Cut Finite Element Methods for PDEs on Surfaces

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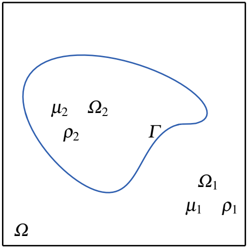
Joint work with: Erik Burman, Thomas Frachon, Peter Hansbo, and Mats G. Larson

BIRS, June 19

Motivation

Multiphase flow simulations



- Main challenge: We need to solve PDEs on dynamically changing geometries. The geometry may undergo strong deformations.

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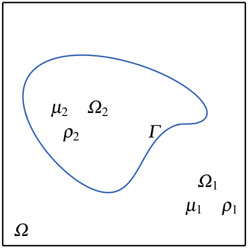

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- Standard finite element methods efficiently solve PDEs in complex geometries but require the mesh to conform to the interface.

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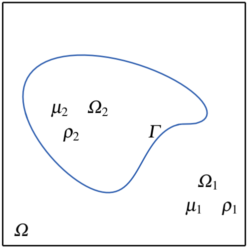
Multiphase flow simulations

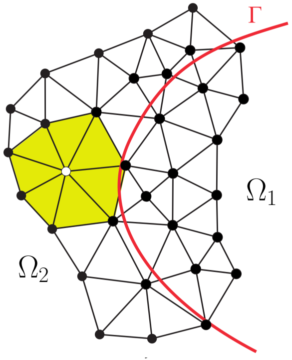


- Main challenge: We need to solve PDEs on dynamically changing geometries. The geometry may undergo strong deformations.
- Standard finite element methods efficiently solve PDEs in complex geometries but require the mesh to conform to the interface.
- CutFEM: avoid re-meshing. The goal is to obtain all properties we have for standard meshed methods but allow for cut elements.

CutFEM

Main characteristics

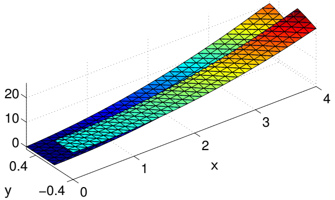
1. The representation of the geometry is separated from the approximation of the PDE. The geometry is allowed to cut through the background mesh in an arbitrary way.

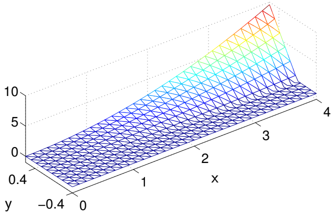


CutFEM

Main characteristics

1. The representation of the geometry is separated from the approximation of the PDE. The geometry is allowed to cut through the background mesh in an arbitrary way.
2. Weak enforcement of boundary and interface conditions



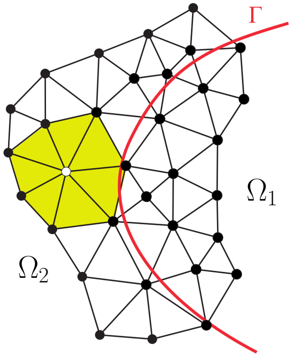


P. Hansbo, M.G. Larson, S. Zahedi, A cut finite element method for a Stokes interface problem, Appl. Numer. Math. **85**, 90 –114 (2014).

CutFEM

Main characteristics

1. The representation of the geometry is separated from the approximation of the PDE. The geometry is allowed to cut through the background mesh in an arbitrary way.
2. Weak enforcement of boundary and interface conditions
3. Stabilization terms added to the weak formulation handle cut elements



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- A. Hansbo and P. Hansbo, An unfitted finite element method, based on Nitsche's method, for elliptic interface problems, Comput. Methods Appl. Mech. Engrg. 191, 5537–5552, (2002)
- M. A. Olshanskii, A. Reusken, and J. Grande, A finite element method for elliptic equations on surfaces. SIAM J. Numer. Anal., 47(5), 3339–3358, (2009).
- E. Burman, P. Hansbo, M. G. Larson, A stabilized cut finite element method for partial differential equations on surfaces: the Laplace-Beltrami operator. Comput. Methods Appl. Mech. Engrg. 285, 188–207 (2015).
- M. Larson, S. Zahedi, Stabilization of high order cut finite element methods on surfaces, https://arxiv.org/pdf/1710.03343.pdf (2017).
- P. Hansbo, M. Larson, S. Zahedi, A cut finite element method for coupled bulk-surface problems on time-dependent domains, Comput. Methods Appl. Mech. Engrg. 307, 96–116 (2016).

The Laplace-Beltrami equation

$$-\Delta_\Gamma u = f \quad \text{on } \Gamma$$

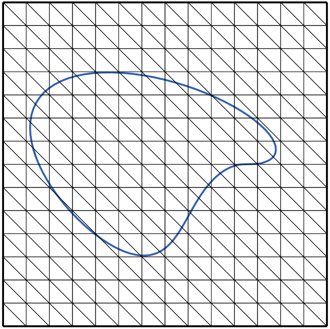
Weak formulation: Find $u \in H^1(\Gamma)$ with $\int_\Gamma u \, ds = 0$ such that

$$\int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v \, ds = \int_\Gamma f v \, ds \qquad \forall v \in H^1(\Gamma)$$

- $U_{\delta_0}(\Gamma)$: open tubular neighborhood of Γ. For $x \in U_{\delta_0}(\Gamma)$: $u^e(x) = u(p(x))$ where $p(x)$ is the closest point projection onto Γ
- $\nabla_\Gamma u = P_\Gamma \nabla u^e$ is the tangential gradient, $P_\Gamma = \boldsymbol{I} - \boldsymbol{n} \otimes \boldsymbol{n}$
- $f \in L^2(\Gamma)$ with $\int_\Gamma f \, ds = 0$, $\partial\Gamma = \emptyset$, $\Gamma \in C^3$
- There exist a unique weak solution u and $\|u\|^2_{H^2(\Gamma)} \leq c\|f\|^2_{L^2(\Gamma)}$

G. Dziuk, Finite elements for the beltrami operator on arbitrary surfaces, in Partial Differential Equations and Calculus of Variations, S. Hildebrandt and R. Leis eds., vol 1357 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, pp. 142–155 (1988)

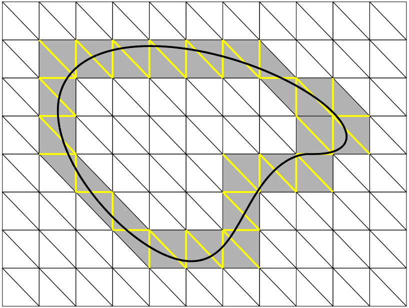
Background mesh and space



- $\mathcal{K}_{0,h}$: a quasiuniform partition of the computational domain Ω into shape regular triangles for $d = 2$ and tetrahedra for $d = 3$ of diameter h.
- $V_{0,h}^{p}$: the space of continuous piecewise polynomials of degree p with average zero defined on the background mesh $\mathcal{K}_{0,h}$.

The active mesh and the finite element space

The stabilized formulation



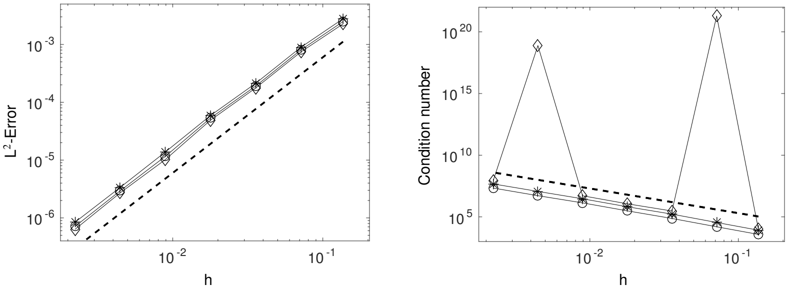
- The active mesh $\mathcal{T}_h$: take the restriction of the background mesh to cut elements, i.e. the grey domain $\mathcal{T}_h$
- The finite element space: $V_h^p = V_{0,h}^p|_{\mathcal{T}_h}$

Example 1: The Laplace-Beltrami equation

Linear elements

Find $u_h \in V_h^1$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \; ds_h + s_h(u_h, v_h) = \int_{\Gamma_h} f_h v_h \; ds_h \qquad \forall v_h \in V_h^1$$



- Diamonds: No stabilization, Circles: With stabilization

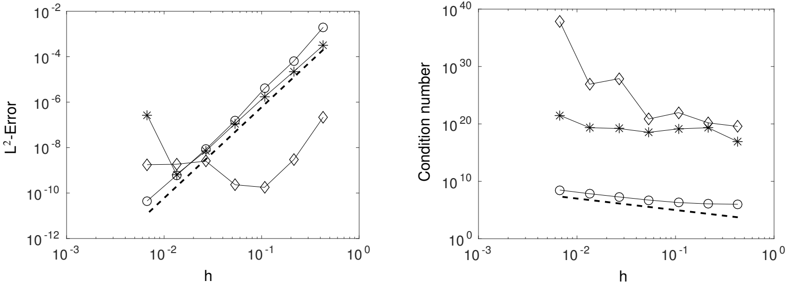
M. A. Olshanskii, A. Reusken, J. Grande, A finite element method for elliptic equations on surfaces. SIAM J. Numer. Anal., **47**(5), 3339–3358, (2009)

Example 1: The Laplace-Beltrami equation

Cubic elements

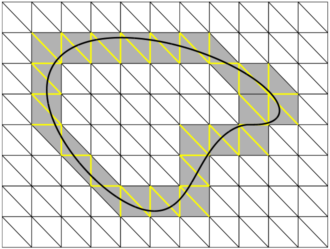
Find $u_h \in V_h^3$ such that

$$\int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} v_h \, ds_h + s_h(u_h, v_h) = \int_{\Gamma_h} f_h v_h \, ds_h \qquad \forall v_h \in V_h^3$$



- Diamonds: No stabilization, Circles: With stabilization

The stabilization term



$$s_h(u_h, v_h) = s_{h,\mathcal{F}}(u_h, v_h) + s_{h,\Gamma}(u_h, v_h)$$

- $s_{h,\mathcal{F}}(u_h, v_h) = \sum_{F \in \mathcal{F}_{S,h}} \sum_{i=1}^{p} c_{F,i} h^{\gamma} ([\partial_n^i u_h]|_F, [\partial_n^i v_h]|_F)_F$
- $s_{h,\Gamma}(u_h, v_h) = \sum_{i=1}^{p} c_{\Gamma,i} h^{\gamma} \left(\partial_n^i u_h, \partial_n^i v_h\right)_{\Gamma_h}$
- $2i - 2 \leq \gamma \leq 2i$

M. G. Larson, S. Zahedi, Stabilization of Higher Order Cut Finite Element Methods on Surfaces
E. Burman, Ghost penalty, C. R. Acad. Sci. Paris, Ser. I **348** (21-22), 1217–1220 (2010)

Optimal error estimates

$$A_h(w,v) = a_h(w,v) + s_h(w,v) = L_h(v)$$
$$a_h(w,v) = \int_{\Gamma_h} \nabla_{\Gamma_h} w \cdot \nabla_{\Gamma_h} v \; ds_h$$
$$L_h(v) = \int_{\Gamma_h} f_h v_h \; ds_h$$

Theorem

Let $u \in H^{p+1}(\Gamma) \cap H^1_0(\Gamma)$ be the exact solution and $u_h \in V_h^p$ the cut finite element approximation. There are constants independent of the mesh size h and of how the surface cuts the background mesh such that the following error bounds hold

$$\|u^e - u_h\|_{A_h} \lesssim h^p \|u\|_{H^{p+1}(\Gamma)} + h^{p+1}\|f\|_{L^2(\Gamma)}$$
$$\|u^e - u_h\|_{L^2(\Gamma_h)} \lesssim h^{p+1} \|u\|_{H^{p+1}(\Gamma)} + h^{p+1}\|f\|_{L^2(\Gamma)}$$

Error Analysis

Energy norm

- A_h is continuous:

$$A_h(w,v) \leq \|w\|_{A_h}\|v\|_{A_h}$$

- A_h satisfies the inf-sup condition:

$$\|w\|_{A_h} \lesssim \sup_{v\in V_h^p\setminus\{0\}} \frac{A_h(w,v)}{\|v\|_{A_h}}$$

- Strang Lemma:
$\|u^e - u_h\|_{A_h} \lesssim \|u^e - \pi_h^p u^e\|_{A_h} + \sup_{v\in V_h^p\setminus\{0\}} \frac{|A_h(u^e,v)-L_h(v)|}{\|v\|_{A_h}}$

Condition number estimate

For $v \in V_h^p$

$$v = \sum_{i=1}^{N} \widehat{v}_i \varphi_i$$

$\mathcal{A}_h$: the stiffness matrix associated with A_h,

$$(\mathcal{A}_h \widehat{v}, \widehat{w})_{\hat{\mathbb{R}}^N} = A_h(v, w) \qquad \forall v, w \in V_h^p$$

Theorem

There is a constant C independent of the mesh size h and of how the surface cuts the background mesh such that the spectral condition number $\kappa(\mathcal{A}_h)$ of the stiffness matrix $\mathcal{A}_h$ satisfies

$$\kappa(\mathcal{A}_h) = \frac{\max_{u \in \hat{\mathbb{R}}^N, \|\widehat{u}\|_{\mathbb{R}^N}=1} (\mathcal{A}_h \widehat{u}, \widehat{u})_{\hat{\mathbb{R}}^N}}{\min_{u \in \hat{\mathbb{R}}^N, \|\widehat{u}\|_{\mathbb{R}^N}=1} (\mathcal{A}_h \widehat{u}, \widehat{u})_{\hat{\mathbb{R}}^N}} \leq C h^{-2}$$

Condition number estimate

Main steps in the proof

- The equivalence between the $\mathbb{R}^N$ norm and the mesh dependent L^2-norm $\|v\|_{L^2(\mathcal{T}_h)} \sim h^{d/2}\|\widehat{v}\|_{\hat{\mathbb{R}}^N}$
- The continuity of A_h:
  $A_h(w, v) \lesssim \|w\|_{A_h}\|v\|_{A_h}, \quad \forall v, w \in H^1_*(\mathcal{T}_h)$
- The coercivity of A_h: $\|v\|^2_{A_h} \lesssim A_h(v, v) \quad \forall v \in V^p_h$
- The inverse inequality: $\|v\|_{A_h} \lesssim h^{-3/2}\|v\|_{L^2(\mathcal{T}_h)} \qquad v \in V^p_h$
- The Poincaré inequality: $\|v\|_{L^2(\mathcal{T}_h)} \lesssim h^{1/2}\|v\|_{A_h} \qquad v \in V^p_h$

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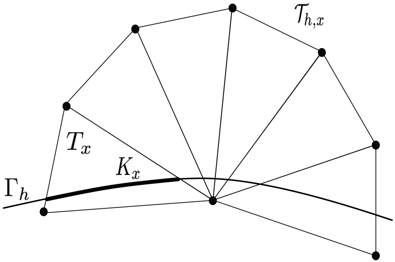


Properties of the stabilization term

P1. For $v \in H^{p+1}(\Gamma) \cap H^1_0(\Gamma)$

$$\|v^e - \pi_h^p v^e\|_{s_h} \lesssim h^p \|v\|_{H^{p+1}(\Gamma)}$$

P2. For $v \in H^{p+1}(\Gamma) \cap H^1_0(\Gamma)$

$$\|v^e\|_{s_h} \lesssim h^p \|v\|_{H^{p+1}(\Gamma)}$$

P3. For $v \in H^2(\Gamma)$,

$$\|\pi_h^p v^e\|_{s_h} \lesssim h \|v\|_{H^2(\Gamma)}$$

P4. For $v \in V_h^p$,

$$\|v\|_{s_h} \lesssim h^{-3/2} \|v\|_{L^2(\mathcal{T}_h)}$$

P5. For $v \in V_h^p$,

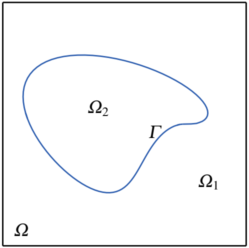
$$\|v\|^2_{L^2(\mathcal{T}_h)} \lesssim h(\|v\|^2_{a_h} + \|v\|^2_{s_h})$$

Time dependent surface PDE

Mathematical model

$$\partial_t u_S + \boldsymbol{\beta} \cdot \nabla u_S + (\mathrm{div}_\Gamma \boldsymbol{\beta}) u_S - k_S \mathrm{div}_\Gamma u_S = f \quad \text{on } \Gamma(t), \quad t \in I$$
$$u_S(0, \boldsymbol{x}) = u_S^0 \quad \text{on } \Gamma(0)$$

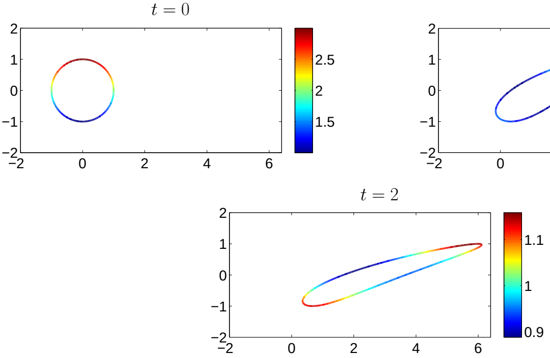
where $\mathrm{div}_\Gamma = \mathrm{tr}((\mathbf{I} - n \otimes n)\nabla)$



- The interface Γ is evolving by $\boldsymbol{\beta}$
- $\int_{\Gamma(t)} f \, ds = 0$ for all $t \geq 0$ and we have $\int_{\Gamma(t)} u_S \, ds = \int_{\Gamma(0)} u_S^0 \, ds$ for all $t \geq 0$

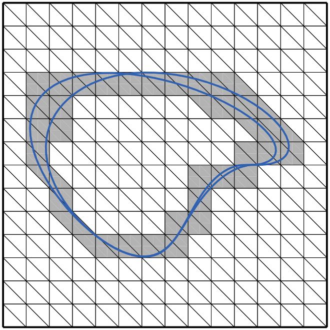
S. Zahedi, A space-time cut finite element method with quadrature in time, Geometrically unfitted finite element methods and applications-proceedings of the UCL Workshop 2016

Example: Deforming interface



- The velocity field $\beta = (\frac{(y+2)^2}{3}, 0)$.
- The initial surfactant concentration $u_S = y/r_0 + 2$.

The active mesh



- The time interval $I = [0, T]$, $0 = t_0 < t_1 < \cdots < t_N = T$, is partitioned into time steps $I_n = (t_{n-1}, t_n]$ of length $k_n = t_n - t_{n-1}$ for $n = 1, 2, \ldots, N$.
- $\mathcal{K}_{S,h}(t) = \{K \in \mathcal{K}_{0,h} \,:\, K \cap \Gamma(t) \neq \emptyset\}$, $\quad \mathcal{N}^n_{S,h} = \bigcup_{t \in I_n} \bigcup_{K \in \mathcal{K}_{S,h}(t)} K$

The finite element space

- On the space-time slab $S_S^n = I_n \times \mathcal{N}_{S,h}^n$ we define the space

$$V_{S,h}^n = P_q(I_n) \otimes V_{0,h}^p|_{\mathcal{N}_{S,h}^n}$$

- Functions $v_h(t,x)$ in $V_{S,h}^n$ take the form

$$v_h(t,x) = \sum_{j=0}^{q} v_{S,j}(x) \left(\frac{t - t_{n-1}}{k_n} \right)^j$$

where $t \in I_n$ and $v_{S,j}(x)$, $j = 0, 1, \cdots, q$ are functions in $V_{0,h}^p|_{\mathcal{N}_{S,h}^n}$ and can be written as

$$v_{S,j}(x) = \sum_i \xi_{ij}^S \varphi_i(\boldsymbol{x})|_{\mathcal{N}_{S,h}^n}$$

The weak form

For $t \in I_n$ and given $u_h(t_{n-1}^-, x)$ the weak formulation is to find $u_h \in V_{S,h}^n$ such that

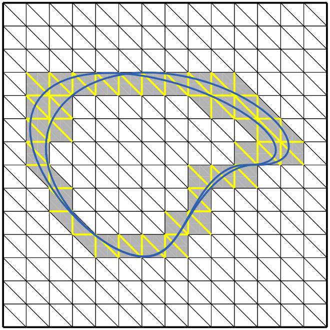
$$A_h^n(u_h, v_h) + s_h^n(u_h, v_h) = L_h^n(v_h), \quad \forall v_h \in V_{S,h}^n$$

Here

$$A_h^n(u, v) = \int_{I_n} (\partial_t u, v)_{\Gamma_h(t)}\, dt + \int_{I_n} a_h(t, u, v)\, dt + ([u], v(t_{n-1}^+, \boldsymbol{x}))_{\Gamma_h(t_{n-1})}$$

$$a_h(t, u, v) = (\boldsymbol{\beta} \cdot \nabla u, v)_{\Gamma_h(t)} + ((\mathrm{div}_\Gamma \boldsymbol{\beta}) u, v)_{\Gamma_h(t)} + (k_S \nabla_\Gamma u, \nabla_\Gamma v)_{\Gamma_h(t)}$$

Stabilization



- $s_h^n(u_h, v_h) = \int_{I_n} s_{h,\mathcal{F}}(u_h, v_h)\, dt + \int_{I_n} s_{h,\mathcal{T}}(u_h, v_h)\, dt$
- $s_{h,\mathcal{F}}(u_h, v_h) = \sum_{F \in \mathcal{F}_{S,h}} \sum_{i=1}^{p} c_{F,i} h^{\gamma} ([\partial_n^i u_h]|_F, [\partial_n^i v_h]|_F)_F$
- $s_{h,\Gamma}(u_h, v_h) = \sum_{i=1}^{p-1} c_{\Gamma,i} h^{\gamma} \left(\partial_n^i u_h, \partial_n^i v_h\right)_{\Gamma_h(t)}$
- $\gamma = 2i$

Space-time CutFEM

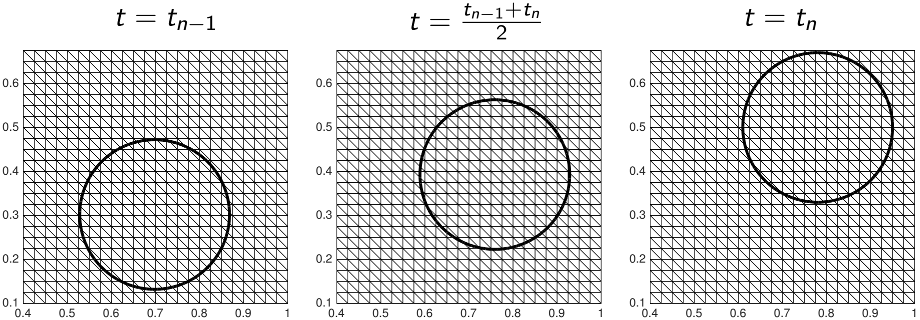
with quadrature in time

- We employ a quadrature formula with weights ω_q and quadrature points t_q, $q = 1, \ldots n_q$, in time

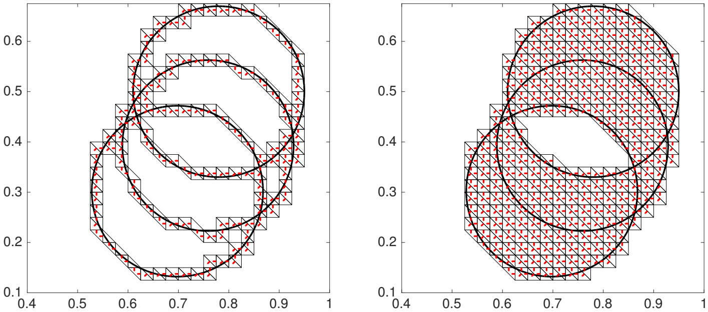
$$\int_{I_n} a_h(t, u, v)\ dt \approx \sum_{q=1}^{n_q} \omega_q a_h(t_q, u(t_q), v(t_q))$$

- Note that this means that the assembly and the geometry computations is only done at the quadrature points t_q.
- Essentially reduces the complexity of the implementation to that of a stationary problem.
- Simpson's quadrature rule: $n_q = 3$, $t_1^n = t_{n-1}$, $t_2^n = \frac{t_{n-1}+t_n}{2}$, $t_3^n = t_n$, $\omega_1^n = \omega_3^n = \frac{k_n}{6}$, and $\omega_2^n = \frac{4k_n}{6}$.

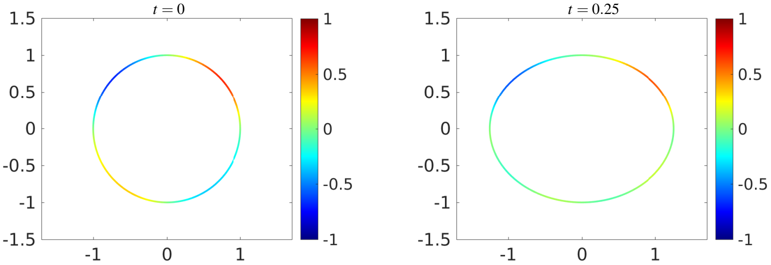
Quadrature in time



Quadrature in time



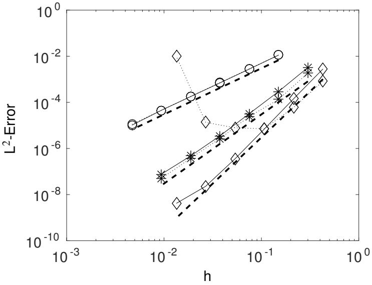
Example: A time dependent surface problem

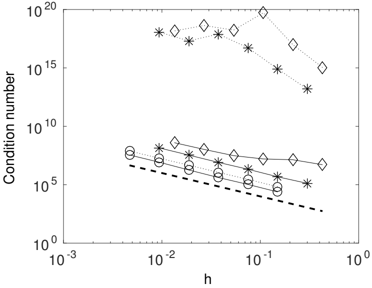


- $\beta = \frac{\pi}{2} \frac{\cos(2\pi t)}{(1+0.25\sin(2\pi t))}(x_1, 0)$
- $k_S = 1$
- Exact solution: $u(x,t) = e^{-4t}x_1x_2 + x_1^3x_2^2$

Example: A time dependent surface problem

Error



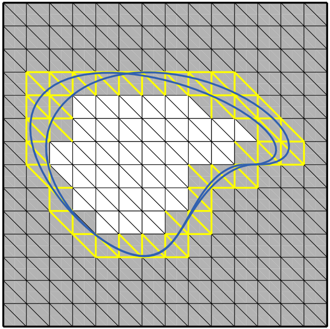


- Circles: p=1, Stars: p=2, Diamonds, p=3

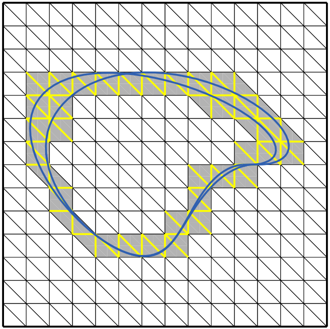
The active mesh

Coupled bulk-surface problem

Bulk



Interface



$$\mathcal{K}_{B,h}(t) = \{K \in \mathcal{K}_{0,h} \; : \; K \cap \Omega_{h,1}(t) \neq \emptyset\}$$

$$\mathcal{K}_{S,h}(t) = \{K \in \mathcal{K}_{0,h} \; : \; K \cap \Gamma_h(t) \neq \emptyset\}$$

$$\mathcal{N}^n_{B,h} = \bigcup_{t \in I_n} \bigcup_{K \in \mathcal{K}_{B,h}(t)} K$$

$$\mathcal{N}^n_{S,h} = \bigcup_{t \in I_n} \bigcup_{K \in \mathcal{K}_{S,h}(t)} K$$

P. Hansbo, M. Larson, S. Zahedi, A cut finite element method for coupled bulk-surface problems on time-dependent domains, Comput. Methods Appl. Mech. Engrg. **307**, 96–116 (2016).

The mean curvature vector

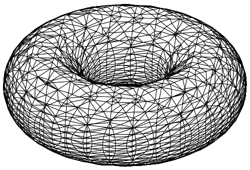
Given the discrete coordinate map $x_{\Gamma_h} : \Gamma_h \ni x \mapsto x \in \mathbf{R}^d$ we want to find the stabilized discrete mean curvature vector $H_h \in [V_h^1]^d$ such that

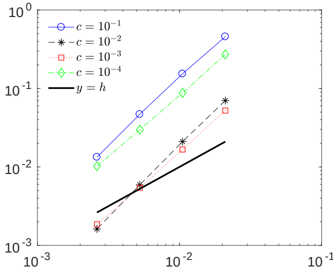
$$(H_h, v_h)_{\Gamma_h} + s_h(H_h, v_h) = (\nabla_{\Gamma_h} x_{\Gamma_h}, \nabla_{\Gamma_h} v_h)_{\Gamma_h},$$

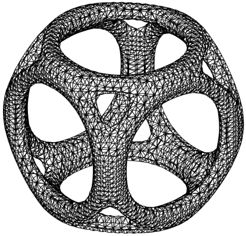
s_h as before with $\gamma = 0$.

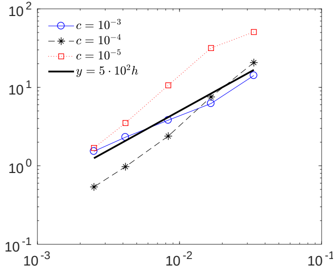
P. Hansbo, M. Larson, S. Zahedi, Stabilized finite element approximation of the mean curvature vector on closed surfaces, SIAM J. Numer. Anal. **53**(4), 1806–1832 (2015).

The mean curvature vector





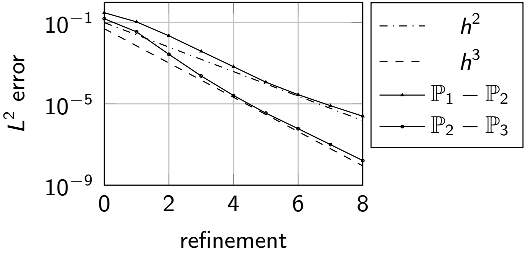




The mean curvature vector

- Interface: piecewise polynomial surface of order p
- Find the stabilized discrete mean curvature vector $H_h \in [V_h^{p-1}]^d$ such that

$$(H_h, v_h)_{\Gamma_h} + s_h(H_h, v_h) = (\nabla_{\Gamma_h} x_{\Gamma_h}, \nabla_{\Gamma_h} v_h)_{\Gamma_h},$$



Conclusions

- Main ideas in CutFEM for Surface PDEs:
  - A fixed partition of the computational domain
  - A finite element space defined on the background mesh
  - An active mesh
  - Restrict the finite element spaces defined on the fixed mesh to the active mesh
  - Stabilization terms
- Space-time CutFEM with quadrature in time a convenient method for problems on evolving domains
- Optimal order error estimates independently of the location of the interface
- The condition number of the stiffness matrix is $\mathcal{O}(h^{-2})$ independently of the location of the interface