Crossing Numbers and Stress of Random Graphs

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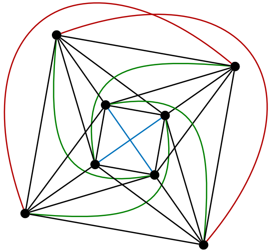
Crossing Numbers

Crossing Number

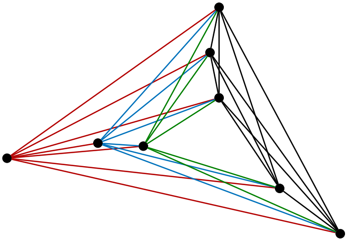
$\mathrm{cr}(G)$



$\mathrm{cr}(K_8) = 18$

Rectilinear Crossing Number

$\overline{\mathrm{cr}}(G)$



$\overline{\mathrm{cr}}(K_8) = 19$

Observation. $\mathrm{cr}(G) \leq \overline{\mathrm{cr}}(G)$

Crossing Number Approximations

There is no PTAS [Cabello 13]

- Is there any constant-factor appriximation for general graphs?
- ...what if the graph has bounded maximum degree Δ?

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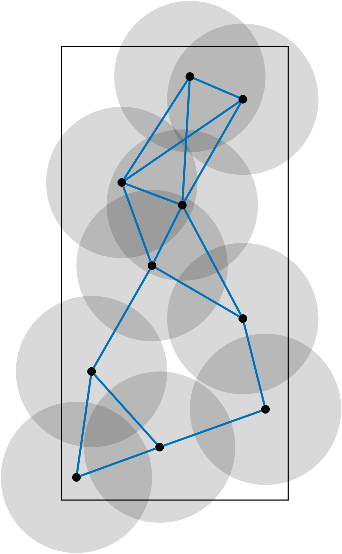
Known approximations:

graph class	bounded Δ	ratio
general	✓	$\mathcal{O}(n^{9/10} \cdot \text{polylog } n)$
$m = \Theta(n^2)$	–	$\mathcal{O}(1)$
bounded **genus**	✓	$\mathcal{O}(1)$
bounded number of graph elements away from planarity	✓	$\mathcal{O}(1)$
bounded **pathwidth**	–	$\mathcal{O}(1)$

Random Geometric Graphs

Geometric Graph (unit-disc/-ball graph):

Given: points $V \subset \mathbb{R}^d$, threshold δ. Connect points iff their Euclidean distance is at most δ.



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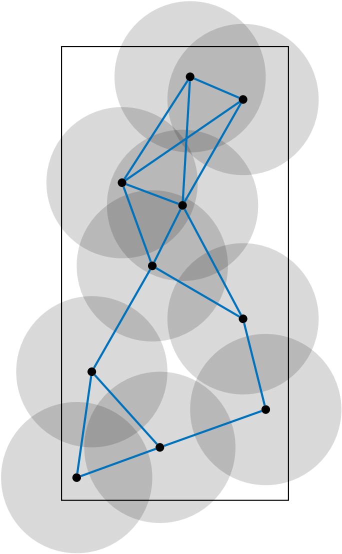

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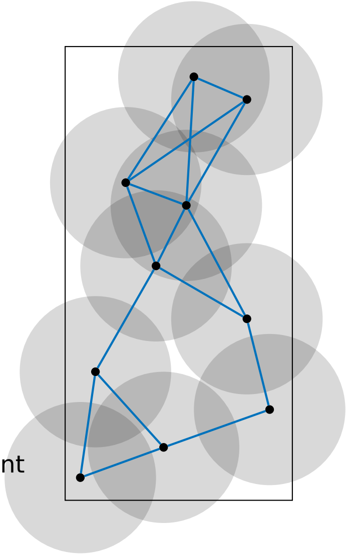
More formally:

- Convex set $W \subset \mathbb{R}^d$ with $\text{vol}_d(W) = 1$
- Poisson process of intensity $t \rightarrow \mathbb{E}n = t$.
- Choose n points $V \subset W$ independently according to uniform distribution.
- Simpler: W is a ball $\rightarrow$ V is rotation invariant
- Consider threshold δ dependent on $t \rightarrow \delta_t$



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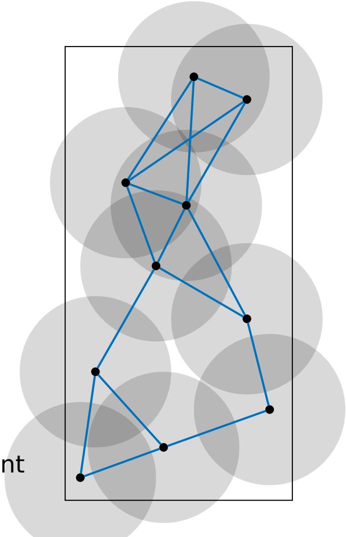

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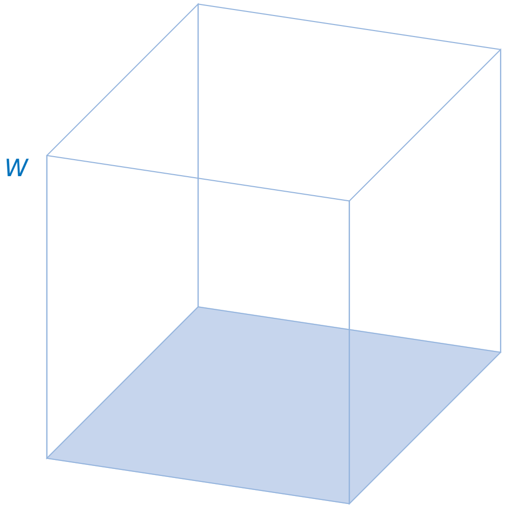
Remark: Poisson simplifies formulae. We can de-Poissonize: simply pick n uniform random independent points in W.

Projection Algorithm



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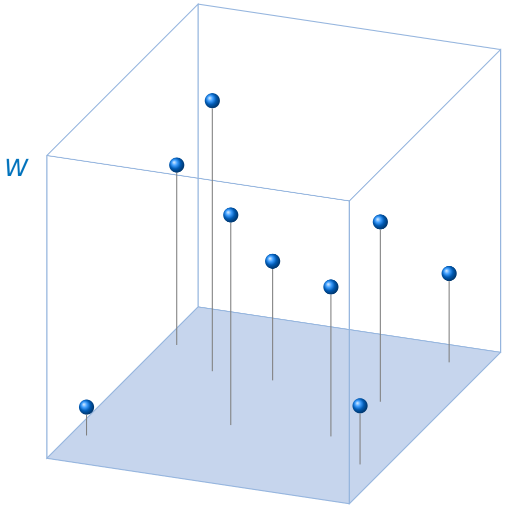

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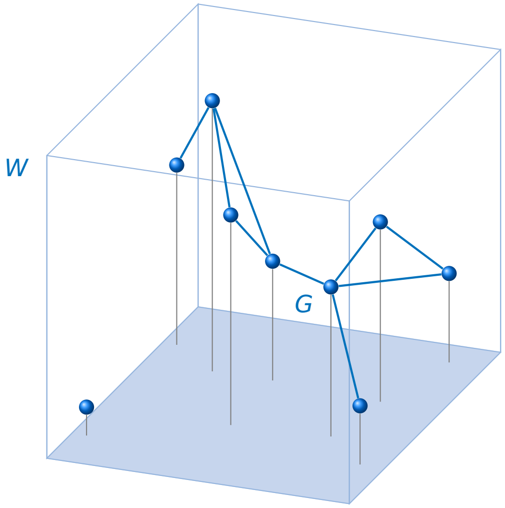

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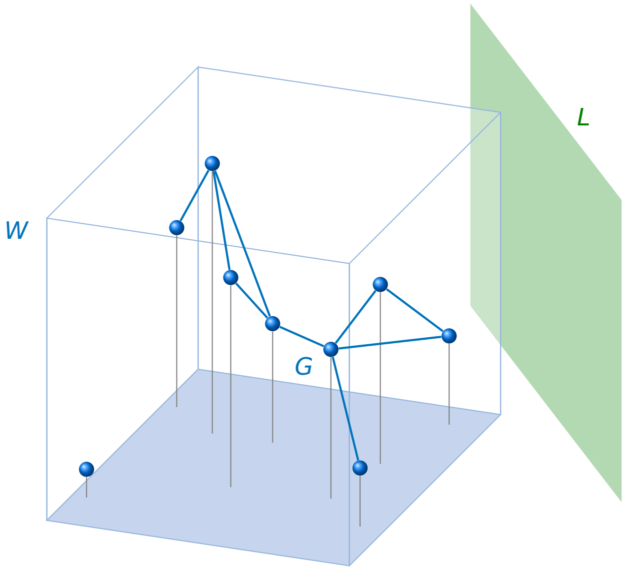

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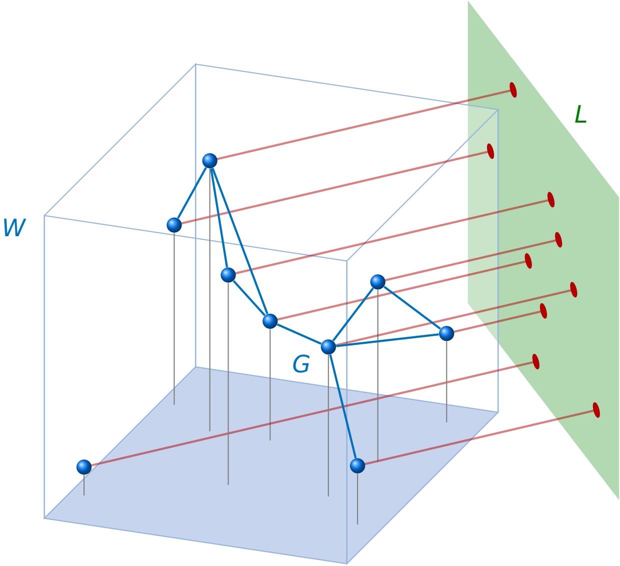

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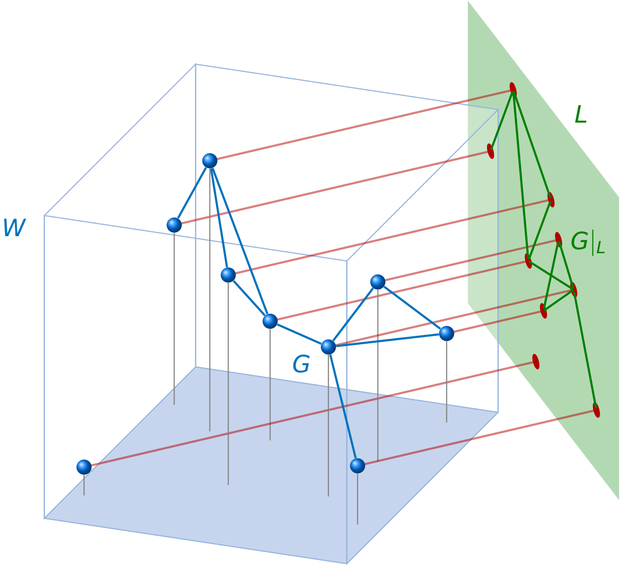

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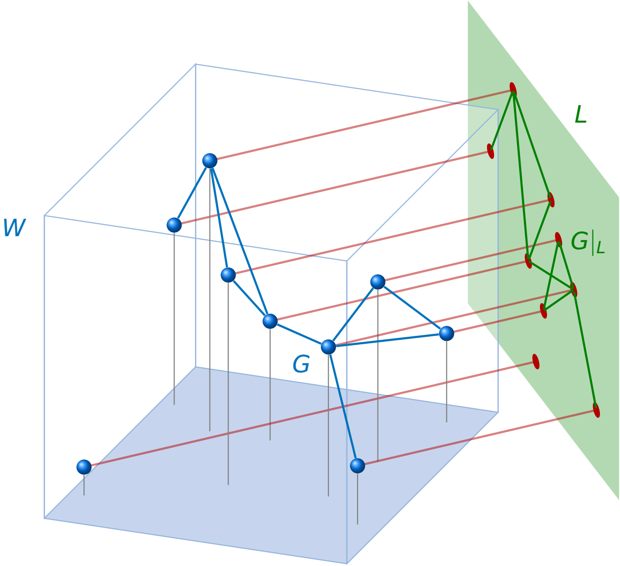

G_0 = abstract graph of G: Relation between $\overline{\text{cr}}(G_0)$ and $\overline{\text{cr}}(G|_L)$?

Stochastic Tools: U-Statistics

measurable, non-negative, real-valued, independent of other points

U-statistic = measure $U(k, f) := \sum_{\mathbf{v} \in V^k_{\neq}} f(\mathbf{v})$

k-tuple of pairwise distinct points

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- of order $k = 2$: Number of edges in G

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- of order $k=4$: Number of crossings in G after projecting onto L

line segment after projection on L

$$\overline{\text{cr}}(G|_L) = \sum_{(v_1,v_2,v_3,v_4)\in V^4_{\neq}} \mathbb{1}([v_1,v_2]|_L \cap [v_3,v_4]|_L \neq \emptyset)/8$$

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Expectation of f: Multivariate Slivnyak-Mecke

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Variance of f: Malliavin calculus for Poisson point processes

Wiener-Itô chaos expansion, assuming f is L^2-integrable

Poincaré inequality

$$t \int_W (\mathbb{E}_V D_v f(V))^2\, dv \quad \leq \quad \mathbb{V}\mathrm{ar}_V\, f(V) \quad \leq \quad t \int_W \mathbb{E}_V (D_v f(V))^2\, dv.$$

where $D_v f(V) := f(V \cup \{v\}) - f(V)$ is an operator measuring the difference when adding a point.

after some calculations... Stochastic Results

Let G_0 be the abstract graph (=no coordinates) of G.
For **any** projection plane L we have:

density

$$\operatorname{cr}(G_0) \le \overline{\operatorname{cr}}(G_0) \le \mathbb{E}_V\, \overline{\operatorname{cr}}(G|_L) = \Theta\left(\frac{m^3}{n^2} \cdot \left(\frac{m}{n^2}\right)^{\frac{2-d}{d}}\right)$$

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- A random geometric graph G in $\mathbb{R}^2$ **is** an **expected constant-factor** approximation for $\mathrm{cr}(G_0)$ and $\overline{\mathrm{cr}}(G_0)$.
- Let d and density m/n^2 fixed. Picking **any** projection plane L for a random geometric graph in $\mathbb{R}^d$ yields an **expected constant-factor** approximation for $\mathrm{cr}(G_0)$ and $\overline{\mathrm{cr}}(G_0)$.

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Details in the paper...

- $\mathbb{V}\text{ar} \ll \mathbb{E}$ by several orders + Law-of-Large-Numbers $\rightarrow$ observed crossing number will be very close to $\mathbb{E}$ w.h.p.

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- We pick L not arbitrary but **random** (uniform): compute $\mathbb{E}_{L,V}\, \overline{\text{cr}}(G|_L)$, $\mathbb{V}\text{ar}_{L,V}\, \overline{\text{cr}}(G|_L)$, LLN
- This is „simpler“ if W is rotation invariant: $\mathbb{V}\text{ar}_L\, \mathbb{E}_V\, \overline{\text{cr}}(G|_L) = 0 \rightarrow$ all L have same chance of being „good“

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- The probability of finding „optimum“ L is only in $\mathcal{O}(t^{-1})$...
  expensive! $\rightarrow$ How to find a good L?

Stress

often $\frac{1}{d_0(v_1,v_2)^2}$ — distance in drawing

$$\text{stress}(G) := \sum_{\substack{v_1,v_2 \in V(G),\\ v_1 \neq v_2}} w(v_1,v_2) \cdot \big(d_0(v_1,v_2) - d_1(v_1,v_2) \big)^2$$

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- $w(v_1, v_2)$: often $\frac{1}{d_0(v_1, v_2)^2}$
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If stress and crossing number positively correlated
$\rightarrow$ MDS yields crossing number approximations?!

Not really (graph-theoretic != our geometric distances), but close.

Stress vs. Crossings

Details in the paper...

- Stress is a **U-statistic**!
- Project a random geometric graph G onto L
  $\rightarrow$ Consider stress w.r.t. $\mathbb{R}^d$-distances as desired distances
  $\rightarrow$ we compute $\mathbb{E}$, $\mathbb{V}\mathrm{ar}$, LLN

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  $\rightarrow$ we compute $\mathbb{E}$, $\mathbb{V}\mathrm{ar}$, LLN
- **Furthermore** we show:
  strictly **positive correlation** between $\mathbb{E}_V\,\overline{\mathrm{cr}}$ and $\mathbb{E}_V\mathrm{stress}$

Yes, in some sense a stress-minim**um** drawing **is** a crossing number approximation!

Wrapping up...

Summary. For a **random geometric graph**,...

1. ...a trivial projection yields an **expected** crossing number approximation **with high probability**.
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- Can we achieve 1 without the disclaimer, i.e., randomized approximation for **any** random geometric input graph?
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Thank you for your attention!