A quick introduction to dessins d’enfants, Belyi’s theorem and Galois action

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Definition of dessin d’enfant (Grothendieck, 80’s)
Definition of dessin d’enfant

A **dessin d’enfant**, or simply a *dessin*, is a pair \((X, D)\) where \(X\) is an oriented compact topological surface, and \(D \subset X\) is a finite **graph** such that:

1. \(D\) is **connected**.
2. \(D\) is **bicoloured**, i.e. the vertices have been given either white or black colour and vertices connected by an edge have different colours.
3. \(X \setminus D\) is the union of finitely many topological discs, which we call **faces** of \(D\).

The **genus** of \((X, D)\) is simply the genus of \(X\).

We consider two dessins \((X_1, D_1)\) and \((X_2, D_2)\) **equivalent** when there exists an orientation-preserving homeomorphism from \(X_1\) to \(X_2\) whose restriction to \(D_1\) induces an isomorphism between the coloured graphs \(D_1\) and \(D_2\).
Warning: A dessin is not an abstract graph

These bicoulored graphs are equal as abstract graphs but they are different as dessins: they even have different genus!
Permutation representation pair of $\mathcal{D}$: $(\sigma_0, \sigma_1)$

![Diagram of a dessin with labeling $(\sigma_0, \sigma_1)$](image)

Cycles of $\sigma_0 \leftrightarrow$ white vertices of $\mathcal{D}$, (length of each cycle = degree of the corresponding vertex).
Cycles of $\sigma_1 \leftrightarrow$ black vertices, etc.
Cycles of $\sigma_1 \sigma_0 \leftrightarrow (1/2$ of the edges of) faces of $\mathcal{D}$
Connectedness of $\mathcal{D} \Rightarrow < \sigma_0, \sigma_1 >$ is a transitive subgroup.
Eg: \( \sigma_0 = (1, 5, 4)(2, 6, 3) \), \( \sigma_1 = (1, 2)(3, 4)(5, 6) \)

\[ \sigma_1 \sigma_0 = (1, 6, 4, 2, 5, 3) \]

\[ 2 - 2g = (\#\{\text{cycles of } \sigma_0\} + \#\{\text{cycles of } \sigma_1\}) - \#\{\text{edges}\} + \#\{\text{cycles of } \sigma_1 \sigma_0\} = 2 + 3 - 6 + 1 = 0 \]

Of course this is nothing but the Euler–Poincaré characteristic of \( X \) corresponding to the polygonal decomposition induced by the dessin.
There is a correspondence between dessins d’enfants and algebraic curves defined over number fields, i.e. curves $C : F(x, y) = 0$ with $F(X, Y) \in \overline{\mathbb{Q}}[X,Y]$. E.G.:

$\sigma_0 = (1, 7, 5, 3)$
$\sigma_1 = (1, 2, 3, 4, 5, 6, 7)$

$\tilde{\sigma}_0 = (4, 6, 8, 2)$
$\tilde{\sigma}_1 = (1, 2, 3, 4, 6, 7)$

These represent the algebraic curves

$y^2 = x(x - 1)(x + \sqrt{2})$ and $y^2 = x(x - 1)(x - \sqrt{2})$
Sketch of the proof of the main result

PART I: To any dessin one can associate a compact Riemann surface (= an algebraic curve) together with a (Belyi) function on it.

PART II: The curves so obtained account for all algebraic curves defined over number fields.

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Then we will discuss the action of the absolute Galois group on dessins with special attention to the case of regular ones.
The triangle decomposition $\mathcal{T}(D)$ associated to $D$
The topological covering of $S^2$ associated to $\mathcal{T}(D)$
The Riemann surface structure associated to $\mathcal{D}$

Let $X^* = X \setminus \{\circ, \bullet, \times\}$.

By construction the map $f^*_{\mathcal{D}} : X^* \to \hat{\mathbb{C}}$ is a local homeo; in fact, a topological cover of $\hat{\mathbb{C}} \setminus \{0, 1, \infty\}$. Therefore $X^*$ inherits the Riemann surface structure of $\hat{\mathbb{C}}$; charts being defined by restriction of $f^*_{\mathcal{D}}$ to small neighborhoods.

By definition, $f^*_{\mathcal{D}} : X^*_{\mathcal{D}} \to \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ is a holomorphic map.

Moreover, by standard extension arguments this complex structure extends to a complex structure on the compact surface $X_{\mathcal{D}}$ and the function $f^*_{\mathcal{D}}$ to a meromorphic function $f_{\mathcal{D}} : X_{\mathcal{D}} \to \hat{\mathbb{C}}$ (i.e. a holomorphic map $f_{\mathcal{D}} : X_{\mathcal{D}} \to \hat{\mathbb{C}}$).

Observation/Definition: $f_{\mathcal{D}} : X_{\mathcal{D}} \to \hat{\mathbb{C}}$ is a Belyi function meaning that it has $\leq 3$ branching values, say $0, 1, \infty$. 
1) Dessins $\mathcal{D} \leftrightarrow$ Belyi pairs $(X_D, f_D) \equiv (C_D, R_D)$

Due to the correspondence between meromorphic functions of compact Riemann surfaces and rational functions of algebraic curves we will sometimes denote $(X_D, f_D)$ by $(C_D, R_D)$ to emphasize the algebraic nature of the Belyi pair.
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This classical correspondence can be established as follows: If \((h, g)\) is a pair of generating functions there is an irreducible polynomial \(F(X, Y) \in \mathbb{C}[X, Y]\) satisfying \(F(h, g) \equiv 0\) and then \(X_D\) can be identified to the curve

\[C_D : F(x, y) = 0\]

via the map

\[\phi : X_D \longrightarrow C_D\]

\[P \longmapsto (h(P), g(P))\]

Under \(\phi\) the pair \((h, g)\) is identified to the coordinate functions \((x, y)\) and the Belyi function \(f_D = R(h, g)\), for some \(R(X, Y) \in \mathbb{C}(X, Y)\), with the rational function \(R(x, y)\) on \(C\).
Properties of the Belyi pair \((X_D, f_D)\)

1. \(\deg(f_D) = \text{card}(f_D^{-1}(1/2)) = \text{number of edges of } D.\)
2. Poles \(\leftrightarrow \{\times \text{'s } \equiv \text{Faces}\};\) Zeros \(\leftrightarrow \{\circ \text{'s }\};\) 1's \(\leftrightarrow \{\bullet \text{'s }\}\)
3. The order of a pole \(\times\) agrees with half the number of edges of the face of which \(\times\) is a center (suitably counted).
4. The multiplicity of \(f_D\) at a vertex \(v = \bullet, \circ\) of \(D\) coincides with the degree of the vertex.
2) dessins $\mathcal{D}_f \leftarrow$ Belyi pairs $(S, f)$

**Proposition**

Let $f : S \to \hat{\mathbb{C}}$ be a Belyi function and set $\mathcal{D}_f = f^{-1}([0, 1])$. Consider $\mathcal{D}_f$ as a bicoloured graph embedded in $S$ whose set of white (resp. black) vertices is $f^{-1}(0)$ (resp. $f^{-1}(1)$). Then

- $\mathcal{D}_f$ is a dessin d’enfant and $\mathcal{D}_{f_\mathcal{D}} = \mathcal{D}$. 

![Graph Diagram](image)
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- $\mathcal{D}_f$ is a dessin d’enfant and $\mathcal{D}_{f_\mathcal{D}} = \mathcal{D}$. 
The correspondences:

\[ \{ \text{Equiv. classes of dessins} \} \longrightarrow \{ \text{Equiv. classes of Belyi pairs} \} \]

\[ (X, D) \quad \longleftrightarrow \quad (X_D, f_D) = (C_D, R_D) \]

\[ (S, D_f) \quad \longleftrightarrow \quad (S, f) \]

are mutually inverse.
Equivalent ways to describe dessins

1) **Dessins** $\mathcal{D}$ of degree $n$.
2) **Belyi pairs** $(C, R)$ with $\text{deg}(R) = n$.
3) **Pairs of permutations** $(\sigma_0, \sigma_1)$ generating a transitive subgroup of $S_n$.
4) **Finite index subgroups** of $\Gamma(2) = \Delta(\infty, \infty, \infty)$.
5) **Finite index subgroups** of $\Delta(l, m, d)$.

The link between 4) (or 5) ) and 3) is established as follows:

Suppose that $l, m, d$ are the orders of $\sigma_0, \sigma_1, \sigma_0\sigma_1$ and consider the homomorphism

$$\rho : \Delta(l, m, d) \rightarrow S_n$$

determined by $x, y, z \rightarrow \sigma_0, \sigma_1, (\sigma_0\sigma_1)^{-1}$
then, the subgroup of $\Delta(l, m, d)$ in question is the preimage of the inertia subgroup of $< \sigma_0, \sigma_1 >$.
The genus of $\mathcal{D}$ equals 0 $\Rightarrow (C_{\mathcal{D}}, f_{\mathcal{D}}) \equiv (\hat{\mathcal{C}}, R(z))$

\[
\begin{align*}
\sigma_0 &= (1, 2, 3, 4) \\
\sigma_1 &= (1)(2)(3)(4)
\end{align*}
\]

1) $\mathcal{D}$ has only one face $\Rightarrow R \equiv f_{\mathcal{D}}$ is a polynomial.
2) $\mathcal{D}$ has 4 edges $\Rightarrow \text{deg}(R) = 4$
3) $\mathcal{D}$ has a white vertex of degree 4 at $z = 0$ $\Rightarrow R(z) = cz^4$
4) 1 is a black vertex $\Rightarrow R(1) = 1 \Rightarrow R(z) = z^4$
The modular function $j(\lambda) = \frac{4}{27} \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}$

It is easy to check that this is a Belyi function and that its corresponding dessin is

- The 3 faces (of degree 4) correspond to the 3 poles 0, 1, $\infty$ (of degree 4/2=2).
- The 2 white vertices (of degree 3) stand for the 2 roots of $(1 - \lambda + \lambda^2)^3 = 0$ (of degree 3).
- The 3 black vertices (of order 2) correspond to the remaining three branch points (of degree 2) one encounters by computing the derivative, namely

$$j'(\lambda) = \frac{(\lambda^2 - \lambda + 1)^2}{\lambda^3(\lambda - 1)^3} (2\lambda - 1)(\lambda^2 - \lambda - 2) = 0$$
A genus one example

From here one can see that the corresponding Belyi pair is
\((C \oplus \mathbb{Z}e^{2\pi i/3}, \wp_3) \sim (y^2 = x^3 - 1, x^3)\)

\((\wp_3 = \text{Weierstrass function})\)
A genus one example

From here one can see that the corresponding Belyi pair is

\((\mathbb{C} \mathbb{Z} \oplus \mathbb{Z} e^{2\pi i/3}, \wp^3) \cong (y^2 = x^3 - 1, x^3)\)

(\(\wp = \text{Weierstrass function}\))
Belyi’s theorem

We now deal with the second part. We wanted to show that there is a correspondence

\[ \{\text{Dessins}\} \leftrightarrow \{\text{Curves defined over } \overline{\mathbb{Q}} \text{ (plus a Belyi function)}\} \]

We already established the correspondence

\[ \{\text{Equiv. classes of dessins}\} \leftrightarrow \{\text{Equiv. classes of Belyi pairs}\} \]

so it remains to show that Belyi curves are defined over \( \overline{\mathbb{Q}} \).

Theorem (Belyi, 1979)

Let \( S \) be a compact Riemann surface. The following statements are equivalent:

(a) \( S \) is defined over \( \overline{\mathbb{Q}} \) (or, equivalently, over a number field).

(b) \( S \) admits a morphism \( f : S \rightarrow \mathbb{P}^1 \) with at most three branching values, i.e. a Belyi function.
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Dessins

PART I: From dessins to Belyi pairs

Examples

PART II: Belyi’s theorem

Galois action

Action of $\text{Gal}(\overline{\mathbb{Q}})$ on triangle curves

(Proof of $\Rightarrow$)

$P(z) = \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$

\[
\left\{ y^2 = x(x - 1) \left( x - \sqrt[2]{\frac{m+n}{m}} \right) \right\} \cup \{ \infty \}
\]

\[ (x,y) \mapsto x \]

\[ \hat{\mathbb{C}} \left\{ 0, 1, \sqrt[2]{\frac{m+n}{m}}, \infty \right\} \]

\[ t \mapsto t^2 \quad (\approx \text{minimal pol.}) \]

\[ u \mapsto 1/u \]

\[ \hat{\mathbb{C}} \left\{ \infty, 1, \frac{m}{m+n}, 0 \right\} \]

\[ z \mapsto \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n \quad (\text{Belyi’s pol.}) \]

\[ \hat{\mathbb{C}} \left\{ \infty, 1, 0 \right\} \]
A complex number $\alpha$ lies in $\overline{\mathbb{Q}} \iff$ the set of complex numbers

$$G_\mathbb{C}(\alpha) = \{ \alpha^\sigma := \sigma(\alpha) \text{ such that } \sigma \in G_\mathbb{C} := Gal(\mathbb{C}) \}$$

is finite.

**Examples:**

- $G_\mathbb{C}(\alpha) = \{ \alpha \} \iff \alpha \in \mathbb{Q}$
- $G_\mathbb{C}(\sqrt{2}) = \{ \sqrt{2}, -\sqrt{2} \}$
- $G_\mathbb{C}(\pi) = \{ \text{transcendental numbers} \}$

If

$$C : F(x, y) = \sum a_{ij} x^i y^j = 0$$

we define

$$C^\sigma : F^\sigma(x, y) = \sum a_{ij}^\sigma x^i y^j = 0 , \quad G_\mathbb{C}(C) = \{ C^\sigma : \sigma \in G_\mathbb{C} \}$$

Then, again: $C$ is defined over $\overline{\mathbb{Q}} \iff G_\mathbb{C}(C)/\text{equiv.}$ is finite.

The analogous statement holds for coverings $(C, R)$ of $\mathbb{P}^1$. 
Proof of $\Leftarrow$)

Consider the following commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\sigma} & C^\sigma \\
\downarrow R & & \downarrow R^\sigma \\
\mathbb{P}^1 & \xrightarrow{\sigma} & \mathbb{P}^1 \\
\end{array}
\]

\[
(\sigma(a, b) = (a^\sigma, b^\sigma))
\]

(i.e. \(R(a, b)^\sigma = R^\sigma(a^\sigma, b^\sigma)\))

It shows that \(\sigma\) sends Belyi pairs of **degree** \(n\) to Belyi pairs of **same degree** \(n\), hence there is an action of \(G_{\mathbb{C}}\) on dessins:

\[
\begin{array}{ccc}
D & \xrightarrow{\sigma} & D^\sigma \\
\downarrow & & \uparrow \\
(C_D, R_D) & \xrightarrow{\sigma} & (C_D^\sigma, R_D^\sigma) \\
\end{array}
\]

Because the degree and the branching is preserved, the set

\[
G_{\mathbb{C}}(C, R) = \{(C^\sigma, R^\sigma) : \sigma \in G_{\mathbb{C}}\} \leftrightarrow G_{\mathbb{C}}(D) = \{D^\sigma : \sigma \in G_{\mathbb{C}}\}
\]

**is finite** \(\Rightarrow\) \(C\) and \(R\) can be defined over a number field.
Once we know that Belyi pairs are defined over $\overline{\mathbb{Q}}$ it makes sense to “restrict” the action of $\text{Gal}(\mathbb{C})$ on dessins

\[
\mathcal{D} \xrightarrow{- - - - - \sigma} \mathcal{D}^\sigma
\]

\[
(C_D, R_D) \xrightarrow{\sigma} (C_D^\sigma, R_D^\sigma)
\]

to an action of $\text{Gal}(\overline{\mathbb{Q}})$ in the hope this action of the absolute Galois group on combinatorial objects might help to understand the nature of the group.

Let us look at a couple of examples
Gal($\overline{\mathbb{Q}}$)-orbits of female and male symbols

\[ f(z) = \frac{z^4(z-1)^2}{(z-b)} \quad \text{where} \quad 36b^2 - 44b + 9 = 0 \]

\[ m(z) = \frac{z^3(z-1)^2}{(z-b)} \quad \text{where} \quad 25b^2 - 28b + 4 = 0 \]
It can be seen that the following two dessins form a complete orbit for the action of $Gal(\overline{Q})$.

They are conjugate by any $\sigma$ satisfying the property

$$\sigma(\sqrt{2}) = -\sqrt{2}.$$
Dessin of \( f : \{ y^2 = x(x - 1)(x - \sqrt{2}) \} \rightarrow \hat{\mathbb{C}} \)

\[
f(x, y) = \frac{-4(x^2 - 1)}{(x^2 - 2)^2}
\]
Regular dessins

\( \mathcal{D} \) is said to be **regular** if \( G = Aut(\mathcal{D}) \) acts transitively on its edges so that \((X_\mathcal{D}, f_\mathcal{D})\) is a Galois cover with covering group \( G \).

\[
\begin{align*}
X_\mathcal{D} & \quad \xymatrix{ \pi \ar[d] & f_\mathcal{D} \ar[l] \ar[d] \\
X_\mathcal{D}/Aut(\mathcal{D}) & \sim \ar[r] & \mathbb{P}^1}
\end{align*}
\]
Regular dessins

\( \mathcal{D} \) is said to be \textbf{regular} if \( G = \text{Aut}(\mathcal{D}) \) acts transitively on its edges so that \((X_\mathcal{D}, f_\mathcal{D})\) is a Galois cover with covering group \( G \).

\[
\begin{align*}
X_\mathcal{D} & \xrightarrow{\pi} X_\mathcal{D}/\text{Aut}(\mathcal{D}) & \sim & \mathbb{P}^1 \\
& \xrightarrow{f_\mathcal{D}} & & \\
\end{align*}
\]

We have already seen an example:

\[
\begin{align*}
\hat{\mathbb{C}} & \xrightarrow{j(\lambda) = \frac{(1-\lambda+\lambda^2)^3}{\lambda^2(\lambda-1)^2}} \hat{\mathbb{C}}/G & \sim & \mathbb{P}^1 \\
& \xrightarrow{\pi} & & \\
\end{align*}
\]

\( (G = \langle z \rightarrow 1 - z, \ z \rightarrow 1/(1 - z) \rangle \cong S_3 \) )
A key point in Grothendieck’s theory of dessins is that the action of $Gal(\overline{\mathbb{Q}})$ on them, or equivalently on Belyi pairs $(C, f)$, is faithful.
A key point in Grothendieck’s theory of dessins is that the action of $\text{Gal} (\overline{\mathbb{Q}})$ on them, or equivalently on Belyi pairs $(C, f)$, is faithful.

In fact, except for the case $g = 0$, the action is already faithful on the set of Belyi curves $C$ of any given genus (disregarding the Belyi function).

For instance if $\sigma \in \text{Gal}(\overline{\mathbb{Q}})$ is non-trivial and $j(\lambda)^\sigma \neq j(\lambda)$ then the elliptic curves

$$C : y^2 = (x - 1)(x - 2)(x - \lambda) \quad \text{and}$$

$$C^\sigma : y^2 = (x - 1)(x - 2)(x - \lambda^\sigma)$$

are non-isomorphic to each other.
Triangle curves

Q. 1: Does $\text{Gal}(\overline{\mathbb{Q}})$ act faithfully on regular dessins $(C, f)$?

Curves arising in this way are called **triangle (or quasiplatonic) curves** i.e. $C$ is a triangle curve if $C/\text{Aut}(C) \equiv \mathbb{P}^1$ and the projection $f : C \rightarrow C/\text{Aut}(C) \equiv \mathbb{P}^1$ ramifies over 3 values.

Q. 1’: Is the action of $\text{Gal}(\overline{\mathbb{Q}})$ faithful on triangle curves?

There are only finitely many triangle curves in each genus

List of triangle curves in low genus

- **Genus 2**: $y^2 = x^6 - x$, $y^2 = x^6 - 1$, $y^2 = x^5 - x$
- **Genus 3**: $y^2 = x^8 - x$, $y^2 = x^7 - x$, $y^2 = x^8 - 1$,
  $y^2 = x^8 - 14x^4 + 1$, $y^3 = x(1 - x^3)$,
  $y^4 = 1 - x^3$, $y^4 = 1 - x^4$, $y^7 = x(1 - x)^2$
- **Genus 4**: 11 curves **all of them again defined over** $\mathbb{Q}$. 
In all previous examples of triangle curves the action of $Gal(\overline{Q})$ is trivial. However:

**Theorem (\text{-, A. Jaikin-Zapirain})**

$Gal(\overline{Q})$ acts faithfully on triangle curves.

In fact on the subset of triangle curves which are Galois covers of a given one, e.g. Fremat’s or Klein’s curve, or any of the previous list.....
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THANK YOU