Some New q-Series Conjectures

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Based on joint works with:
Matthew C. Russell, Debajyoti Nandi

1. Preliminaries

Partitions

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Pochhamer symbols:

$$(a;\, q)_j = (1-a)(1-aq)\dots(1-aq^{j-1})$$
$$(q;\, q)_j = (1-q)(1-q^2)\dots(1-aq^j)$$
$$(a;\, q)_\infty = (1-a)(1-aq)\dots$$
$$(a_1, a_2, a_3, \dots;\, q)_t = (a_1; q)_t(a_2; q)_t(a_3; q)_t \dots.$$

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Rogers-Ramanujan identities

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↑ partitions with $\equiv 1, 4 \pmod 5$

Generating functions:

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Analytic sum-side

Product-side

Partition-theoretic sum-side

Sum-sides

$1 + 3 + 6 + 8$

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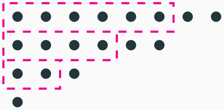

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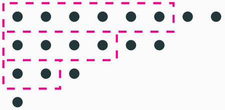

⇝ Remove the 2-staircase ⇝

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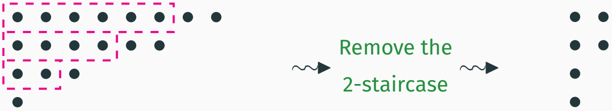

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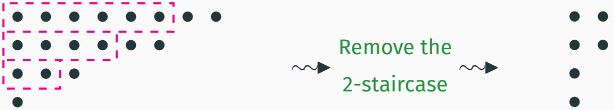
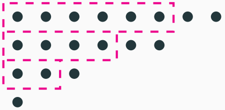


RR partitions
of length 4

Sum-sides

$1 + 3 + 6 + 8$



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RR partitions of length 4 ⟸ Bijection ⟹ Partitions of length 4

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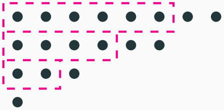

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RR partitions of length 4 $\Longleftarrow$ Bijection $\Longrightarrow$ Partitions of length 4

$$\frac{x^4 \cdot q^4}{(q;q)_4}$$

Sum-sides

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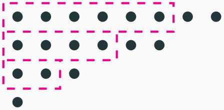

$\rightsquigarrow$ Remove the 2-staircase $\rightsquigarrow$

$1+1+2+2$

RR partitions of length 4 $\Longleftarrow$ Bijection $\Longrightarrow$ Partitions of length 4

$\leftsquigarrow$ Put back 2-staircase $\leftsquigarrow$ $\dfrac{x^4 \cdot q^4}{(q;q)_4}$

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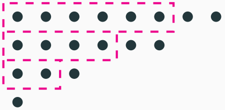

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RR partitions of length 4 $\Longleftarrow$ Bijection $\Longrightarrow$ Partitions of length 4

$$\frac{x^4 q^{2+4+6} \cdot q^4}{(q;q)_4}$$ $\leftsquigarrow$ Put back 2-staircase $\leftsquigarrow$ $$\frac{x^4 \cdot q^4}{(q;q)_4}$$

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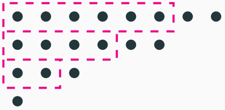

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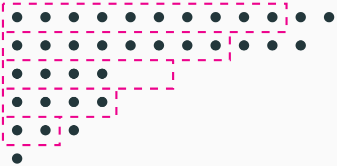
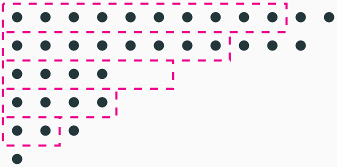
$$\sum_{\ell \geq 0} \frac{x^\ell q^{\ell(\ell-1)} \cdot q^\ell}{(q;q)_\ell} \quad \leftsquigarrow \text{2-staircase} \rightsquigarrow \quad \sum_{\ell \geq 0} \frac{x^\ell \cdot q^\ell}{(q;q)_\ell}$$

Jagged partitions



$1 + 3 + 4 + 4 + 11 + 12$

Jagged partitions



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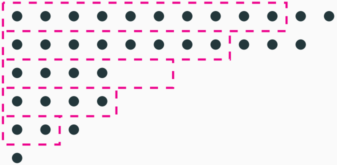

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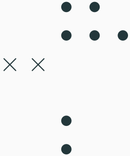

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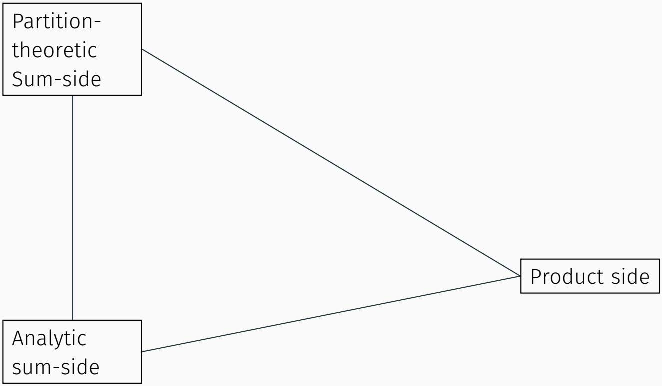
Jagged partition

Pick a side


Partition-theoretic Sum-side
Product side
Analytic sum-side

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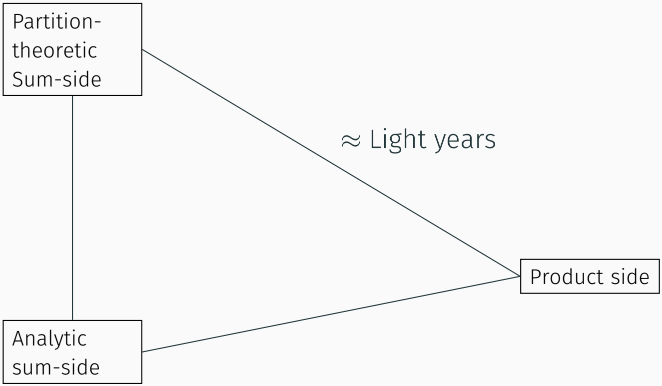
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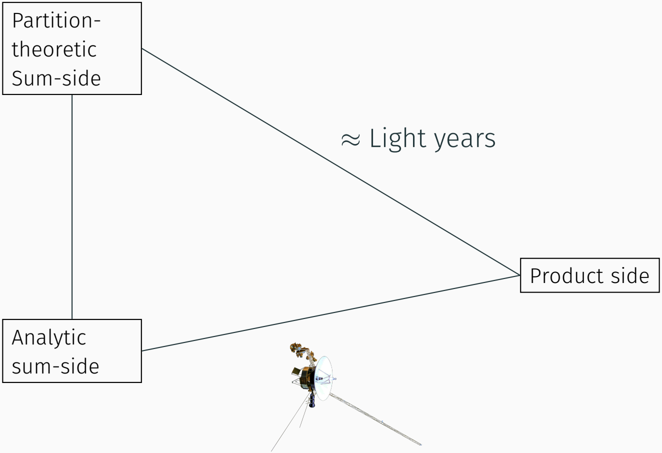
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Rogers-Ramanujan 2

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Rogers-Ramanujan 2

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Partitions of n whose adjacent parts differ by at least 2 and whose smallest part is at least 2

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A few generalizations

Andrews-Gordon: $\pmod 7$

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Andrews-Gordon: $\pmod{7}$

Partitions of n where parts satisfy distance at least 2 at distance 2

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- In general, there are such identities for all odd moduli:
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- For even moduli, one more condition on partition-theoretic sum-sides

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partitions whose parts are $\not\equiv 0, \pm 3 \pmod 7$.

- Analytic sum-side: $\displaystyle\sum_{n_1,n_2\geq 0} \frac{q^{n_1^2+2n_2^2+2n_1n_2}}{(q;q)_{n_1}(q;q)_{n_2}}$.
- Deform the q exponent by a linear term
  $\rightsquigarrow$ Two other identities : $\not\equiv 0, \pm i \pmod 7$, $i = 1, 2$.
  Guess the partition sum-sides $\rightsquigarrow$ Differ in "initial conditions".
- In general, there are such identities for all odd moduli: $\pmod{2k+1}$.
- For even moduli, one more condition on partition-theoretic sum-sides $\rightsquigarrow$ Andrews-Bressoud identities.

2. Context

[1] RR and principal subspaces

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$$W = \mathcal{A} / \left(\mathcal{A}\langle r_{-2}, r_{-3}, r_{-4}, \ldots \rangle \right)$$

Definition (actually, a Theorem of Calinescu-Lepowsky-Milas): Principal Subspace

This is the principal subspace of **level** 1 “vacuum module” of $\widehat{\mathfrak{sl}}_2$.

- W has a basis of monomials satisfying difference-2 conditions. (proved by several people in different contexts)

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- Higher level for $\mathfrak{sl}_2$: Andrews-Gordon identities
- Change $\mathfrak{sl}_2$: Noncommutative algebras. [Work of Butorac, Capparelli, Calinescu, Georgiev, Lepowsky, Milas, Penn, Primc, Trupčević, Sadowski,...]

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The homology of this complex captures “relations” amongst the elements $r_{-2}, r_{-3}, r_{-4}, \ldots$.

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Conjecture (Gorsky-Oblomkov-Rasmussen '12)

$\mathrm{Kh}(T(n,\infty))$ is dual to the homology of the Koszul complex determined by the elements $r_{-2},\dots,r_{-n-1}$. (Note: their gradings are different than ours.)

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$$\chi(L(\lambda)) = \left(e^{-\lambda}\,\mathrm{ch}(L(\lambda))\right)\Big|_{e^{-\alpha_0},\ldots,e^{-\alpha_t} \mapsto q}.$$

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Theorem

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Second factor: character of the “vacuum space”.

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Many other ways to mine identities:

- Meurman and Primc: Look at the entire modules, not just vacuum spaces.
  $\rightsquigarrow$ new identities found by Meurman-Primc, Siladić (proved by Dousse), Primc-Šikić.
- Beyond principal specializations: Analytic sum-sides are given by Hall-Littlewood polynomials. (Ole Warnaar)

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Proofs by Andrews, Andrews-Alladi-Gordon, Tamba-Xie, Capparelli, Meurman-Primc, Dousse-Lovejoy.

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Conjecture: Nandi (2014), a slight reformulation due to A. Sills

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Number of partitions of n into parts $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ is the same as number of partitions $n = \pi_1 + \cdots + \pi_t$ with:

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The point

It is getting harder to implement Z-algebras to mine new identities.

3. Experimental strategies: Sums to products

The key: Euler’s algorithm

Given a power series

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$$f(q) = 1 + q\mathbb{N}[[q]],$$

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Moreover, $a_1, a_2 \dots a_N$ are completely determined by the expansion of f up to coefficient of q^N.

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Inverse Euler Transform

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Mod 9 asymmetric conjectures

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$\mathbf{I}_4$ Partitions of n satisfying difference at least 3 at distance 2 and

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Recall:

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I_4 Partitions of n satisfying difference at least 3 at distance 2 and two consecutive parts differ by 0 or 1 $\Rightarrow$ their sum is $\equiv 2 \pmod 3$ and

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Mod 12 asymmetric conjectures

Condition 😈$_t$

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I_5 Partitions of n satisfying **Condition** 😈$_1$

Mod 12 asymmetric conjectures

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Matthew C. Russell in his Rutgers Ph.D. Thesis found companions I_{4a}, I_{5a}, I_{6a} whose products sides involved “negatives” of the residues of the asymmetric product sides.

Difference on the partition-theoretic sum-sides: only in the initial condition.

Sums to products: II

- Need a better search space to inch closer to Nandi's conditions

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$$3 + 3 + 2 + 2 < 3 + 3 + 3 + 1$$

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$$\begin{aligned} 3+3+2+2 &< 3+3+3+1 < 4+2+2+2 < 4+3+2+1 \\ &< 4+4+1+1 \end{aligned}$$

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$$3+3+2+2 < 3+3+3+1 < 4+2+2+2 < 4+3+2+1$$
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$$
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\end{aligned}
$$

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- Look closer into Lepowsky-Wilson's mechanism
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$(2, 2t + 1)$ already found by Andrews.

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4. Experimental strategies: Products to sum

Look at those **level** 2 modules for $A_{\mathbf{odd}}^{(2)}$ that are contained in the (vector space) tensor product of inequivalent level 1 modules.

$A_9^{(2)}$

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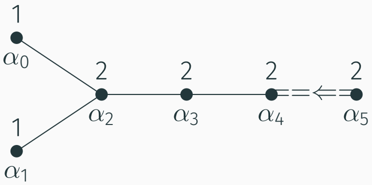
unless one is ready to do some extremely tedious algebraic computations


1
α0
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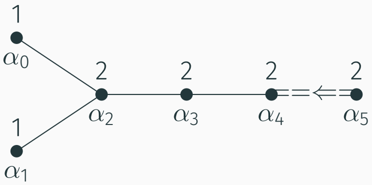
α5
1
α1



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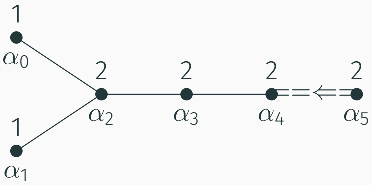

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Should have partition-theoretic sum-sides differing only in initial conditions.

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$(q^4, q^5, q^6, q^7, q^8; q^{12})_\infty^{-1}$	**Condition** 😎 and no $1, 2$ or 3 s

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Remove 2-staircase $\rightsquigarrow$ Jagged partitions

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⇝ Vary this term!

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