On some 1-dimensional flux limited diffusion equations.

S. Moll

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From classical diffusion to flux limited one.

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- **P. Rosenau:** (1992)

$$u_t = \nu \Delta u \iff u_t + \operatorname{div}(uV_u) = 0,$$

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Let c be the maximal speed of transport of the media.

$$V_u = -\nu \frac{\nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla u|^2}}.$$

Flux limited diffusion equations.

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Diffusion of heat in a neutral gas: $\nu = \nu(u) \sim \nu_1 u^{\frac{1}{2}}$ and $c = c(u) \sim c_1 u^{\frac{1}{2}}$.

$$(PMRHE) \qquad u_t = \operatorname{div}\left(\frac{\nu_1 u^{\frac{3}{2}} Du}{\sqrt{u^2 + \frac{\nu_1^2}{c_1^2}|Du|^2}}\right).$$

Mass transportation interpretation

Y. Brenier (2001)

$$u_t = \operatorname{div}\left(u \nabla k^* [\nabla(F'(u))]\right)$$

$$k(z) := \begin{cases} c^2\left(1 - \sqrt{1 - \frac{|z|^2}{c^2}}\right) & \text{if } \ |z| \le c \\ \\ +\infty & \text{if } \ |z| > c. \end{cases}$$

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If $F(x) = \nu(\log(x) - 1)x$ (the Boltzmann entropy) $\Rightarrow (RHE)$. If $F(x) = \frac{x^M}{M-1}$ (the Tsallis entropy) $\Rightarrow$:

$$(SLPME) \qquad u_t = \frac{M}{M-1}\mathrm{div}\left(\frac{u\nabla u^{M-1}}{\sqrt{1+\frac{M^2}{(M-1)^2c^2}|\nabla u^{M-1}|^2}}\right).$$

Flux limited diffusion equations. Known results

- Existence and uniqueness of entropy solutions for the Cauchy problem (F. Andreu, V.Caselles, J.M.Mazón (2005), V. Caselles (2011).
- Propagation of the support (F. Andreu, V.Caselles, J.M.Mazón, S.M (2006), L. Giacomelli (2015).
- Vertical contact angle, Rankine-Hugoniot condition, speed of propagation of discontinuity fronts (V. Caselles 2011).
- Waiting time phenomenon (L. Giacomelli (2015), J. Calvo, J. Campos, V. Caselles, J. Soler, O. Sánchez (2015), L. Giacomelli, S.M., F. Petitta (2017)).

Two flux limited porous medium equations.

$$(PMRHE) \qquad u_t = \operatorname{div}\left(\frac{\nu u^m Du}{\sqrt{u^2 + \frac{\nu^2}{c^2}|Du|^2}}\right).$$

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  From now on, $\nu \equiv c \equiv 1$.
- Different scaling for large gradients:

$$u_t \sim (u^m)_x\,, \text{ for } u_x >> 1 \quad \text{resp.} \quad u_t \sim u_x \text{ for } (u^{M-1})_x >> 1$$

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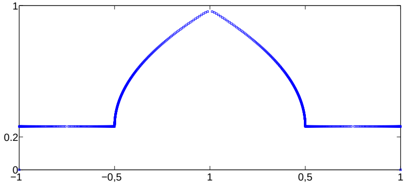
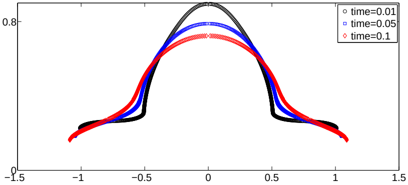
- (Caselles, 2011) R. H. condition:

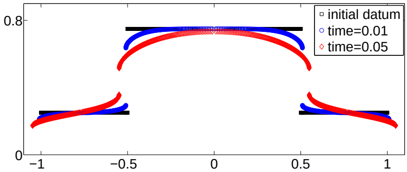
$$v = \frac{(u^+)^m - (u^-)^m}{\ldots} \quad \text{resp.} \quad v = 1$$

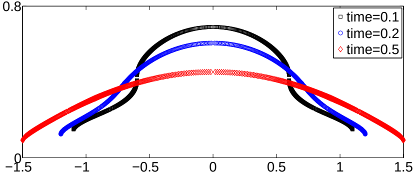
Expectations: Numerics J.A.Carrillo, V. Caselles, S.M (2013)

- Smoothing? .









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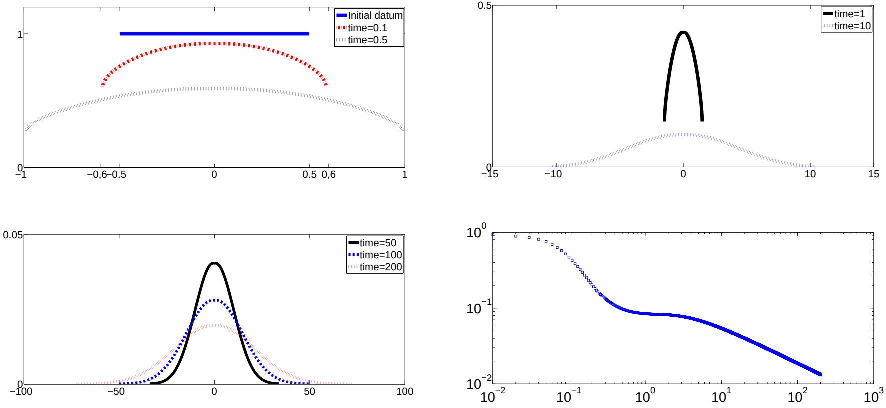
- Asymptotics?



1-D simple observation.

Preservation of mass within jump discontinuities:
Suppose that, in $t \in [t_0, t_1]$ $u^+(t, a(t)) < u^-(t, a(t))$ and $u^-(t, b(t)) < u^+(t, b(t)$. Then, recall the vertical contact angle and R-H-condition.

$$a(t) = a_{t_0} + t\,, \quad b(t) = b_{t_0} + t$$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} u(t,x)\,dx = u(t,b(t))^- - u(t,a(t))^+ + \int_{a(t)}^{b(t)} u_t(t,x)dx$$

$$= u(t,b(t))^- - u(t,a(t))^+$$

$$+u(t,b(t))^- \mathrm{sign}(u_x(t,b(t)))^- - u(t,a(t))^+ \mathrm{sign} u_x(t,a(t))^+ = 0$$

Therefore, a natural change of variable is mass variable (Lagrangian coordinates).

The 1-D case: dual formulation and large solutions.

Let u a probability density governed by an evolution equation with finite speed of propagation and mass conservative:

$$u_t = \mathbf{a}(u, u_x)_x$$

and with initial datum u_0 with compact support and mass equal to 1.
Consider a diffeomorphism $\Phi : [0,1] \to \mathbb{R}$ given by $\int_{-\infty}^{\Phi(\eta,t)} u(t,x)\,dx = \eta$.
Then, taking $v(t,\eta) := \frac{1}{u(\Phi(\eta,t))}$ we get

$$v_t = \left(-v\,\mathbf{a}(\frac{1}{v}, \frac{-v_\eta}{v^3})\right)_\eta, \quad \text{in } [0,T]\times[0,1]$$

Suppose that $u = 0$ at the boundary of its support. Then, $v(t,\eta) = +\infty$ in $[0,T]\times\{0,1\}$.

Example

For PMRHE: $u_t = \left(u^m \frac{u_x}{\sqrt{u^2 + |u_x|^2}} \right)_y$,

$$\begin{cases} v_t = \left(v^{1-m} \frac{v_\eta}{\sqrt{v^4 + |v_\eta|^2}} \right)_\eta & \text{in } [0,T] \times [0,1] \\ u(t,\eta) = +\infty & \text{on } [0,T] \times \{0,1\} \end{cases}$$

Example

For SLPME:

$$\begin{cases} v_t = \left(\frac{v_\eta}{\sqrt{v^{2(M+1)} + |v_\eta|^2}} \right)_\eta & \text{in } [0,T] \times [0,1] \\ u(t,\eta) = +\infty & \text{on } [0,T] \times \{0,1\} \end{cases}$$

$$
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v_t = \left(\dfrac{v_\eta}{\sqrt{\varphi(v)^2 + |v_\eta|^2}} \right)_\eta & \text{in } [0,T] \times [0,1] \\[2ex]
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- Equivalence of problems for (RHE) and regularity properties for locally Lipschitz initial data on their support. (J. A. Carrillo, V. Caselles, S.M (2013).
- Regularity properties for locally Lipschitz but a finite set of points (J. Calvo, J. Campos, V. Caselles, J. Soler, O. Sánchez (2015))

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In the model cases, $\varphi(v) \geq \varphi(\frac{1}{\|u_0\|_\infty}) > 0$. The, one expects that the PDE behaves as the Time–Dependent Minimal Surfaces Equation:

$$v_t = \left(\frac{v_\eta}{\sqrt{1 + |v_\eta|^2}} \right)_\eta$$

S.M., F. Smarrazzo (2018)

$$(NP)\begin{cases} u_t = \operatorname{div}\left(\mathbf{a}(u,\nabla u)\right) & \text{in } \Omega\times[0,T] \\ \mathbf{a}(u,\nabla u)\cdot\nu = 0 & \text{in } \partial\Omega\times[0,T] \\ u(0)=u_0 & \text{in } \Omega \end{cases}$$

- (H_1) $\mathbf{a}$ Lipschitz , $\mathbf{a}(z,0)=0$ and $|\mathbf{a}(z,\xi)|\le 1$

$$\mathbf{a}(z,\xi)=\nabla_\xi f(z,\xi) \quad \text{for all } (z,\xi)\in\mathbb{R}\times\mathbb{R}^N\,,$$

  $f\in C^1(\mathbb{R}\times\mathbb{R}^N)$, convex with respect to ξ
- (H_2) $C_0|\xi| - D_0 \le f(z,\xi)\le C_1\big(|\xi|+|z|+1\big)$
- $(H3)$ $f^0(z,\xi):=\lim_{t\to 0^+} tf\left(z,\dfrac{\xi}{t}\right)=|\xi| \qquad (z\in\mathbb{R},\ \xi\in\mathbb{R}^N)\,.$

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Apart from the previous models, it also includes doubly nonlinear equations with linear growth s.a:

$$(u^q)_t = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\,, \quad q>1.$$

- [F. Andreu, V. Caselles, J. Mazón (2004,2005)] Given $u_0 \in L^1(\Omega)$, there exists a unique entropy solution to (PN).

Theorem

Given $u_0 \in BV(\Omega)$ ($L^2(\Omega)$), there exists a unique **strong** *solution to (PN) such that:*

(i) $u \in L^\infty(0,T;BV(\Omega))$ *and* $u_t \in L^2(Q_T)$ *(*$\in L^2(\Omega \times (\tau,T))$*).*

(ii) *if* $u_0 \in BV(\Omega) \cap L^2(\Omega)$*, then* $u \in C([0,T];L^2(\Omega))$ *and*

$$\frac{1}{2}\|u(t_2)\|^2_{L^2(\Omega)} + \int_{t_1}^{t_2} h(u(s),Du(s))(\Omega)\,ds = \frac{1}{2}\|u(t_1)\|^2_{L^2(\Omega)}$$

(iii) *if* $u_0 \in BV(\Omega) \cap L^2(\Omega)$*, for* $a.e.\ t \in (0,T)$

$$\big(\mathbf{a}(u(t),\nabla u(t)),Du(t)\big) = h(u(t),Du(t)) \quad \textit{in } \mathcal{M}(\Omega)\,.$$

(here $h(z,\xi) := \mathbf{a}(z,\xi)\cdot\xi$*.)*

Theorem

If u is a strong solution, then

$$[\mathbf{a}(u(t), \nabla u(t), \nu_{u(t)}] = 1\,, \quad \mathcal{H}^{N-1} - \text{a.e. in } J_{u(t)}$$

If $N = 1$, then

$$D^s_{\pm} u(t) = D^s_{\pm} u(t) | \{x \in \Omega : \mathbf{a}(u(t), \nabla u(t)) = \pm 1\} \quad \text{a.e. } t \in [0, T]$$

Rough idea. Pseudoparabolic approximation.

$$(NP)_\varepsilon \begin{cases} u_t = \operatorname{div}(\mathbf{a}(u,\nabla u)) + \sqrt{\varepsilon}\Delta u + \varepsilon \Delta u_t & \text{in } \Omega \times [0,T] \\ (\mathbf{a}(u,\nabla u) + \sqrt{\varepsilon}\nabla u + \varepsilon \nabla u_t)\cdot \nu = 0 & \text{in } \partial\Omega \times [0,T] \\ u(0) = u_0 \in H^1(\Omega) & \text{in } \Omega \end{cases}$$

$$\updownarrow$$

$$\begin{cases} u_t = \mathcal{L}(u) \ \text{ in } \ (0,T)\,, \\ u(0) = u_0 \in H^1(\Omega)\,. \end{cases}$$

Here, for every $u \in H^1(\Omega)$, $w_u := \mathcal{L}(u)$ is the unique weak solution in $H^1(\Omega)$ of the Neumann problem

$$\begin{cases} -\varepsilon \Delta w + w = \operatorname{div}(\mathbf{a}(u,\nabla u) + \sqrt{\varepsilon}\nabla u) & \text{in } \ \Omega\,, \\ \frac{\partial w}{\partial \nu} = -[a(u,\nabla u) + \sqrt{\varepsilon}\nabla u, \nu] & \text{on } \ \partial\Omega\,, \end{cases}$$

A piori estimates

$$
\begin{aligned}
&\int_\Omega f(u(x,t),\nabla u(x,t))\,dx + \frac{\sqrt{\varepsilon}}{2}\int_\Omega |\nabla u(x,t)|^2\,dx + \\
&+\varepsilon\int_0^t\!\!\int_\Omega |\nabla u_t|^2\,dxds + \frac{1}{2}\int_0^t\!\!\int_\Omega u_t^2\,dxds \leq \\
&\leq \frac{\beta^2}{2}t\,|\Omega| + \int_\Omega f(u_0(x),\nabla u_0(x))\,dx + \frac{\sqrt{\varepsilon}}{2}\int_\Omega |\nabla u_0|^2\,dx
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Hence, $\{u_\varepsilon\}$ bdd in $L^\infty(0,T;W^{1,1}(\Omega)) \cap W^{1,1}(Q_T) \cap W^{1,2}(0,T;L^1(\Omega))$.
Then,

A piori estimates

$$\int_\Omega f(u(x,t),\nabla u(x,t))\,dx + \frac{\sqrt{\varepsilon}}{2}\int_\Omega |\nabla u(x,t)|^2\,dx +$$
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Then, $\exists u \in C([0,T];L^1(\Omega) \cap L^\infty_w(0,T;BV(\Omega))$ s.t.

$$u_{\varepsilon_k} \to u \quad \text{in } L^1(Q_T)\,,$$

$$u_{\varepsilon_k}(t) \to u(t) \quad \text{in } L^1(\Omega) \text{ for every } t\in[0,T]\,, \text{ and}$$

$$u_{\varepsilon_k t} \rightharpoonup u_t \quad \text{in } L^2(Q_T)\,.$$

There exists $\mathbf{z} \in L^\infty(Q_T; \mathbb{R}^N)$ such that

$$\mathbf{a}(u_{\varepsilon_k}, \nabla u_{\varepsilon_k}) \xrightharpoonup{*} \mathbf{z} \quad \text{in } L^\infty(Q_T; \mathbb{R}^N)\,.$$

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$$\mathbf{a}(u_{\varepsilon_k}, \nabla u_{\varepsilon_k}) \overset{*}{\rightharpoonup} \mathbf{z} \quad \text{in } L^\infty(Q_T;\mathbb{R}^N)\,.$$

Moreover, $\mathbf{z}(t) \in X(\Omega)$, $\operatorname{div}(\mathbf{z}(t)) = u_t(t)$ in $\mathcal{D}'(\Omega)$ and $[\mathbf{z}(t), \nu] = 0$ for $a.e.$ $t \in [0, T]$.

Finally, for $a.e.$ $(x,t) \in Q_T$ there holds

$$\mathbf{z}(x,t) = \mathbf{a}(u(x,t), \nabla u(x,t))\,.$$

$\square$

Case $N = 1$. S. M., F. Smarrazzo, in preparation.

$$\begin{cases} u_t = \left(\dfrac{u_x}{\sqrt{\varphi^2(u) + u_x^2}} \right)_x & \text{in } [0,T] \times [a,b] \\ \left(\dfrac{u_x}{\sqrt{\varphi^2(u) + u_x^2}} \right)(a,b) = (z_1, z_2) \in [-1,1]^2 & \\ u(0) = u_0 & \text{in } [a,b] \end{cases}$$

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\end{cases}
$$

Theorem

For any $u_0 \in BV(\Omega)$, there exists a unique strong solution. Moreover,

$$
|(u_x(t))^s_\pm| \le |((u_0)_x)^s_\pm \quad \text{for all } t \in [0,T]
$$

Sketch of proof

- Take the pseudoparabolic approximation and write the equation for $u_{\varepsilon x}$: $v_\varepsilon := \mathbf{a}(u_\varepsilon, u_{\varepsilon,x}) + \sqrt{\varepsilon} + \varepsilon u_{\varepsilon,x,t}$,

$$u_{\varepsilon,x,t} = v_{\varepsilon,x,x} + b.c.$$

- Obtain entropy inequalities for $u_{\varepsilon,x}$.

$$\int\int_Q G(u, u_x)\psi_t - \int\int g(v)\psi_x v_x + g'(v)v_x^2\psi$$

$$\geq - \int_\Omega G(u_0, u_{0x})\psi(x,0) - \int_Q H^g(x,t)\psi$$

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- $u_{\varepsilon,x} \to u_x^a$ a.e. in Q.
- $v_\varepsilon \to \mathbf{a}(u, u_x^a)$ a.e. in Q.
- Pass to the limit in the entropy formulation for u_x ($g \in C^1(\mathbb{R})$, $g' \geq 0$, close to $-\chi_{]-\infty,1]}$ and finally with $g = -\chi_{]-\infty,1]}$:

$$\int_\Omega u_x^a(x,t)\rho(x) + \leq -\int_\Omega u_{0,x}^a \rho + \int_0^t \int_\Omega z_x \rho_x + + \langle u_{0,x,s}^-, \rho \rangle_\Omega$$

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$$\int_\Omega u_x^a(x,t)\rho(x) + \leq -\int_\Omega u_{0,x}^a \rho + \int_0^t \int_\Omega z_x \rho_x + + \langle u_{0,x,s}^-, \rho \rangle_\Omega$$

- Together with weak formulation...

$$\langle u_{x,s}^+(\cdot,t), \rho\rangle_\Omega - \langle u_{x,s}^-(\cdot,t), \rho\rangle_\Omega \leq \langle {u_0}_{x,s}^+, \rho\rangle_\Omega$$

- From energy estimates, obtain $u_{\varepsilon,x}(\cdot,t) \overset{*}{\rightharpoonup} u_x(\cdot,t) \quad \text{in } \mathcal{M}(\Omega)$
- $u_{\varepsilon,x} \to u_x^a$ a.e. in Q.
- $v_\varepsilon \to \mathbf{a}(u, u_x^a)$ a.e. in Q.
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- Symmetric argument for $u_{x,s}^+$

$\square$

Asymptotics. S. M., in preparation.

$$\mathcal{A} := \{u \in \mathcal{S}(\mathbb{R}) : \left\|\frac{u_x}{u}\right\|_\infty < +\infty\,,\ \left\|\left(\frac{u_x}{u}\right)_x\right\|_\infty < +\infty\,,\ u(x) \geq \lambda e^{-\beta\frac{|x^2|}{2}}\}$$

Theorem (Caselles, '11)

Let $u_0 \in \mathcal{A}$. Then, the entropy solution u to (RHE) satisfies:

(i) $u(t) \in \mathcal{S}(\mathbb{R})$, *for all* $t > 0$

(ii) $\left\|\frac{u_x(t)}{u(\cdot,t)}\right\|_\infty \leq \left\|\frac{u_{0x}}{u_0}\right\|_\infty$, *for all* $t > 0$.

Basic estimates. $u_t = \frac{u_{xx}}{\left(1+\left(\frac{u_x}{u}\right)^2\right)^{\frac{3}{2}}} + \frac{u\frac{u_x^4}{u^4}}{\left(1+\left(\frac{u_x}{u}\right)^2\right)^{\frac{3}{2}}}$

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Lemma

There exist $\alpha_1, \alpha_2, K_1, K_2 > 0$ such that

$$K_1 G_1 * u_0 \leq u \leq K_2 G_2 * u_0,$$

with G_i being the fundamental solutions of the heat equations $u_t = \alpha_i u_{xx}$.

Perform the following change of variables,

$$v(\tau, y) = e^{\tau} u(\frac{e^{2\tau} - 1}{2}, e^{\tau} y) \,.$$

Then,

$$v_\tau = \left(yv + \nu \frac{v \nabla v}{\sqrt{v^2 + e^{-2\tau} |\nabla v|^2}} \right)_y \,. \tag{1}$$

Take v_1 as the solution to (1) and v_2 the solution of the linear Fokker-Planck equation with $u_0 \in \mathcal{A}$ as initial datum.

Theorem

There exist $K > 0$ such that

$$W_2(v_1(\tau)\mathcal{L}^1, v_2(\tau)\mathcal{L}^1) \leq K e^{-\frac{\tau}{6}}$$

THANK YOU VERY MUCH!