The Heat Flow on Metric Random Walk Spaces

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(i) Ergodicity

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(ii) Fuctional Inequalities and Curvature
Poincare Inequality and its relation with the Isoperimetrical Inequality and Bakry-Emery Curvature
Log-Sobolev Inequality and Its relation with concentration of measures

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(iii) Transport Inequalities and its relation with Bakry-Emery and Ollivier-Ricci Curvature

Metric random walk spaces

Let (X, d) be a Polish metric space equipped with its Borel σ-algebra.

A random walk m on X is a family of probability measures m_x on X for each $x \in X$ satisfying
(i) the measures m_x depend measurably on the point $x \in X$,
(ii) each measure m_x has finite first moment, i.e. for some (hence any) $z \in X$, and for any $x \in X$ one has $\int_X d(z,y)dm_x(y) < +\infty$.

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Definition

A metric random walk space $[X, d, m]$ is a Polish metric space (X, d) equipped with a random walk m.

Let $[X, d, m]$ be a metric random walk space. A Radon measure ν on X is invariant for the random walk $m = (m_x)$ if

$$d\nu(x) = \int_{y \in X} d\nu(y)dm_y(x).$$

Metric random walk spaces

The measure ν is said to be reversible if moreover, the detailed balance condition

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Example

Let $(\mathbb{R}^N, d, \mathcal{L}^N)$, with d the Euclidean distance and $\mathcal{L}^N$ the Lebesgue measure. Let $J : \mathbb{R}^N \to [0, +\infty[$ be a measurable, nonnegative and radially symmetric function verifying $\int_{\mathbb{R}^N} J(z)dz = 1$. In $(\mathbb{R}^N, d, \mathcal{L}^N)$ we can give the following random walk, starting at x,

$$m^J_x(A) := \int_A J(x-y)d\mathcal{L}^N(y) \quad \forall A \subset \mathbb{R}^N \text{ borelian.}$$

Applying Fubini's Theorem it easy to see that the Lebesgue measure $\mathcal{L}^N$ is an invariant and reversible measure for this random walk.

Example

Let $K : X \times X \to \mathbb{R}$ be a Markov kernel on a countable space X, i.e.,

$$K(x,y) \geq 0, \quad \forall x,y \in X, \qquad \sum_{y \in X} K(x,y) = 1 \quad \forall x \in X.$$

Then, for

$$m_x^K(A) := \sum_{y \in A} K(x,y),$$

$[X, d, m^K]$ is a metric random walk for any metric d on X. Basic Markov chain theory guarantees the existence of a unique stationary probability measure (also called steady state) π on X, that is,

$$\sum_{x \in X} \pi(x) = 1 \quad \text{and} \quad \pi(y) = \sum_{x \in X} \pi(x) K(x,y) \qquad \forall y \in X.$$

We say that π is reversible for K if the following detailed balance equation

$$K(x,y)\pi(x) = K(y,x)\pi(y)$$

Example

A weighted discrete graph $G = (V(G), E(G))$ is a graph of vertices $V(G)$ and edges $E(G)$ such that each edge $(x, y) \in E(G)$ (we will write $x \sim y$ if $(x, y) \in E(G)$) is assigned a positive weight $w_{xy} = w_{yx}$. We consider that $w_{xy} = 0$ if $(x, y) \notin E(G)$.

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A finite sequence $\{x_k\}_{k=0}^n$ of vertices on a graph is called a path if $x_k \sim x_{k+1}$ for all $k = 0, 1, ..., n-1$. The length of a path is defined as the number, n, of edges in the path.

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A graph $G = (V(G), E(G))$ is called connected if, for any two vertices $x, y \in V$, there is a path connecting x and y, that is, a path $\{x_k\}_{k=0}^{n}$ such that $x_0 = x$ and $x_n = y$.

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If $G = (V(G), E(G))$ is connected then define the graph distance $d_G(x, y)$ between any two distinct vertices x, y as the minimum of the lengths of the paths connecting x and y.

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For each $x \in V(G)$ we define the following probability measure

$$m_x^G = \frac{1}{d_x} \sum_{y \sim x} w_{xy} \, \delta_y \quad \text{with} \quad d_x := \sum_{y \sim x} w_{xy} \, .$$

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If $G = (V(G), E(G))$ is a locally finite weighted connected graph, we have that $[V(G), d_G, (m_x^G)]$ is a metric random walk space.
Furthermore, the measure ν_G defined as

$$\nu_G(A) := \sum_{x \in A} d_x, \quad A \subset V(G)$$

is an invariant and reversible measure for this random walk.

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Assume that balls in X have finite measure and that $\text{Supp}(\mu) = X$. Given $\epsilon > 0$, the ϵ-step random walk on X, starting at point x, consists in randomly jumping in the ball of radius ϵ around x, with probability proportional to μ; namely

$$m_x^{\mu,\epsilon} := \frac{\mu \llcorner B(x,\epsilon)}{\mu(B(x,\epsilon))}.$$

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Note that μ is an invariant and reversible measure for the metric random walk $[X, d, m^{\mu,\epsilon}]$.

Given a metric random walk space $[X, d, m]$ with invariant and reversible measure ν for m, and given a ν-measurable set $\Omega \subset X$ with $\nu(\Omega) > 0$, if we define, for $x \in \Omega$,

$$m_x^{\Omega}(A) := \int_A dm_x(y) + \left(\int_{X \setminus \Omega} dm_x(y) \right) \delta_x(A) \quad \forall A \subset \Omega \text{ borelian},$$

we have that $[\Omega, d, m^{\Omega}]$ is a metric random walk space and it easy to see that $\nu \llcorner \Omega$ is reversible for m^{Ω}.

Metric random walk spaces

Given a metric random walk space $[X, d, m]$, geometers will think of m_x as a replacement for the notion of ball around x, and probabilists will rather think of this data as defining a Markov chain whose transition probability from x to y in n steps is

$$dm_x^{*n}(y) := \int_{z \in X} dm_z(y) dm_x^{*(n-1)}(z) \tag{1}$$

where $m_x^{*1} = m_x$.

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where $m_x^{*1} = m_x$.

We have

$$\int_{y \in X} dm_x^{*n}(y) = \int_{z \in X} \left(\int_{y \in X} dm_z(y) \right) dm_x^{*(n-1)}(z) = \int_{z \in X} dm_x^{*(n-1)}(z) = 1.$$

Hence, $[X, d, m^{*n}]$ is also a metric random walk space. Moreover, if ν is invariant and reversible for m, then also ν is invariant and reversible for m^{*n}.

Ollivier-Ricci curvature

In Riemannian geometry, positive Ricci curvature is characterized by the fact that "small balls are closer, in the 1-Wasserstein distance, than their centers are". In the framework of metric random walk spaces, inspired by this, Y. Ollivier [Y. Ollivier, J. Funct. Anal. (2009)] introduces the concept of coarse Ricci curvature changing the ball by the measures m_x

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Let (X, d) a Polish metric space and $\mathcal{M}^+(X)$ the positive Radon measures on X. Fix $\mu, \nu \in \mathcal{M}^+(X)$ satisfying $\mu(X) = \nu(X)$. The Monge-Kantorovich problem is the minimization problem

$$\min \left\{ \int_{X \times X} d(x,y)\, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\},$$

where $\Pi(\mu,\nu) := \{\text{Radon measures } \gamma \text{ in } X \times X : \pi_0 \# \gamma = \mu, \pi_1 \# \gamma = \nu\}$, with $\pi_t(x,y) := x + t(y-x)$. The elements $\gamma \in \Pi(\mu,\nu)$ are called transport plans between μ and ν, and a minimizer γ^* an optimal transport plan.

Ollivier-Ricci curvature

For $1 \leq p < \infty$, the p-Wasserstein distance between μ, ν is defined as

$$W_p^d(\mu,\nu) := \left(\min \left\{ \int_{X\times X} d(x,y)^p \, d\gamma(x,y) \, : \, \gamma \in \Pi(\mu,\nu) \right\} \right)^{\frac{1}{p}}.$$

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Definition

On a given metric random walk space $[X, d, m]$, for any two distinct points $x, y \in X$, the Ollivier-Ricci curvature of $[X, d, m]$ along (x, y) is defined as

$$\kappa_m(x,y) := 1 - \frac{W_1^d(m_x, m_y)}{d(x,y)},$$

where

$$W_1^d(m_x, m_y) = \min \left\{ \int_{X \times X} d(u,v) \, d\gamma(u,v) \, : \, \gamma \in \Pi(m_x, m_y) \right\}.$$

The Ollivier-Ricci curvature of $[X, d, m]$ is defined by

$$\kappa_m := \inf_{\substack{x,\, y \,\in\, X \\ x \neq y}} \kappa_m(x,y).$$

The heat flow

Let $[X, d, m]$ be a metric random walk space with invariant measure ν for m. For a function $u : X \to \mathbb{R}$ we define its nonlocal gradient $\nabla u : X \times X \to \mathbb{R}$ as

$$\nabla u(x, y) := u(y) - u(x) \quad \forall\, x, y \in X,$$

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For a function $\mathbf{z} : X \times X \to \mathbb{R}$, its m-divergence $\mathrm{div}_m \mathbf{z} : \mathbb{R}^N \to \mathbb{R}$ is defined as

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The averaging operator on $[X, d, m]$ is defined as

$$M_m f(x) := \int_X f(y) dm_x(y),$$

and the Laplace operator as $\Delta_m := M_m - I$, i.e.,

$$\Delta_m f(x) = \int_X f(y) dm_x(y) - f(x) = \int_X (f(y) - f(x)) dm_x(y).$$

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$$\Delta_m f(x) = \mathrm{div}_m(\nabla f)(x)$$

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If the invariant measure ν is reversible, the following integration by parts formula is straightforward:

$$\int_X f(x)\Delta_m g(x) d\nu(x) = -\frac{1}{2}\int_{X\times X}(f(y)-f(x))(g(y)-g(x))dm_x(y)d\nu(x)$$

for $f,g \in L^2(X,\nu)\cap L^1(X,\nu)$.

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for $f,g \in L^2(X,\nu)\cap L^1(X,\nu)$.

In $L^2(X,\nu)$ we consider the symmetric form given by

$$\mathcal{E}_m(f,g) = -\int_X f(x)\Delta_m g(x) d\nu(x) = \frac{1}{2}\int_{X\times X} \nabla f(x,y)\nabla g(x,y) dm_x(y) d\nu(x),$$

with domain for both variables $D(\mathcal{E}_m) = L^2(X,\nu)\cap L^1(X,\nu)$, which is a linear and dense subspace of $L^2(X,\nu)$.

Theorem

Let $[X, d, m]$ be a metric random walk space with invariant and reversible measure ν for m. Then, $-\Delta_m$ is a non-negative self-adjoint operator in $L^2(X, \nu)$ with associated closed symmetric form $\mathcal{E}_m$, which, moreover, is a Markovian form.

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By the above Theorem, we have that if $(T_t^m)_{t \geq 0}$ is the strongly continuous semigroup associated with $\mathcal{E}_m$, then $(T_t^m)_{t \geq 0}$ is a positivity preserving (i.e., $T_t^m f \geq 0$ if $f \geq 0$) Markovian semigroup (i.e., $0 \leq T_t^m f \leq 1$ ν-a.e. whenever $f \in L^2(X, \nu)$, $0 \leq f \leq 1$ ν-a.e.).

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From now on we denote $e^{t\Delta_m} := T_t^m$ and we call to $\{e^{t\Delta_m} : t \geq 0\}$ the heat flow on the metric random walk space $[X, d, m]$ with invariant and reversible measure ν for m. For every $u_0 \in L^2(X, \nu)$, $u(t) := e^{t\Delta_m} u_0$ is the unique solution of the heat equation

$$\begin{cases} \frac{du}{dt} = \Delta_m u(t) \quad \text{in} \quad (0, +\infty) \times X, \\ u(0) = u_0, \end{cases} \tag{2}$$

The heat flow

in the sense that $u \in C([0,+\infty) : L^2(X,\nu)) \cap C^1((0,+\infty) : L^2(X,\nu))$ and verifies (2), or equivalently,

$$\begin{cases} \dfrac{du}{dt}(t,x) = \displaystyle\int_X (u(t)(y) - u(t)(x)) dm_x(y) \quad \text{in} \quad (0,+\infty) \times X, \\ u(0) = u_0. \end{cases}$$

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Associated with $\mathcal{E}_m$ we define the energy functional $\mathcal{H}_m : L^2(X,\nu) \to [0,+\infty]$ as

$$\mathcal{H}_m(f) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{X\times X} (f(x) - f(y))^2 dm_x(y) d\nu(x) & \text{if } f \in L^2(X,\nu) \cap L^1(X,\nu \\ +\infty, & \text{else.} \end{cases}$$

Note that for $f \in D(\mathcal{H}_m) = L^2(X,\nu) \cap L^1(X,\nu)$, we have

$$\mathcal{H}_m(f) = -\int_X f(x)\Delta_m f(x) d\nu(x).$$

The heat flow

$\partial \mathcal{H}_m = -\Delta_m$. Consequently $-\Delta_m$ is a maximal monotone operator in $L^2(X,\nu)$. Moreover, $-\Delta_m$ is completely accretive operator and then

$$\|e^{t\Delta_m} u_0\|_{L^p(X,\nu)} \leq \|u_0\|_{L^p(X,\nu)} \quad \forall u_0 \in L^p(X,\nu) \cap L^2(X,\nu), \quad 1 \leq p \leq +\infty,$$

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Theorem

Let $[X,d,m]$ be a metric random walk with invariant and reversible measure ν. Let $u_0 \in L^2(X,\nu) \cap L^1(X,\nu)$. Then,

$$e^{t\Delta_m}u_0(x) = e^{-t}\left(u_0(x) + \sum_{n=1}^{\infty} \int_X u_0(y)\,dm_x^{*n}(y)\frac{t^n}{n!}\right)$$

$$= e^{-t}\sum_{n=0}^{\infty} \int_X u_0(y)\,dm_x^{*n}(y)\frac{t^n}{n!},$$

where $\int_X u_0(y)\,dm_x^{*0}(y) = u_0(x)$.

Infinite speed of propagation and ergodicity

The infinite speed of propagation of the heat flow $(e^{t\Delta_m})_{t\geq 0}$, that is:

$$e^{t\Delta_m}u_0 > 0 \quad \text{for all } t > 0 \quad \text{whenever } \ 0 \leq u_0 \in L^2(X,\nu), \ u_0 \not\equiv 0.$$

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Let $[X, d, m]$ be a metric random walk with invariant measure ν. For a ν measurable set D, we set

$$N_D^m = \{x \in X \, : \, m_x^{*n}(D) = 0 \; \forall n \in \mathbb{N}\}.$$

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Definition

A metric random walk space $[X,d,m]$ with invariant measure ν is called random-walk-connected or *r-connected* if for any $D \subset X$ with $0 < \nu(D) < +\infty$ we have that $\nu(N_D^m) = 0$.

Infinite speed of propagation and ergodicity

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$$e^{t\Delta_m}u_0 > 0 \quad \text{for all } t > 0 \quad \text{whenever } \; 0 \leq u_0 \in L^2(X,\nu), \; u_0 \not\equiv 0.$$

Let $[X,d,m]$ be a metric random walk with invariant measure ν. For a ν measurable set D, we set

$$N_D^m = \{x \in X \, : \, m_x^{*n}(D) = 0 \; \forall n \in \mathbb{N}\}.$$

Definition

A metric random walk space $[X,d,m]$ with invariant measure ν is called random-walk-connected or *r-connected* if for any $D \subset X$ with $0 < \nu(D) < +\infty$ we have that $\nu(N_D^m) = 0$.

Theorem

Let $[X,d,m]$ be a metric random walk with invariant and reversible measure ν. The space is r-connected if and only if for any non-null $0 \leq u_0 \in L^2(X,\nu)$, we have $e^{t\Delta_m}u_0 > 0$ ν-a.e., for all $t > 0$.

Infinite speed of propagation and ergodicity

Theorem

Let $[V(G), d_G, (m^G_x)]$ *be the random walk associated whith the locally finite weighted connected graph* $G = (V(G), E(G))$*. Then* $[V(G), d_G, m^G]$ *with* ν_G *is strong* r*-connected, that is,* $N^m_D = \emptyset$*, which is equivalent to*

$$e^{t\Delta_m} u_0(x) > 0 \quad \textit{for all } x \in X, \textit{ and for all } t > 0\,.$$

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Example

Take $([0,1], d)$ with d the euclidean distance and let C the Cantor set. Let μ be the Cantor distribution. We denote $\eta := \mathcal{L}^1 \llcorner [0,1]$ and define the random walk

$$m_x := \begin{cases} \eta & \text{if } \ x \in [0,1] \setminus C \\ \\ \mu & \text{if } \ x \in C \end{cases}$$

Then $\nu = \eta + \mu$ is invariant and reversible.
$m^{*n}_x(C) = 0$ for every $x \in (0,1) \setminus C$ and for every $n \in \mathbb{N}$ so that $\nu(N^m_C \setminus C) \geq \nu((0,1) \setminus C) = 1 > 0$ and therefore the space $[(0,1), d, m]$ is not r-connected. Its Ollivier-Ricci curvature is $\kappa = -\infty$

Example

Let $\Omega = \Big( ]-\infty, 0] \cup [\frac{1}{2}, +\infty[\Big) \times \mathbb{R}^{N-1}$ and consider the metric random walk space $[\Omega, d, m^{J,\Omega}]$, with d the Euclidean distance and $J(x) = \frac{1}{|B_1(0)|}\chi_{B_1(0)}$. It is easy to see that this space with reversible and invariant measure $\nu = \mathcal{L} \llcorner \Omega$ is r-connected but (Ω, d) is not connected. Its Ollivier-Ricci curvature is $\kappa < 0$

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For $\Omega = \left( ]-\infty, 0] \cup [2, +\infty[\right) \times \mathbb{R}^{N-1}$, neither $[\Omega, d, m^{J,\Omega}]$ with $\nu = \mathcal{L}\llcorner\Omega$ is r-connected, nor (Ω, d) is connected. Its Ollivier-Ricci curvature is $\kappa < 0$

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Theorem

Let $[X, d, m]$ be a metric random walk space with invariant measure ν such that $\nu(X) < +\infty$. Assume that the Ollivier-Ricci curvature $\kappa > 0$. Then, $[X, d, m]$ with ν is r-connected

Infinite speed of propagation and ergodicity

Definition

Let $[X, d, m]$ be a metric random walk space with invariant probability measure ν. A Borel set $B \subset X$ is said to be invariant with respect to the random walk m if $m_x(B) = 1$ whenever x is in B.

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Definition

Let $[X, d, m]$ be a metric random walk with invariant measure ν. We say that Δ_m is ergodic if $\Delta_m u = 0$ implies that u is constant (being this constant 0 if ν is not finite).

Infinite speed of propagation and ergodicity

Theorem

Let $[X, d, m]$ be a metric random walk with invariant measure ν such that $\nu(X) < +\infty$. Then,

$$\Delta_m \text{ is ergodic } \Leftrightarrow [X, d, m] \text{ is random walk connected.}$$

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we introduce the *m*-total variation of a function $u : X \to \mathbb{R}$ as

$$TV_m(u) := \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y) d\nu(x).$$

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We define the concept of *m*-perimeter of a ν-measurable subset $E \subset X$ as

$$P_m(E) := TV_m(\chi_E) = \int_E \int_{X \setminus E} dm_x(y) d\nu(x).$$

where the last equality is consequence of the reversibility of ν

Infinite speed of propagation and ergodicity

In the particular case of a graph $[V(G), d_G, m^G]$, the definition of perimeter of a set $E \subset V(G)$ is given by

$$|\partial E| := \sum_{x \in E, y \in V \setminus E} w_{xy}.$$

Then we have that

$$|\partial E| = P_{m^G}(E) \quad \text{for all } E \subset V(G). \tag{3}$$

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Let $[X, d, m]$ be a metric random walk with invariant and reversible measure ν and assume that $\nu(X) < +\infty$. The following facts are equivalent:

1. Δ_m *is ergodic;*
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Functional inequalities and curvature

Let $[X, d, m]$ be a metric random walk space with invariant and reversible measure ν In this section we will assume that ν is a probability.

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We denote the mean value of $f \in L^1(X, \nu)$ (or the expected value of f) by

$$\nu(f) = \mathbb{E}_\nu(f) = \int_X f(x) d\nu(x).$$

And, for $f \in L^2(X, \nu)$, we denote its variance by

$$\mathrm{Var}_\nu(f) := \int_X (f(x) - \nu(f))^2 d\nu(x) = \frac{1}{2} \int_{X \times X} (f(x) - f(y))^2 d\nu(y) d\nu(x).$$

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Definition

The spectral gap of $-\Delta_m$ is defined as

$$\mathrm{gap}(-\Delta_m) := \inf\left\{ \frac{\mathcal{H}_m(f)}{\mathrm{Var}_\nu(f)} \ : \ f \in D(\mathcal{H}_m),\ \mathrm{Var}_\nu(f) \neq 0 \right\}$$

$$= \inf\left\{ \frac{\mathcal{H}_m(f)}{\|f\|_2^2} \ : \ f \in D(\mathcal{H}_m),\ \|f\|_2 \neq 0,\ \int_X f d\nu = 0 \right\}.$$

Definition

We say that (m, ν) satisfies a Poincaré inequality if there exists $\lambda > 0$ such that

$$\lambda \mathrm{Var}_\nu(f) \le \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu),$$

or equivalently,

$$\lambda \|f\|^2_{L^2(X,\nu)} \le \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu) \text{ with } \nu(f) = 0.$$

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Note that when $\mathrm{gap}(-\Delta_m) > 0$, (m,ν) satisfies a Poincaré inequality with $\lambda = \mathrm{gap}(-\Delta_m)$,

$$\mathrm{gap}(-\Delta_m)\mathrm{Var}_\nu(f) \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X,\nu),$$

being the spectral gap the best constant in the Poincaré inequality.

Functional inequalities and curvature

Example

Let $V(G) = \{x_3, x_4, x_5 \ldots, x_n \ldots\}$ be a weighted linear graph with

$$w_{x_{3n},x_{3n+1}} = \frac{1}{n^3},\ \ w_{x_{3n+1},x_{3n+2}} = \frac{1}{n^2},\ \ w_{x_{3n+2},x_{3n+3}} = \frac{1}{n^3}.$$

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Remark

Let $[X, d, m]$ be a metric random walk space with invariant and reversible probability measure ν. Y. Ollivier, under the assumption that

$$\int\int\int d(y,z)^2 dm_x(y) dm_x(z) d\nu(x) < +\infty,$$

proves that if the Ollivier-Ricci curvature $\kappa_m > 0$ and the space is ergodic, then (m, ν) satisfies the Poincaré inequality

$$\kappa_m \mathrm{Var}_\nu(f) \leq \mathcal{H}_m(f) \quad \text{for all } f \in L^2(X, \nu),$$

and, consequently, $\kappa_m \leq \mathrm{gap}(-\Delta_m)$.

Functional inequalities and curvature

Observe that the Poincaré inequality, given only for characteristic functions, implies that there exists $\lambda > 0$ such that

$$\lambda\, \nu(D)\big(1 - \nu(D)\big) \leq P_m(D) \quad \text{for all } \nu\text{−mesasurable sets } D,$$

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In a weighted graph $G = (V(G), E(G))$ the Cheeger constant is defined as

$$h_G := \inf_{D \subset V(G)} \frac{|\partial D|}{\min\{\nu_G(D), \nu_G(V(G) \setminus D)\}}. \tag{5}$$

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The following relation between the Cheeger constant and the first positive eigenvalue $\lambda_1(G)$ of the graph Laplacian Δ_{m^G} is wel-known:

$$\frac{h_G^2}{2} \le \lambda_1(G) \le 2h_G. \tag{6}$$

Functional inequalities and curvature

Let $[X, d, m]$ be a metric random walk space with invariant and reversible probability measure ν. We define its Cheeger constant as

$$h_m(X) := \inf \left\{ \frac{P_m(D)}{\min\{\nu(D), \nu(X \setminus D)\}} \ : \ D \subset X, \ 0 < \nu(D) < 1 \right\},$$

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Recall that, given a function $u : X \to \mathbb{R}$, we say that $\mu \in \mathbb{R}$ is a median of u with respect to a measure ν if

$$\nu(\{x \in X \ : \ u(x) < \mu\}) \leq \frac{1}{2}\nu(X), \quad \nu(\{x \in X \ : \ u(x) > \mu\}) \leq \frac{1}{2}\nu(X).$$

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Theorem

If $[X,d,m]$ is a metric random walk space with invariant and reversible probability measure ν, then

$$h_m(X) = \inf\left\{ TV_m(u) \ : \ \|u\|_1 = 1,\ 0\in \mathrm{med}_\nu(u)\right\}.$$

Functional inequalities and curvature

Theorem

Let $[X, d, m]$ be a metric random walk space with invariant and reversible probability measure ν. The following Cheeger inequality holds

$$\frac{h_m^2}{2} \leq \operatorname{gap}(-\Delta_m) \leq 2h_m.$$

Theorem

Let $[X, d, m]$ be a metric random walk space with invariant and reversible probability measure ν. The following statements are equivalent:

1. (m, ν) *satisfies a Poincaré inequality,*
2. $\operatorname{gap}(-\Delta_m) > 0$,
3. (m, ν) *satisfies an isoperimetric inequality,*
4. $h_m(X) > 0$.

Functional inequalities and curvature

Theorem

The following statements are equivalent:

(i) *There exists* $\lambda > 0$ *such that*

$$\lambda \mathrm{Var}_\nu(f) \leq \mathcal{H}_m(f) \quad \textit{for all } f \in L^2(X,\nu).$$

(ii) *For every* $f \in L^2(X,\nu)$

$$\|e^{t\Delta_m} f - \nu(f)\|_{L^2(X,\nu)} \leq e^{-\lambda t}\|f - \nu(f)\|_{L^2(X,\nu)} \quad \textit{for all } t \geq 0.$$

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Theorem

Let $[X,d,m]$ *be a metric random walk space with invariant-reversible probability measure* ν*. Assume that* Δ_m *is ergodic. Then*

$$\mathrm{gap}(-\Delta_m) = \sup\left\{ \lambda \geq 0 \, : \, \lambda \mathcal{H}_m(f) \leq \int_X (-\Delta_m f)^2 d\nu \ \ \forall f \in L^2(X,\nu) \right\}. \tag{7}$$

Functional inequalities and curvature

Let $[X, d, m]$ be a metric random walk space with invariant-reversible probability measure ν. For $\mu \ll \nu$ with $\frac{d\mu}{d\nu} = f$, we will write $\mu = f\nu$. Let $0 \leq \mu \in \mathcal{M}(X)$, $\mu \ll \nu$, we define the relative entropy of μ with respect to ν by

$$\mathrm{Ent}_\nu(\mu) := \begin{cases} \displaystyle\int_X f \log f d\nu - \nu(f) \log\left(\nu(f)\right) & \text{if } \mu = f\nu, \ f \geq 0, \ f \log f \in L^1(X, \nu), \\ +\infty, & \text{otherwise,} \end{cases}$$

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For $0 \leq u_0 \in L^2(X,\nu)$ let $u(t) = e^{t\Delta_m} u_0$. Then, , we have

$$\frac{d}{dt}\mathrm{Ent}_\nu(u(t)) = \int_X \Delta_m u(t)(\log u(t) + 1) d\nu = \int_X \Delta_m u(t) \log u(t) d\nu.$$

Hence,

$$\frac{d}{dt}\mathrm{Ent}_\nu(u(t)) = -\mathcal{E}_m(u(t), \log u(t)).$$

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$$\mathcal{F}_m(f) = -\mathcal{E}_m(f, \log f) = -\int_X \log f \Delta_m f d\nu.$$

Functional inequalities and curvature

We have that the time-derivative of the entropy along the heat flow verifies

$$\frac{d}{dt}\mathrm{Ent}_\nu(e^{t\Delta_m}u_0) = -\mathcal{F}_m(e^{t\Delta_m}u_0). \tag{8}$$

We call $\mathcal{F}_m$ the modified Fisher information, which, due to (8), corresponds to the entropy production funtional of the heat flow $(e^{t\Delta_m})_{t\geq 0}$.

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The Fisher-Donsker-Varadhan information of a probability measure μ on X with respect to ν is defined by

$$I_\nu(\mu) := \begin{cases} 2\mathcal{H}_m(\sqrt{f}) & \text{if } \mu = f\nu, \ f \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

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We have that the time-derivative of the entropy along the heat flow verifies

$$\frac{d}{dt}\mathrm{Ent}_\nu(e^{t\Delta_m}u_0) = -\mathcal{F}_m(e^{t\Delta_m}u_0). \tag{8}$$

We call $\mathcal{F}_m$ the modified Fisher information, which, due to (8), corresponds to the entropy production funtional of the heat flow $(e^{t\Delta_m})_{t\geq 0}$.

The Fisher-Donsker-Varadhan information of a probability measure μ on X with respect to ν is defined by

$$I_\nu(\mu) := \begin{cases} 2\mathcal{H}_m(\sqrt{f}) & \text{if } \mu = f\nu,\ f \geq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

In the continuous setting,we have that $I_\nu(f\nu) = \mathcal{F}_m(f)$,

Definition

We say that (m, ν) satisfies a logarithmic-Sobolev inequality if there exists $\lambda > 0$ such that

$$\lambda \mathrm{Ent}_\nu(f) \leq \mathcal{H}_m(\sqrt{f}) \quad \text{for all } f \in L^1(X,\nu)^+, \tag{9}$$

or, equivalently,

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We denote

$$\mathrm{LS}(m, \nu) := \inf \left\{ \frac{\mathcal{H}_m(\sqrt{f})}{\mathrm{Ent}_\nu(f)} \ : \ 0 \neq \mathrm{Ent}_\nu(f) < +\infty \right\}$$

$$= \inf \left\{ \frac{\mathcal{H}_m(f)}{\mathrm{Ent}_\nu(f^2)} \ : \ 0 \neq \mathrm{Ent}_\nu(f^2) < +\infty \right\}.$$

Functional inequalities and curvature

Definition

We say that (m, ν) satisfies a modified logarithmic-Sobolev inequality if there exists $\lambda > 0$ such that

$$\lambda \mathrm{Ent}_\nu(f) \leq \mathcal{F}_m(f) \quad \text{for all } f \in D(\mathcal{F}_m). \tag{10}$$

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Theorem

The following statements are equivalent:

(i) *There exists $\lambda > 0$ such that*

$$\lambda \mathrm{Ent}_\nu(f) \leq \mathcal{F}_m(f) \quad \textit{for all} \ \ f \in D(\mathcal{F}_m). \tag{11}$$

(ii) *For every $0 \leq f \in L^2(X, \nu)$*

$$\mathrm{Ent}_\nu(e^{t\Delta_m} f) \leq \mathrm{Ent}_\nu(f)\, e^{-\lambda t} \quad \forall t \geq 0. \tag{12}$$

Functional inequalities and curvature

Theorem

Let $[X, d, m]$ be a metric random walk with invariant-reversible probability measure ν, and assume that the constants $LS(m, \nu)$, $MLS(m, \nu)$ and $\mathrm{gap}(-\Delta_m)$ are positive. Then

$$2LS(m, \nu) \leq \frac{1}{2} MLS(m, \nu) \leq \mathrm{gap}(-\Delta_m).$$

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Corollary

Let $[X, d, m]$ be a metric random walk space with invariant-reversible probability measure ν. If there is a $\lambda > 0$ satisfying the logarithmic-Sobolev inequality

$$\lambda \mathrm{Ent}_\nu(f) \leq \mathcal{H}_m(\sqrt{f}) \quad \text{for all } \ f \in L^1(X, \nu)^+,$$

then, for every $f \in L^2(X, \nu)$, we have

$$\mathrm{Ent}_\nu(e^{t\Delta_m} f) \leq \mathrm{Ent}_\nu(f)\, e^{-4\lambda t} \quad \text{for all } t \geq 0,$$

and

$$\| e^{t\Delta_m} f - \nu(f) \|_{L^2(X,\nu)} \leq \| f - \nu(f) \|_{L^2(X,\nu)} e^{-\frac{\lambda t}{2}} \quad \text{for all } t \geq 0.$$

Functional inequalities and curvature

To study the Bakry-Émery curvature condition in our context note that $\mathcal{E}_m$ admits a Carré du champ Γ defined by

$$\Gamma(f,g)(x) = \frac{1}{2}\Big(\Delta_m(fg)(x) - f(x)\Delta_m g(x) - g(x)\Delta_m f(x)\Big) \quad \text{for } f,g \in L^2(X,\nu).$$

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According to Bakry and Émery , we define the Ricci curvature operator Γ_2 by iterating Γ:

$$\Gamma_2(f,g) = \frac{1}{2}\Big(\Delta_m\Gamma(f,g) - \Gamma(f,\Delta_m g) - \Gamma(\Delta_m f, g)\Big),$$

which is well defined for $f,g \in L^2(X,\nu)$.

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which is well defined for $f,g \in L^2(X,\nu)$.
We write, for $f \in L^2(X,\nu)$,

$$\Gamma(f) := \Gamma(f,f) = \frac{1}{2}\Delta_m(f^2) - f\Delta_m f$$

and

$$\Gamma_2(f) := \Gamma_2(f,f) = \frac{1}{2}\Delta_m\Gamma(f) - \Gamma(f,\Delta_m f).$$

Definition

The operator Δ_m satisfies the Bakry-Émery curvature condition $BE(K,n)$ for $n \in (1,+\infty)$ and $K \in \mathbb{R}$ if

$$\Gamma_2(f) \geq \frac{1}{n}(\Delta_m f)^2 + K\Gamma(f) \quad \forall\, f \in L^2(X,\nu).$$

The constant n is called the dimension of the operator Δ_m, and K is called the lower bound of the Ricci curvature of the operator Δ_m. If there exists $K \in \mathbb{R}$ such that

$$\Gamma_2(f) \geq K\Gamma(f) \quad \forall\, f \in L^2(X,\nu),$$

then it is said that the operator Δ_m satisfies the Bakry-Émery curvature condition $BE(K,\infty)$.

Functional inequalities and curvature

Integrating the Bakry-Émery curvature condition $BE(K,n)$ we have

$$\int_X \Gamma_2(f)\,d\nu \geq \frac{1}{n}\int_X (\Delta_m f)^2\,d\nu + K\int_X \Gamma(f)\,d\nu.$$

Now, this inequality can be rewritten as

$$\int_X (\Delta_m f)^2\,d\nu \geq \frac{1}{n}\int_X (\Delta_m f)^2\,d\nu + K\mathcal{H}_m(f),$$

or, equivalently, as

$$K\frac{n}{n-1}\mathcal{H}_m(f) \leq \int_X (\Delta_m f)^2\,d\nu. \tag{13}$$

Similarly, integrating the Bakry-Émery curvature condition $BE(K,\infty)$ we have

$$K\mathcal{H}_m(f) \leq \int_X (\Delta_m f)^2\,d\nu. \tag{14}$$

We call the inequalities (13) and (14) the integrated Bakry-Émery curvature conditions.

Functional inequalities and curvature

Theorem

Let $[X, d, m]$ be a metric random walk with invariant-reversible probability measure ν. Assume that Δ_m is ergodic. Then,

(1) *Δ_m satisfies an integrated Bakry-Émery curvature condition $BE(K, n)$ with $K > 0$ if and only if a Poincaré inequality with constant $K\frac{n}{n-1}$ is satisfied.*

(2) *Δ_m satisfies an integrated Bakry-Émery curvature condition $BE(K, \infty)$ with $K > 0$ if and only if a Poincaré inequality with constant K is satisfied.*

Therefore, if Δ_m satisfies the Bakry-Émery curvature condition $BE(K, n)$ with $K > 0$, we have

$$\mathrm{gap}(-\Delta_m) \geq K\frac{n}{n-1}. \tag{15}$$

In the case that Δ_m satisfies the Bakry-Émery curvature condition $BE(K, \infty)$ with $K > 0$, we have

$$\mathrm{gap}(-\Delta_m) \geq K. \tag{16}$$

Example

Consider the non weighted linear graph G with vertices $V(G) = \{a, b, c\}$ (that is, the positive weights are $w_{a,b} = w_{b,c} = 1$) We have that this graph Laplacian satisfies

$$BE\left(1 - \frac{2}{n}, n\right) \quad \text{for any} \quad n > 1,$$

being $K = 1 - \frac{2}{n}$ the best constant for a fixed $n > 1$.
Now, $\text{gap}(-\Delta) = 1$, therefore we have that Δ satisfies the *integrated Bakry-Émery curvature condition* $BE(K, n)$ with $K = 1 - \frac{1}{n} > 1 - \frac{2}{n}$

Theorem

Let $[X, d, m]$ *be a metric random walk space with invariant-reversible probability measure* ν, *assume that* $\nu(X) < +\infty$ *and let* $T_t = e^{t\Delta_m}$ *be the heat semigroup. Then,* Δ_m *satisfies the Bakry-Émery curvature condition* $BE(K, \infty)$ *with* $K > 0$ *if, and only if,*

$$\Gamma(T_t f) \leq e^{-2Kt} T_t(\Gamma(f)) \quad \forall\, t > 0,\ \forall\, f \in L^2(X, \nu).$$

Concentration of measures

Let (X, d, ν) be a metric measure space with $\nu(X) < +\infty$. For simplicity we assume that ν is a probability measure. We introduce the concentration function

$$\alpha_{(X,d,\nu)}(r) := \sup \left\{ 1 - \nu(A_r), \ A \subset X, \ \nu(A) \geq \frac{1}{2} \right\},$$

where $A_r := \{x \in X \ : \ d(x, A) < r\}$. We say that ν has normal concentration on (X, d) if there exist $C, c > 0$ such that, for every $r > 0$,

$$\alpha_{(X,d,\nu)}(r) \leq C \exp(-cr^2).$$

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For $x \in X$ we define

$$\Theta(x) := \frac{1}{2} \left(W_2^d(\delta_x, m_x) \right)^2 = \frac{1}{2} \int_X d(x,y)^2 dm_x(y),$$

and

$$\Theta_m := \operatorname{supess}_{x \in X} \Theta(x).$$

Since $\Theta(x) \leq \frac{1}{2} \left(\operatorname{diam}(\operatorname{supp}(m_x)) \right)^2$, if $\operatorname{diam}(X)$ is finite, we have $\Theta_m \leq \frac{1}{2}(\operatorname{diam}(X))^2$.

Concentration of measures

Theorem

Let $[X, d, m]$ be a metric random walk space with invariant-reversible probability measure ν and assume that Θ_m is finite. If (m, ν) satisfies the logarithmic-Sobolev inequality

$$\beta \, \mathrm{Ent}_\nu(f^2) \leq \mathcal{H}_m(f) \quad \text{for all } 0 \leq f \in D(\mathcal{H}_m), \quad \beta > 0, \tag{17}$$

then

$$\alpha_{(X,d,\nu)}(r) \leq \exp\left(-\frac{\beta r^2}{16\Theta_m}\right). \tag{18}$$

Transport inequalities

Theorem

Let $[X, d, m]$ be a metric random walk space with invariant-reversible probability measure ν and assume that Θ_m is finite. If Δ_m satisfies the Bakry-Émery curvature condition $BE(K, \infty)$ with $K > 0$, then ν satisfies the transport-information inequality

$$W_1^d(\mu, \nu) \leq \frac{\sqrt{\Theta_m}}{K} \sqrt{I_\nu(\mu)}, \quad \text{for all probability measures } \mu \ll \nu.$$

Transport inequalities

Theorem

Let $[X, d, m]$ be a metric random walk space with invariant-reversible probability measure ν and assume that Θ_m is finite. Then the transport-information inequality

$$W_1^d(\mu,\nu) \leq \frac{1}{K}\sqrt{I_\nu(\mu)}, \quad \text{for all probability measures } \mu \ll \nu,$$

implies the transport-entropy inequality

$$W_1^d(\mu,\nu) \leq \sqrt{\frac{\sqrt{2\Theta_m}}{K}\operatorname{Ent}_\nu(\mu)}, \quad \text{for all probability measures } \mu \ll \nu.$$

Transport inequalities

Example

Let $\Omega = [-1, 0] \cup [2, 3]$ and consider the metric random walk space $[\Omega, d, m^{J,\Omega}]$, with d the Euclidean distance in $\mathbb{R}$ and $J(x) = \frac{1}{2}\chi_{[-1,1]}$. An invariant and reversible probability measure for $m^{J,\Omega}$ is $\nu := \frac{1}{2}\mathcal{L}^1 \llcorner \Omega$. ν satisfies a transport-entropy inequality. However, ν does not satisfy a transport-information inequality, since if ν satisfies a transport-information inequality, then ν must be ergodic. Now it is easy to see that $[\Omega, d, m^{J,\Omega}]$ is not r-connected and then, ν is not ergodic.

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Theorem

Let $[X, d, m]$ be a metric random walk space with invariant-reversible probability measure ν. If $\kappa_m > 0$, then the following transport-information inequality holds

$$W_1^d(\mu,\nu)^2 \leq \frac{1}{\kappa_m} I_\nu(\mu).$$

Reference

J.M. Mazón, M. Solera and J. Toledo.
The Heat Flow on Metric Random Walk Spaces
arXiv:1806.01215v1 [math.AP] 4 Jun 2018

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THANKS FOR YOUR ATTENTION