if $\lambda = 1$).
Previous results for graphs

Theorem (Kirkman 1847)

If $K_n$ is triangle-divisible, then there exists a Steiner triple system, i.e. a triangle decomposition of $K_n$. 
Theorem (Kirkman 1847)

If $K_n$ is triangle-divisible, then there exists a Steiner triple system, i.e. a triangle decomposition of $K_n$.

Theorem (Wilson 1975)

For $n$ large, every $F$-divisible $K_n$ has an $F$-decomposition.
(n, q, r, \lambda)-design = set of distinct copies of K_q^{(r)} in K_n^{(r)} such that every edge of K_n^{(r)} is covered exactly \lambda times

**Theorem (Teirlinck 1987)**

For every r, there exist infinitely many nontrivial (n, r + 1, r, \lambda)-designs, where \lambda = (r + 1)!^{r+1}.

**Theorem (Kuperberg, Lovett and Peled 2013\textsuperscript{+})**

There exists an absolute constant C such that whenever q \geq Cr there are infinitely many nontrivial (n, q, r, \lambda)-designs (for some (large) \lambda).

**Question:** What about decompositions, i.e. case \lambda = 1?
Relaxation: aim for an ‘approximate decomposition’
(i.e. an almost perfect packing of edge disjoint $K_{q(r)}$)

Conjecture (Erdős and Hanani, 1963)

There exists a $K_{q(r)}$-packing in $K_n^{(r)}$ covering all but $o(n^r)$ of the edges of $K_n^{(r)}$ (as $n \to \infty$).
**The Rödl nibble**

**Relaxation:** aim for an ‘approximate decomposition’ (i.e. an almost perfect packing of edge disjoint $K_q^{(r)}$)

**Conjecture (Erdős and Hanani, 1963)**

*There exists a $K_q^{(r)}$-packing in $K_n^{(r)}$ covering all but $o(n^r)$ of the edges of $K_n^{(r)}$ (as $n \to \infty$).*

**Theorem (Rödl, 1985)**

*The conjecture is true.*

Proof: ‘Rödl nibble’ or ‘semirandom method’ (also very important ingredient in our proof)
Theorem (Keevash 2014\textsuperscript{+})

For any fixed $q, r, \lambda$, there exist $(n, q, r, \lambda)$-designs. More precisely, if $n \gg q, \lambda$ and $K_n^{(r)}$ is $(q, r, \lambda)$-divisible, then there exists an $(n, q, r, \lambda)$-design.

- can actually replace $K_n^{(r)}$ by any dense quasirandom $r$-graph
- proof is based on algebraic and probabilistic arguments.

We generalize this beyond the quasi-random setting, using combinatorial and probabilistic arguments.
from now on restrict to case $\lambda = 1$, results also extend to $\lambda > 1$

$\delta_{r-1}(G) = \text{minimum degree of an } (r-1)\text{-tuple of vertices}$

**Theorem (Glock, Kühn, Lo, Osthus 2016+)**

*For all $q > r \geq 2$, there exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let*

$$c_{q,r}^\diamond := \frac{r!}{3 \cdot 14^r q^{2r}}.$$

*If $G$ is an $n$-vertex $r$-graph with $\delta_{r-1}(G) \geq (1 - c_{q,r}^\diamond)n$, then $G$ has a $K_q^{(r)}$-decomposition whenever it is $K_q^{(r)}$-divisible.*
The decomposition threshold

Previous result leads to notion of decomposition threshold $\delta_{q,r}$:

**Definition**

Let $\delta_{q,r}$ be the smallest $\delta \in [0,1]$ satisfying the following:
for all large enough $n$, every $K_q^{(r)}$-divisible $r$-graph $G$ on $n$ vertices
with $\delta(G) \geq (\delta + o(1))n$ has a $K_q^{(r)}$-decomposition.

- Keevash $\Rightarrow \delta_{q,r} < 1$
- GKLO $\Rightarrow \delta_{q,r} \leq 1 - c_{q,r}^\circ \approx 1 - q^{-2r}$.
- Lower bound construction:
  $\delta_{q,r} \geq 1 - c_rq^{-r+1}\log q \approx 1 - q^{-r+1}$.

graph case $r = 2$ has received much attention – see later.
Main result: supercomplexes

Previous result follows from our main result on designs in ‘supercomplexes’.

Theorem (Glock, Kühn, Lo, Osthus 2016+)

\[
\text{If } n \gg q, \lambda \text{ and } G \text{ is a } (q, r, \lambda)\text{-divisible supercomplex on } n \text{ vertices,}
\]

\[
\text{then } G \text{ has a } (q, r, \lambda)\text{-design.}
\]

(+ generalisation to dense quasirandom r-graphs)

The conditions of being a supercomplex depend mainly on the distribution of \( q \)-cliques, which should be ‘random-like’.
Main result: supercomplexes

Previous result follows from our main result on designs in ‘supercomplexes’.

Theorem (Glock, Kühn, Lo, Osthus 2016\textsuperscript{+})

If $n \gg q, \lambda$ and $G$ is a $(q, r, \lambda)$-divisible supercomplex on $n$ vertices, then $G$ has a $(q, r, \lambda)$-design.

(+ generalisation to dense quasirandom $r$-graphs)

The conditions of being a supercomplex depend mainly on the distribution of $q$-cliques, which should be ‘random-like’.

Examples of supercomplexes

- complete $r$-graphs
- quasirandom $r$-graphs, in particular ‘typical’ $r$-graphs
- $k$-partite graphs where $k \geq q + 6$
Existence of $F$-designs for arbitrary $F$

so far: considered designs/decompositions into cliques

What about decompositions into arbitrary hypergraphs $F$?

$F$-decomposition = decomposition of edge set of $G$ into copies of $F$
Existence of $F$-designs for arbitrary $F$

**so far:** considered designs/decompositions into cliques
What about decompositions into arbitrary hypergraphs $F$?

$F$-decomposition = decomposition of edge set of $G$ into copies of $F$

**Theorem (Glock, Kühn, Lo, Osthus 2017$^+$)**

Suppose $F$ is an $r$-graph and suppose that $K_n^{(r)}$ is $F$-divisible, where $n \gg |F|$. Then $K_n^{(r)}$ has an $F$-decomposition. (generalisation to dense quasirandom $r$-graphs)

• answers question of Keevash
• graph case $r = 2$ is due to Wilson
• can replace $K_n^{(r)}$ by any dense quasirandom $r$-graph $G$
• can prove design version with $\lambda > 1$
• effective minimum degree version if $F$ is ‘weakly regular’
Special case:

**Theorem (Glock, Kühn, Lo, Osthus 2017⁺)**

Suppose $G$ is a large quasi-random graph and $F$ is fixed with
(i) $e(F)$ divides $e(G)$;
(ii) $\text{hcf}\{\text{degrees of } F\}$ divides $\text{hcf}\{\text{degrees of } G\}$.

Then $G$ has an $F$-decomposition.

**Theorem (Archdeacon)**

If graph $G$ has a decomposition into $K_4$'s, $K_5$'s and $K_6$'s, then $G$ has a self-dual embedding.

**Corollary (Glock, Kühn, Lo, Osthus 2017⁺)**

Almost every graph has a self-dual embedding.
Proof sketch: Absorption

Suppose we seek a $K^{(r)}_q$-decomposition of an $r$-graph $G$

**iterative absorption approach**

Split up the absorbing process into many steps which gradually make leftover smaller and smaller.

$\Rightarrow$ final leftover $L$ has bounded size and lies within prescribed set $X$

$\Rightarrow$ only boundedly many possibilities $H_1, \ldots, H_s$ for leftover $L$
Proof sketch: Absorption

Suppose we seek a $K_q^{(r)}$-decomposition of an $r$-graph $G$

iterative absorption approach

Split up the absorbing process into many steps which gradually make leftover smaller and smaller.

⇒ final leftover $L$ has bounded size and lies within prescribed set $X$
⇒ only boundedly many possibilities $H_1, \ldots, H_s$ for leftover $L$
⇒ suffices to find an ‘exclusive absorber’ $A_i$ for each $H_i$, i.e.
• $A_i \cup H_i$ has a $K_q^{(r)}$-decomposition
• $A_i$ has a $K_q^{(r)}$-decomposition
Recall:
An exclusive absorber $A$ for a potential leftover graph $H$ satisfies
- $A \cup H$ has a $K_q^{(r)}$-decomposition
- $A$ has a $K_q^{(r)}$-decomposition
We construct exclusive absorbers out of ‘transformers’.
Ignore divisibility.

**Definition**

An $r$-graph $T$ is an $(H_1, H_2)$-transformer if both $H_1 \cup T$ and $T \cup H_2$ have $K_q^{(r)}$-decompositions.

**Aim:** transform leftover $H_1$ step by step into $r$-graph which is trivially decomposable
General Idea:

- construct absorber as concatenation of transformers
- show that each $H$ can be transformed into ‘canonical graph’ $C$ which only depends on $e(H)$
- by transitivity this implies that each $H$ can be transformed into a disjoint union $J$ of $K_q^{(r)}$, which is trivially decomposable
Conjecture (Nash-Williams 1970)

Every large $K_3$-divisible graph $G$ on $n$ vertices with $\delta(G) \geq 3n/4$ has a $K_3$-decomposition.

**Extremal example:** blow up each vertex of $C_4$ to a $K_m$ ($m$ odd and divisible by 3).

Each triangle has at least one edge in one of the four cliques but less than a third of the edges lie inside the cliques.
Open question: the decomposition threshold for graphs

Conjecture (Nash-Williams 1970)
Every large $K_3$-divisible graph $G$ on $n$ vertices with $\delta(G) \geq 3n/4$ has a $K_3$-decomposition.

- true if $\delta(G) \geq (0.9 + o(1))n$ (Barber, Kühn, Lo, Osthus & Dross)
- showing that $\frac{3n}{4}$ guarantees ‘fractional decomposition’ or approx. decomposition would suffice
- conjectured threshold for $K_q$-decompositions: $\frac{qn}{q+1}$, partial results by Barber, Glock, Kühn, Lo, Montgomery, Osthus
- similar questions in partite setting, partial results by BKLMOT (applications to completions of partially filled latin squares)