Infrared problem and adiabatic limit in perturbative quantum field theory

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Relativistic perturbative QFT in Minkowski spacetime

- Wightman and Green functions:

$$\mathrm{W}(B_1(x_1),\ldots,B_m(x_m)) = (\Omega|B_{1,\mathrm{int}}(x_1)\ldots B_{m,\mathrm{int}}(x_m)\Omega) \in \mathcal{S}'(\mathbb{R}^{4m})[\![e]\!],$$
$$\mathrm{G}(B_1(x_1),\ldots,B_m(x_m)) = (\Omega|\,\mathrm{T}(B_{1,\mathrm{int}}(x_1),\ldots,B_{m,\mathrm{int}}(x_m))\Omega) \in \mathcal{S}'(\mathbb{R}^{4m})[\![e]\!]$$

(Ω – the vacuum state, $B_1,\ldots,B_m$ – polynomials in the basic fields and their derivatives, e – the coupling constant).

- Scattering operator: $S = \mathrm{Texp}\left(\mathrm{i}e\int \mathrm{d}^4x\,\mathcal{L}(x)\right) \in L(\mathcal{D})[\![e]\!], \quad \mathcal{D}\subset\mathcal{H}$.

  S-matrix elements: $(\Psi_1|S\Psi_2) \in \mathbb{C}[\![e]\!]$ for $\Psi_1,\Psi_2\in\mathcal{D}\subset\mathcal{H}$.

- Differential cross section:

$$\sigma(p_1,\ldots,p_k;p'_1,\ldots,p'_l) = (2\pi)^4\delta(p_1+\ldots+p_k-p'_1-\ldots-p'_l)$$
$$\times\,(\text{kinematical factor})\times|\mathcal{M}_{\text{connected}}(p_1,\ldots,p_k;p'_1,\ldots,p'_l)|^2,$$

  where the invariant matrix element $\mathcal{M}(p_1,\ldots,p_k;p'_1,\ldots,p'_l)$ is given by

$$(p_1,\ldots,p_k|S|p'_1,\ldots,p'_l)$$
$$= (2\pi)^4\delta(p_1+\ldots+p_k-p'_1-\ldots-p'_l)\left(1+\mathrm{i}\mathcal{M}(p_1,\ldots,p_k;p'_1,\ldots,p'_l)\right).$$

- Interacting field operators: $B_{\mathrm{int}}(x)\in\mathcal{S}'(\mathbb{R}^4,L(\mathcal{D}))[\![e]\!], \quad \mathcal{D}\subset\mathcal{H}$.
- Algebra of interacting fields: an abstract algebra $\mathfrak{F}$ over the ring $\mathbb{C}[\![e]\!]$.
- Vacuum state: Poinacré-invariant real, normalized and positive functionals $\mathfrak{F}\to\mathbb{C}[\![e]\!]$.

Relativistic perturbative QFT in Minkowski spacetime

Ultraviolet problem (short distance/large energy)

- Difficulties in defining the time-ordered products.
- **Completely solved by renormalization techniques**.

Infrared problem (large distance/small energy)

- *Standard solution*: Introduce some infrared regularization and show that the regularization can be removed.

Infrared regularizations

- Green functions:
  Bogoliubov, Parasiuk, Hepp: $\frac{1}{k^2-m^2+\mathrm{i}0} \rightsquigarrow \frac{1}{k^2-m^2+\mathrm{i}\epsilon}, \quad \epsilon > 0.$
  Zimmerman, Lowenstein: $\frac{1}{k^2-m^2+\mathrm{i}0} \rightsquigarrow \frac{1}{k^2-m^2+\mathrm{i}\epsilon(\vec{k}^2+m^2)}, \quad \epsilon > 0.$
- Inclusive cross sections:
  Yennie, Frautschi, Sura: give photons a positive mass.
  Weinberg: introduce a lower bound on the photon momenta.
- S-matrix, interacting fields, Wightman and Green functions:
  Bogoliubov, Epstein, Glaser: $e \rightsquigarrow eg(x)$, where $g \in \mathcal{S}(\mathbb{R}^4)$.
  The function g is called **the switching function** and the above infrared regularization is called **the adiabatic cutoff**.

Theory with *adiabatic cutoff*

- Scattering operator:

$$S(g) = \mathrm{Texp}\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\mathcal{L}(x)\right)$$
$$= \sum_{n=0}^{\infty} \frac{\mathrm{i}^n e^n}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n\, g(x_1)\ldots g(x_n)\ \mathrm{T}(\mathcal{L}(x_1),\ldots,\mathcal{L}(x_n)). \quad (1)$$

- Retarded interacting field operators:

$$B_{\mathrm{ret}}(g;x) = (-\mathrm{i})\frac{\delta}{\delta h(x)} S(g)^{-1}S(g;h), \quad (2)$$

where

$$S(g;h) = \mathrm{Texp}\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\mathcal{L}(x) + \mathrm{i}\int \mathrm{d}^4x\, h(x)B(x)\right). \quad (3)$$

- Time-ordered products of interacting fields:

$$\mathrm{T}(B_{1,\mathrm{ret}}(g;x_1),\ldots,B_{m,\mathrm{ret}}(g;x_m)) = (-\mathrm{i})^m \frac{\delta}{\delta h_1(x_1)}\ldots\frac{\delta}{\delta h_m(x_m)} S(g)^{-1}S(g;h)\Big|_{h=0}, \quad (4)$$

where

$$S(g;h) = \mathrm{Texp}\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\mathcal{L}(x) + \mathrm{i}\int \mathrm{d}^4x \sum_{j=1}^{m} h_j(x)B_j(x)\right). \quad (5)$$

- Wightman and Green functions:

$$\mathrm{W}(g;B_1(x_1),\ldots,B_m(x_m)) = (\Omega|B_{1,\mathrm{ret}}(g;x_1)\ldots B_{m,\mathrm{ret}}(g;x_m)\Omega), \quad (6)$$
$$\mathrm{G}(g;B_1(x_1),\ldots,B_m(x_m)) = (\Omega|\,\mathrm{T}(B_{1,\mathrm{ret}}(g;x_1),\ldots,B_{m,\mathrm{ret}}(g;x_m))\Omega). \quad (7)$$

I. Algebraic adiabatic limit:

construction of the algebra of interacting fields

II. Weak adiabatic limit:

construction of the Wightman and Green functions
and the vacuum state on the algebra of interacting fields

III. Strong adiabatic limit:

construction of the S-matrix and interacting fields

Algebraic adiabatic limit – abstract algebra of interacting fields

- The construction of the net of local abstract algebras of interacting fields [Brunetti, Fredenhagen (2000)].
- Let $\mathfrak{F}_g(\mathcal{O})$ be the algebra generated by

$$\{B_{\mathrm{ret}}(g;h)\ :\ B \in \mathcal{F},\ h \in \mathcal{D}(\mathbb{R}^4),\ \mathrm{supp}\, h \subset \mathcal{O}\}. \tag{8}$$

- For any bounded region $\mathcal{O}$ in the Minkowski space we set

$$\mathcal{G}_{\mathcal{O}} = \{g \in \mathcal{D}(\mathbb{R}^4)\ :\ g \equiv 1 \text{ on a neighborhood of } J^+(\mathcal{O}) \cap J^-(\mathcal{O})\}. \tag{9}$$

- If $g, g' \in \mathcal{G}_{\mathcal{O}}$, then the algebras $\mathfrak{F}_g(\mathcal{O})$ and $\mathfrak{F}_{g'}(\mathcal{O})$ are unitarily equivalent.
  $\Rightarrow$ There is a unique abstract algebra $\mathfrak{F}(\mathcal{O})$ of interacting fields localized in $\mathcal{O}$.
- The net $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$ satisfies the Haag-Kastler axioms in the sense of formal power series [Fredenhagen, Rejzner (2015)].
- The generalization to models with gauge symmetries: construction of the algebras of interacting *observables* in QED [Dütsch, Fredenhagen (1999)] and non-abelian Yang-Mills theories [Hollands (2008)].

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Weak adiabatic limit – Wightman and Green functions

Adiabatic limit

For any $g \in \mathcal{S}(\mathbb{R}^N)$ such that $g(0) = 1$ we define a one-parameter family of switching functions:

$$g_\epsilon(x) = g(\epsilon x) \quad \text{for} \quad \epsilon > 0. \tag{10}$$

We have $\lim_{\epsilon \searrow 0} g_\epsilon(x) = 1$ pointwise.

In the limit $\epsilon \searrow 0$ the interaction is **turned on/off adiabatically**.

Weak adiabatic limit

$$\mathrm{W}(B_1(x_1), \ldots, B_m(x_m)) = \lim_{\epsilon \searrow 0} (\Omega | B_{1,\mathrm{ret}}(g_\epsilon; x_1) \ldots B_{m,\mathrm{ret}}(g_\epsilon; x_m)\Omega), \tag{11}$$

$$\mathrm{G}(B_1(x_1), \ldots, B_m(x_m)) = \lim_{\epsilon \searrow 0} (\Omega | \mathrm{T}(B_{1,\mathrm{ret}}(g_\epsilon; x_1), \ldots, B_{m,\mathrm{ret}}(g_\epsilon; x_m))\Omega). \tag{12}$$

Existence of the weak adiabatic limit:

- purely massive models [Epstein, Glaser (1973)],
- QED and the massless φ^4 theory [Blanchard, Seneor (1975)],
- all models with interaction vertices of dimension 4 [Duch (2018)].

Weak adiabatic limit – Wightman and Green functions

Properties of the Wightman functions

- Poincaré covariance,
- relativistic spectral condition:

$$\mathrm{supp}\ \mathrm{W}(\tilde{B}_1(p_1),\ldots,\tilde{B}_m(p_m)) \subset \left\{ \sum_{j=1}^m p_j = 0,\ \forall_k \sum_{j=1}^k p_j \in \overline{V}^+ \right\}, \tag{13}$$

- Hermiticity: $\overline{\mathrm{W}(B_1(x_1),\ldots,B_m(x_m))} = \mathrm{W}(B_m^*(x_m),\ldots,B_1^*(x_1))$,
- local (anti)commutativity: if x_k and x_{k+1} are spatially-separated, then

$$\mathrm{W}(\ldots,B_k(x_k),B_{k+1}(x_{k+1}),\ldots) = \pm\,\mathrm{W}(\ldots,B_{k+1}(x_{k+1}),B_k(x_k),\ldots). \tag{14}$$

- positive definiteness condition (in models without vector fields),
- interacting field equations: e.g. in the massless φ^4 theory it holds

$$\Box_x\,\mathrm{W}(\ldots,\varphi(x),\ldots) = \frac{\lambda}{3!}\,\mathrm{W}(\ldots,\varphi^3(x),\ldots). \tag{15}$$

Properties of the Green functions

- Poincaré covariance,
- symmetry (or graded-symmetry in the presence of fermionic fields) under permutations of the arguments,
- causality: for non-coinciding points the Green functions are expressed in terms of the Wightman functions.

Weak adiabatic limit – vacuum state

Algebra of interacting fields

- Retarded field $B_{\mathrm{ret}}(g;f)$ in the *algebraic adiabatic limit* are denoted by $B_{\mathrm{ret}}(\cdot;f)$.
- Abstract algebra $\mathfrak{F}$ of interacting fields is generated by $B_{\mathrm{ret}}(\cdot;f)$.
- An arbitrary element of $\mathfrak{F}$ will be denoted by $\mathbf{B}(\cdot)$.

States in perturbative algebraic QFT

A linear functional $\sigma : \mathfrak{F} \to \mathbb{C}[\![e]\!]$ which satisfies the following conditions:

- normalized: $\sigma(\mathbb{1}) = 1$,
- real: $\sigma(\mathbf{B}(\cdot)^*) = \overline{\sigma(\mathbf{B}(\cdot))}$,
- positive: $\sigma(\mathbf{B}(\cdot)^* \mathbf{B}(\cdot)) \geqslant 0$.

A formal power series $a \in \mathbb{C}[\![e]\!]$ is non-negative iff there exists $b \in \mathbb{C}[\![e]\!]$ such that $a = \bar{b}b$.

An example of a state: $\mathfrak{F}(\mathcal{O}) \ni \mathbf{B}(\cdot) \mapsto \sigma_\Psi(\mathbf{B}(\cdot)) = (\Psi|\mathbf{B}(g)\Psi) \in \mathbb{C}[\![e]\!]$, where $\mathcal{O} \subset \mathbb{R}^4$ is bounded, $g \in \mathcal{G}_\mathcal{O}$ and $\Psi \in \mathcal{D}_0$. States of this type can be also defined in QED [Dütsch, Fredenhagen (1999)] and non-abelian Yang-Mills theories [Hollands (2008)].

Definition of vacuum state (in models with gauge symmetry proof of positivity missing)

A unique linear functional $\sigma : \mathfrak{F} \to \mathbb{C}[\![e]\!]$ such that

$$\sigma(B_{1,\mathrm{ret}}(\cdot;h_1)\dots B_{n,\mathrm{ret}}(\cdot;h_n)) = \mathrm{W}(B_1(h_1),\dots,B_n(h_n)) \tag{16}$$

for any polynomials $B_1,\dots,B_n$ and any $h_1,\dots,h_n \in \mathcal{D}(\mathbb{R}^4)$ is a Poincaré-invariant state.

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Scattering matrix

Scattering matrix in QM in short-range potentials

$$S = \operatorname*{w-lim}_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} U_{\mathrm{fr}}(-t_2) U(t_2 - t_1) U_{\mathrm{fr}}(t_1), \tag{17}$$

where $H = H_{\mathrm{fr}} + eH_{\mathrm{int}}$, $U_{\mathrm{fr}}(t) = \exp(-\mathrm{i}tH_{\mathrm{fr}})$, $U(t) = \exp(-\mathrm{i}tH)$. We also have

$$S = \operatorname*{w-lim}_{\substack{t_1 \to -\infty \\ t_2 \to +\infty}} \mathrm{Texp}\left(-\mathrm{i}e \int_{t_1}^{t_2} \mathrm{d}t \; H^I_{\mathrm{int}}(t)\right), \tag{18}$$

where $H^I_{\mathrm{int}}(t) = U_{\mathrm{fr}}(-t) H_{\mathrm{int}} U_{\mathrm{fr}}(t)$.

Scattering matrix with adiabatic cutoff in QFT

The standard definition due to Bogoliubov:

$$\begin{aligned} S(g) &= \mathrm{Texp}\left(\mathrm{i}e \int \mathrm{d}^4x \, g(x) \mathcal{L}(x)\right) \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{i}^n e^n}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \, g(x_1) \ldots g(x_n) \, \mathrm{T}(\mathcal{L}(x_1), \ldots, \mathcal{L}(x_n)), \end{aligned} \tag{19}$$

where the switching function $g \in \mathcal{S}(\mathbb{R}^4)$. The **physical scattering matrix** is defined as the adiabatic limit of $S(g)$ if this limit exists.

Strong adiabatic limit – scattering operator and interacting fields

- Construction of the physical scattering matrix and the physical interacting fields:

$$S\Psi = \lim_{\epsilon\searrow 0} S(g_\epsilon)\Psi, \quad C_{\mathrm{ret}}(f)\Psi = \lim_{\epsilon\searrow 0} C_{\mathrm{ret}}(g_\epsilon; f)\Psi \quad \text{for all} \quad \Psi \in \mathcal{D}_1. \tag{20}$$

- Strong adiabatic limit exists in all purely massive theories in which one particles states are kinematically stable [Epstein, Glaser (1976)], [Duch (in preparation)].
- Because of **the infrared problem** the strong adiabatic limit does not exist in the standard sense in most theories with massless particles, e.g. in QED.
- In models with long-range interactions *the evolution of the system is substantially different from the free evolution even long after or before the collision of particles*.
  $\Rightarrow$ The standard scattering theory is not applicable.
- Standard solution: **inclusive cross section** [Yennie, Frautschi, Suura (1961)], [Weinberg (1965)].
- Another solution: **modified scattering matrix** [Dollard (1964)], [Kulish, Faddeev (1970)], [Morchio, Strocchi (2016)].

The rest of the talk: rigorous formulation of **the modified scattering theory in perturbative quantum electrodynamics** [Duch (in preparation)].

The action:

$$S = \int \mathrm{d}^4 x \, \Big(\mathcal{L}^{\mathrm{gf}}_{\mathrm{fr}}(x) + e g(x) \mathcal{L}(x) \Big), \tag{21}$$

$$\mathcal{L}^{\mathrm{gf}}_{\mathrm{fr}}(x) = \overline{\psi}(x)(\mathrm{i}\partial\!\!\!/ - m)\psi(x) - \frac{1}{2}(\partial_\mu A_\nu(x))(\partial^\mu A^\nu(x)), \tag{22}$$

$$\mathcal{L}(x) = J^\mu(x) A_\mu(x), \qquad J^\mu(x) = \overline{\psi}(x)\gamma^\mu \psi(x). \tag{23}$$

Notation:

- $\mathcal{H}$ – Fock space (which is a Krein space),
- ψ – Dirac spinor field describing electrons with mass $m > 0$,
- A_μ – real vector field describing massless photons,
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ – the electromagnetic field strength tensor,
- C, $\bar{C}$ – ghosts,
- Q_{BRST} – free BRST charge.

Modified scattering matrix

Modified scattering matrix in QM, long-range potentials (e.g. Coulomb potential)

$$S_{\mathrm{mod}} = \underset{\substack{t_1\to-\infty \\ t_2\to+\infty}}{\text{w-lim}}\; U_{\mathrm{D}}(0,t_2)U(t_2-t_1)U_{\mathrm{D}}(t_1,0), \tag{24}$$

where $H = H_{\mathrm{fr}} + eH_{\mathrm{int}}, \quad H_{\mathrm{D}}(t) = H_{\mathrm{fr}} + eH_{\mathrm{D,int}}(t)$. We also have

$$\begin{aligned} S_{\mathrm{mod}} = \underset{\substack{t_1\to-\infty \\ t_2\to+\infty}}{\text{w-lim}}\; \overline{\mathrm{T}}\mathrm{exp}\left(+\mathrm{i}e\int_0^{t_2}\mathrm{d}t\; H^I_{\mathrm{D,int}}(t)\right) \\ \times\, \mathrm{Texp}\left(-\mathrm{i}e\int_{t_1}^{t_2}\mathrm{d}t\; H^I_{\mathrm{int}}(t)\right) \times \overline{\mathrm{T}}\mathrm{exp}\left(+\mathrm{i}e\int_{t_1}^{0}\mathrm{d}t\; H^I_{\mathrm{D,int}}(t)\right), \end{aligned} \tag{25}$$

where $H^I_{\mathrm{int}}(t) = U_{\mathrm{fr}}(-t)H_{\mathrm{int}}U_{\mathrm{fr}}(t)$ and $H^I_{\mathrm{D,int}}(t) = U_{\mathrm{fr}}(-t)H_{\mathrm{D,int}}U_{\mathrm{fr}}(t)$.

Modified scattering matrix with adiabatic cutoff in QFT (preliminary version)

$$\begin{aligned} S_{\mathrm{mod}}(g) = \overline{\mathrm{T}}\mathrm{exp}\left(-\mathrm{i}e\int \mathrm{d}^4x\; g(x)\mathcal{L}_{\mathrm{out}}(x)\right) \\ \times\, \mathrm{Texp}\left(\mathrm{i}e\int \mathrm{d}^4x\; g(x)\mathcal{L}(x)\right) \times \overline{\mathrm{T}}\mathrm{exp}\left(-\mathrm{i}e\int \mathrm{d}^4x\; g(x)\mathcal{L}_{\mathrm{in}}(x)\right) \in L(\mathcal{D})[\![e]\!] \end{aligned} \tag{26}$$

- Separation of IR and UV problem. UV problem only in defining the Bogolibov S-matrix.
- Dollard modifiers have to be defined in such a way that:
  (1) they are well-defined as elements of $L(\mathcal{D})[\![e]\!]$ for any $g \in \mathcal{S}(\mathbb{R}^4)$,
  (2) **adiabatic limit of $S_{\mathrm{mod}}(g)$ exists**.

Asymptotic interaction in QED

The standard interaction vertex:

$$\mathcal{L}(x) = J^\mu(x)\, A_\mu(x), \quad J^\mu(x) = :\overline{\psi}\gamma^\mu\psi(x):, \tag{27}$$

where

$$A_\mu(x) = \int \mathrm{d}\mu_0(k) \left(a^*_\mu(k) \exp(\mathrm{i}k\cdot x) + a_\mu(k)\exp(-\mathrm{i}k\cdot x)\right), \tag{28}$$

$$\psi_a(x) = \sum_{\sigma=1,2} \int \mathrm{d}\mu_m(p) \left(b^*(\sigma,p)u_a(\sigma,p)\exp(\mathrm{i}p\cdot x) + d(\sigma,p)v_a(\sigma,p)\exp(-\mathrm{i}p\cdot x)\right). \tag{29}$$

Asymptotic interaction vertices [Kulish, Faddeev (1970)]

$$\mathcal{L}_{\mathrm{out/in}}(\eta;x) = J^\mu_{\mathrm{out/in}}(\eta;x)\, A_\mu(x), \tag{30}$$

where the asymptotic currents $J^\mu_{\mathrm{out/in}}(\eta;x)$ are given by

$$J^\mu_{\mathrm{out/in}}(\eta;x) = \int \mathrm{d}\mu_m(p)\, j^\mu_{\mathrm{out/in}}(\eta,p;x)\, \rho(p). \tag{31}$$

Charge density in momentum space: $\rho(p) = \sum_{\sigma=1,2}(b^*(p,\sigma)b(p,\sigma) - d^*(p,\sigma)d(p,\sigma))$.

Out/In part of the current of a point particle: $j^\mu_{\mathrm{out/in}}(p;x) = \frac{p^\mu}{m}\int_{\mathbb{R}} \mathrm{d}\tau\, \theta(\pm\tau)\, \delta\left(x - \tau\frac{p}{m}\right)$.

Charge distribution: $\eta \in \mathcal{S}(\mathbb{R}^4)$, $\int \mathrm{d}^4x\, \eta(x) = 1$.

UV-regularized current: $j^\mu_{\mathrm{out/in}}(\eta,p;x)$ is the convolution of $j^\mu_{\mathrm{out/in}}(p;x)$ with η.

Asymptotic interaction in QED

- Let v be future directed unit time-like four-vector and Ψ be a state in the Fock space whose wave functions belong to the Schwartz space. It holds

$$\lim_{\lambda\to+\infty} \lambda^3 \left(\Psi|J^\mu(\lambda v)\Psi\right) = \lim_{\lambda\to+\infty} \lambda^3 (\Psi|J^\mu_{\text{out}}(\eta;\lambda v)\Psi), \tag{32}$$

$$\lim_{\lambda\to-\infty} \lambda^3 \left(\Psi|J^\mu(\lambda v)\Psi\right) = \lim_{\lambda\to-\infty} \lambda^3 (\Psi|J^\mu_{\text{in}}(\eta;\lambda v)\Psi). \tag{33}$$

- The Dollard modifiers $S^{\text{as}}_{\text{out/in}}(\eta, g)$ are given by

$$\overline{\mathrm{T}}\exp\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x) J^\mu_{\text{out/in}}(\eta;x) A_\mu(x)\right) = \exp\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\, J^\mu_{\text{out/in}}(\eta;x) A_\mu(x)\right)$$
$$\times \exp\left(\mathrm{i}\frac{e^2}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, g(x)g(y)\, g_{\mu\nu} D_0^D(x-y)\, {:}J^\mu_{\text{out/in}}(\eta;x) J^\nu_{\text{out/in}}(\eta;y){:}\right) \in L(\mathcal{D})[\![e]\!].$$

- The first factor is responsible for the generation of clouds of photons which always surround electrons/positrons.
- The second factor is the relativistic Coulomb phase.
- Because in general the asymptotic outgoing and incoming currents are not conserved (*if the total charge is nonzero*) the above expression is **not formally gauge invariant**.

Dollard modifiers and modified S-matrix with adiabatic cutoff

Definition of modified S-matrix with adiabatic cutoff

$$S_{\mathrm{mod}}(\eta,\mathrm{v},g) = R(\eta,\mathrm{v},g)\, S^{\mathrm{as}}_{\mathrm{out}}(\eta,g)\, S(g)\, S^{\mathrm{as}}_{\mathrm{in}}(\eta,g)\, R(\eta,\mathrm{v},g)^{-1} \in L(\mathcal{D})[\![e]\!] \tag{34}$$

where $S(g)$ is the Bogoliubov S-matrix, $S^{\mathrm{as}}_{\mathrm{out/in}}(\eta,g)$ are the Dollard modifiers and

$$R(\eta,\mathrm{v},g) = \exp\left(\mathrm{i}e\int \mathrm{d}^4x\, g(x)\, Q j^{\mu}_{\mathrm{in}}(\eta, m\mathrm{v};x) A_{\mu}(x)\right). \tag{35}$$

The modified S-matrix with adiabatic cutoff $S_{\mathrm{mod}}(\eta,\mathrm{v},g)$

- is well defined as an element of $L(\mathcal{D})[\![e]\!]$,
- is **formally gauge invariant**,
- depends on:
  - v – unit future-directed timelike four-vector – determines charge sector,
  - $\eta \in \mathcal{S}(\mathbb{R}^4)$ – charge distribution – determines cloud of photons in state $b^*(p,\sigma)\Omega$,
  - $g \in \mathcal{S}(\mathbb{R}^4)$ – switching function – infrared regularization.

Modified S matrix in QED

Domain in Fock space

$$\mathcal{D} = \mathrm{span}_{\mathbb{C}}\Big\{\int \mathrm{d}\mu_m(p_1)\ldots\mathrm{d}\mu_m(p_n)\mathrm{d}\mu_0(k_1)\ldots\mathrm{d}\mu_0(k_m)$$
$$h(\vec{p}_1,\ldots,\vec{p}_n,k_1^0,\hat{k}_1,\ldots,k_m^0,\hat{k}_m)\, b^*(p_1)\ldots b^*(p_n)a^*(k_1)\ldots a^*(k_m)\Omega\Big\}, \quad (36)$$

where $h \in \mathcal{S}(\mathbb{R}^{3n}\times(\mathbb{R}\times S^2)^m)$ and h vanishes if the momenta of charged particles are close to each other.

Conjecture: existence of adiabatic limit of modified S-matrix in QED

There exists a renormalization scheme such that for all $\Psi,\Psi' \in \mathcal{D}$, all $\eta \in \mathcal{S}(\mathbb{R}^4)$, such that $\int \mathrm{d}^4x\,\eta(x) = 1$, and all four-velocities v the limit

$$(\Psi|S_{\mathrm{mod}}(\eta,\mathrm{v})\Psi') = \lim_{\epsilon\searrow 0}(\Psi|S_{\mathrm{mod}}(\eta,\mathrm{v},g_\epsilon)\Psi') \quad (37)$$

exists in each order of perturbation theory and defines the physical S-matrix $S_{\mathrm{mod}}(\eta,\mathrm{v}) \in L(\mathcal{D},\mathcal{D}^{\#})[\![e]\!]$. It holds

$$S_{\mathrm{mod}}(\eta',\mathrm{v}) = V(\eta',\eta,\mathrm{v})S_{\mathrm{mod}}(\eta,\mathrm{v})V(\eta,\eta',\mathrm{v}), \quad (38)$$

$$[Q_{\mathrm{BRST}}, S_{\mathrm{mod}}(\eta,\mathrm{v})] = 0. \quad (39)$$

There is explicit formula for the intertwiners $V(\eta',\eta,\mathrm{v})$.
In sectors with zero total charge $S_{\mathrm{mod}}(\eta,\mathrm{v})$ is v-independent.

Conjecture holds true in the first and the second order of perturbation theory.

Modified interacting fields in QED

Modified retarded interacting fields with adiabatic cutoff

$$B_{\mathrm{ret,mod}}(\eta,\mathrm{v},g;h) = R(\eta,\mathrm{v},g)\, S_{\mathrm{in}}^{\mathrm{as}}(\eta,g)^{-1}\, B_{\mathrm{ret}}(g;h)\, S_{\mathrm{in}}^{\mathrm{as}}(\eta,g)\, R(\eta,\mathrm{v},g)^{-1} \tag{40}$$

where B is a polynomial in the basic fields and their derivatives, $h \in \mathcal{S}(\mathbb{R}^4)$, $B_{\mathrm{ret}}(g;h)$ is the Bogoliubov retarded field and $S_{\mathrm{in}}^{\mathrm{as}}(\eta,\mathrm{v},g)$ is the incoming Dollard modifier.

Conjecture: existence of adiabatic limit of modified interacting fields in QED

There exists a renormalization scheme such that for all $\Psi,\Psi' \in \mathcal{D}$, all $\eta \in \mathcal{S}(\mathbb{R}^4)$, such that $\int \mathrm{d}^4x\, \eta(x) = 1$, and all four-velocities v the limit

$$(\Psi|B_{\mathrm{ret,mod}}(\eta,\mathrm{v};h)\Psi') = \lim_{\epsilon\searrow 0}\, (\Psi|B_{\mathrm{ret,mod}}(\eta,\mathrm{v},g_\epsilon;h)\Psi') \tag{41}$$

exists in each order of the perturbation theory and defines the interacting retarded field $B_{\mathrm{ret,mod}}(\eta,\mathrm{v};h) \in L(\mathcal{D},\mathcal{D}^{\#})[\![e]\!]$. It holds

$$B_{\mathrm{ret,mod}}(\eta',\mathrm{v};h) = V(\eta',\eta,\mathrm{v})B_{\mathrm{ret,mod}}(\eta,\mathrm{v};h)V(\eta,\eta',\mathrm{v}). \tag{42}$$

Moreover, if $B_{\mathrm{ret}}(\cdot;h)$ is in the kernel of the interacting BRST differential, then

$$[Q_{\mathrm{BRST}}, B_{\mathrm{ret,mod}}(\eta,\mathrm{v};h)] = 0. \tag{43}$$

Conjecture holds true for $A^{\mu}_{\mathrm{ret,mod}}$, $F^{\mu\nu}_{\mathrm{ret,mod}}$, $\psi_{\mathrm{ret,mod}}$, $\overline{\psi}_{\mathrm{ret,mod}}$, $J^{\mu}_{\mathrm{ret,mod}}$ in the first order of perturbation theory.

Physical interpretation

- **Non-zero asymptotic flux of the electric field** in sectors with nonzero electric charge.

$$F^{\mu\nu}_{\mathrm{ret,mod}}(x) \sim eQ \frac{x^\mu \mathrm{v}^\nu - x^\nu \mathrm{v}^\mu}{((x\cdot \mathrm{v})^2 - x^2)^{3/2}} + O(e^2) \quad \text{as} \quad |\vec{x}| \to \infty \tag{44}$$

  The long-range tail of $F^{\mu\nu}_{\mathrm{ret,mod}}$ coincides with the electromagnetic field of a particle of charge eQ moving with the four-velocity $\mathrm{v} \;\Rightarrow\; \mathrm{v}$ **determines the sector**.

- **LSZ limit** of the electromagnetic field

$$F^{\mathrm{LSZ}}_{\mu\nu}(x) = 2\mathrm{i} \int \mathrm{d}\mu_0(k) \left(k_{[\mu} a^*_{\nu]}(\eta, \mathrm{v}, k) \exp(\mathrm{i}k\cdot x) - \mathrm{h.c.} \right) + O(e^2), \tag{45}$$

  where $a^\#_\mu(\eta, \mathrm{v}, k) = a^\#_\mu(k) - eJ_\mu(\eta, \mathrm{v}, k)$.

  Operators $\varepsilon^\mu(k,s)\, a^\#_\mu(\eta, \mathrm{v}, k)$, $s = 1, 2$ are responsible for **creation and annihilation of physical photons** (up to possible higher order corrections).

  States $b^*(p,\sigma)\Omega$ contain irremovable clouds of photons.

- Modified S-matrix and modified retarded fields are **covariant** with respect to the following **representation of the translation group** $U_{\mathrm{mod}}(\eta, \mathrm{v}; a) = V(\eta, \eta_a, \mathrm{v})U(a)$ which is not unitarily equivalent to the standard Fock representation.

- Joint **spectrum of the energy-momentum operators** contains
  - a unique vacuum state Ω,
  - one-particle massless states $\varepsilon^\mu(k,s)\, a^*_\mu(\eta, \mathrm{v}, k)\Omega$
  - but no one-particle massive states $\Rightarrow$ **electrons/positrons are infraparticles**.

Summary

Method of adiabatic switching of the interaction can be used to construct perturbatively physically relevant objects in QFT:

- Construction of the Wightman and Green functions in all models with interaction vertices of dimension 4. pAQFT framework: Definition of the vacuum state.
- Construction of the scattering operator and the interacting fields in models with only massive fields.
- Definition of the matrix elements of the modified scattering operator and modified interacting fields in QED. *Proof of the existence of the adiabatic limit in low orders of the perturbation theory.*