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JOINT WORK WITH G. SAVARÉ

BIRS WORKSHOP ON “ENTROPIES, THE GEOMETRY OF NONLINEAR FLOWS, AND THEIR APPLICATIONS”

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Some preliminaries: $\lambda$-convexity and slopes

Let $(X, d)$ be a complete metric space.

We consider a lower semicontinuous (l.s.c.) functional $\phi : X \to (-\infty, +\infty]$ with nonempty domain (i.e. $\phi$ is proper – taken for granted from now on)

$$\text{Dom}(\phi) := \{ x \in X : \phi(x) < +\infty \}.$$

Given $\lambda \in \mathbb{R}$, we say that $\phi$ is (geodesically) $\lambda$-convex if for every $x_0, x_1 \in \text{Dom}(\phi)$ there exists a (minimal, constant speed) geodesic $x_\vartheta : [0, 1] \to X$ such that

$$\phi(x_\vartheta) \leq (1 - \vartheta)\phi(x_0) + \vartheta\phi(x_1) - \frac{\lambda}{2} \vartheta(1 - \vartheta)d^2(x_1, x_0) \quad \forall \vartheta \in [0, 1].$$

In particular, in this case $\text{Dom}(\phi)$ is a geodesic space.
If $\phi$ is $\lambda$-convex, one can show that the functional $x \mapsto \phi(x) - \frac{\lambda}{2} d^2(x, o)$ is linearly bounded from below for all $o \in X$:

$$\phi(x) \geq \frac{\lambda}{2} d^2(x, o) - \ell_o d(x, o) - c_o \quad \forall x \in X,$$

for some $\ell_o, c_o \geq 0$.

The metric slope $|\partial \phi|$ is defined for all $x \in \text{Dom}(\phi)$ by

$$|\partial \phi|(x) := \limsup_{y \to x} \frac{(\phi(x) - \phi(y))^+}{d(x, y)},$$

with $|\partial \phi|(x) := +\infty$ if $x \in X \setminus \text{Dom}(\phi)$ and $|\partial \phi|(x) := 0$ if $x \in \text{Dom}(\phi)$ is isolated.

If $\phi$ is $\lambda$-convex then $|\partial \phi|$ coincides with the (l.s.c.) global $\lambda$-slope:

$$\mathcal{L}_\lambda[\phi](x) := \sup_{y \neq x} \frac{(\phi(x) - \phi(y) + \frac{\lambda}{2} d^2(x, y))^+}{d(x, y)}.$$
EVI and Gradient Flows

First we want to give a meaning to $\dot{u} = -\partial\phi(u)$ in our metric framework.

Evolution Variational Inequalities (EVI) [Ambrosio-Gigli-Savaré ’05]

A continuous curve $u: t \in (0, +\infty) \mapsto u_t \in \text{Dom}(\phi)$ is a solution to EVI$_\lambda(X, d, \phi)$ if

$$\frac{1}{2} \frac{d^+}{dt} d^2(u_t, v) + \frac{\lambda}{2} d^2(u_t, v) \leq \phi(v) - \phi(u_t) \quad \forall t > 0, \forall v \in \text{Dom}(\phi).$$

Here

$$\frac{d^+}{dt} \zeta(t) := \limsup_{h \downarrow 0} \frac{\zeta(t + h) - \zeta(t)}{h} \quad \text{(upper right Dini derivative)}.$$ 

Gradient Flows (GF)

A $\lambda$-Gradient Flow of $\phi$ is a family of continuous maps $S_t : \overline{\text{Dom}(\phi)} \to \overline{\text{Dom}(\phi)}$, $t \geq 0$, such that for every $u_0 \in \text{Dom}(\phi)$ there hold

$$S_{t+h}(u_0) = S_h(S_t(u_0)) \quad \forall t, h \geq 0, \quad \lim_{t \downarrow 0} S_t(u_0) = S_0(u_0) = u_0,$$

the curve $t \mapsto S_t(u_0)$ is a solution of EVI$_\lambda(X, d, \phi)$.
A classical example: Hilbert spaces

Let \((X, \langle \cdot, \cdot \rangle)\) be a Hilbert space, with \(d(x, y) := |x - y| = \sqrt{\langle x - y, x - y \rangle}\). Let \(\phi : X \to (-\infty, +\infty]\) be a l.s.c. \(\lambda\)-convex functional. In other words, \(x \mapsto \phi(x) - \frac{\lambda}{2} |x|^2\) is a convex functional in the usual sense.

Then [Brézis '73] a continuous curve \(u: t \in (0, +\infty) \mapsto u_t \in \text{Dom}(\phi)\) is a solution to \(\text{EVI}_\lambda(X, d, \phi)\) if and only if \(u\) is locally Lipschitz and

\[
\dot{u}_t \in -\partial \phi(u_t) \quad \text{for a.e. } t > 0
\]

(for every \(t > 0\) if we use right derivatives), where

\[
w \in \partial \phi(u) \iff \langle w, v - u \rangle + \frac{\lambda}{2} |v - u|^2 \leq \phi(v) - \phi(u) \quad \forall v \in X,
\]

i.e. \(\partial \phi\) is the subgradient of \(\phi\). In this case,

\[
|\partial \phi|(u) := \min\{|w|: w \in \partial \phi(u)\}.
\]
A more elaborate example: drift diffusion with nonlocal interaction

Let $\mathcal{X} := \mathcal{P}_2(\mathbb{R}^d)$ be the space of Borel probability measures, with finite quadratic moment, endowed with the Wasserstein distance $W_2$.

We consider the following functional on $\mathcal{X}$:

$$\phi(\mu) := \int_{\mathbb{R}^d} \varrho \log \varrho \, dx + \int_{\mathbb{R}^d} V \, d\mu + \frac{1}{2} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \mathcal{W}(x - y) \, d\mu(y) \right) \, d\mu(x) \quad \text{if} \quad \mu \equiv \varrho \mathcal{L}^d,$$

$$\phi(\mu) := +\infty \quad \text{if} \quad \mu \ll \mathcal{L}^d,$$

i.e. the sum of internal, potential and interaction energy. Here $V : \mathbb{R}^d \to \mathbb{R}$ is a l.s.c. convex function and $\mathcal{W} : \mathbb{R}^d \to \mathbb{R}^+$ is a $C^1(\mathbb{R}^d)$, even and convex function satisfying a suitable “doubling” condition.

Then [Carrillo-McCann-Villani ’03, Ambrosio-Gigli-Savaré ’05] the functional $\phi$ admits a GF in $\mathcal{X}$, which is given by solutions to the drift-diffusion (with interaction) equation

$$\partial_t \varrho_t = \Delta \varrho_t + \text{div} \left[ \varrho_t (\nabla V + \nabla \mathcal{W} \ast \varrho_t) \right] \quad \text{in} \ \mathbb{R}^d, \quad \lim_{t \to 0} \varrho_t \mathcal{L}^d = \mu_0 \quad \text{in} \ \mathcal{P}_2(\mathbb{R}^d).$$
Main properties of solutions to EVI

Theorem

Let $\phi: X \to (-\infty, +\infty]$ be a l.s.c. functional and $\lambda \in \mathbb{R}$. Let $u, u^1, u^2 \in C^0([0, +\infty); X)$ be solutions to $\text{EVI}_\lambda(X, d, \phi)$. The following properties hold:

- **$\lambda$-contraction and uniqueness:**
  $$d(u^1_t, u^2_t) \leq e^{-\lambda(t-s)}d(u^1_s, u^2_s) \quad \forall \ 0 \leq s < t < +\infty.$$

  In particular, for each $u_0 \in \text{Dom}(\phi)$ there is at most one solution s.t. $\lim_{t \downarrow 0} u_t = u_0$.

- **Regularizing effects:**
  - $u$ is locally Lipschitz in $(0, +\infty)$ and $u_t \in \text{Dom}(|\partial \phi|) \subset \text{Dom}(\phi)$ for all $t > 0$;
  - the map $t \in [0, +\infty) \mapsto \phi(u_t)$ is nonincreasing and (locally) semi-convex;
  - the map $t \in [0, +\infty) \mapsto e^{\lambda t}|\partial \phi|(u_t)$ is nonincreasing and right continuous.

- **A priori estimates:** for every $v \in \text{Dom}(\phi)$ and $t > 0$
  $$\frac{e^{\lambda t}}{2}d^2(u_t, v) + E_\lambda(t)(\phi(u_t) - \phi(v)) + \frac{(E_\lambda(t))^2}{2}|\partial \phi|^2(u_t) \leq \frac{1}{2}d^2(u_0, v),$$
  where $E_\lambda(t) := \int_0^t e^{\lambda s} \, ds$. 

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Theorem (continued)

- **Right, left limits and energy identity:** for every $t > 0$ the right limits

\[
|\dot{u}_{t+}| := \lim_{h \downarrow 0} \frac{d(u_{t+h}, u_t)}{h}, \quad \frac{d}{dt} \phi(u_{t+}) := \lim_{h \downarrow 0} \frac{\phi(u_{t+h}) - \phi(u_t)}{h}
\]
exist finite, satisfy

\[
\frac{d}{dt} \phi(u_{t+}) = -|\dot{u}_{t+}|^2 = -|\partial \phi|^2(u_t) = -\mathcal{L}_X^2[\phi](u_t) \quad \forall t > 0
\]

and define a right-continuous map. In particular, the functional

\[x \mapsto \phi(x) - \frac{\lambda}{2} d^2(x, o)\] is linearly bounded from below for all $o \in X$. 

Minimizing Movements (MM)

Given $\tau > 0$, we consider the *quadratically-perturbed* functional

$$
\Phi(\tau, U, V) := \frac{1}{2\tau} d^2(U, V) + \phi(V) \quad \forall U, V \in X.
$$

We say that $\{U^n_\tau\}_{n \in \mathbb{N}}$ is a **discrete minimizing sequence** if

$$
U^n_\tau \in \text{Argmin}_{V \in X} \Phi(\tau, U^{n-1}_\tau, V) \quad \forall n \in \mathbb{N} \setminus \{0\},
$$

i.e. $U^n_\tau$ satisfies

$$
\frac{1}{2\tau} d^2(U^{n-1}_\tau, U^n_\tau) + \phi(U^n_\tau) \leq \frac{1}{2\tau} d^2(U^{n-1}_\tau, V) + \phi(V) \quad \forall V \in X.
$$

The corresponding **discrete minimizing movement** is the piecewise-constant interpolant

$$
\overline{U}_\tau(t) := U^n_\tau \quad \text{if } t \in ((n - 1)\tau, n\tau], \quad \overline{U}_\tau(0) = U^0_\tau \approx u_0.
$$

Following [De Giorgi ’93, Almgren-Taylor-Wang ’93, Jordan-Kinderlehrer-Otto ’98], the MM method can be used to *construct* the gradient flow of $\phi$. However, **without coercivity** assumptions on $\phi$, one cannot hope to have **exact** minimizers.
Ekeland's variational principle and relaxed MM

**Ekeland's variational principle**

Let $\Phi : X \to (-\infty, +\infty]$ be a l.s.c. functional bounded from below. Then for every $U \in \text{Dom}(\Phi)$ and every $\eta > 0$ there exists $U_\eta \in \text{Dom}(\Phi)$ s.t.

$$
\Phi(U_\eta) \leq \Phi(U) - \eta \, \text{d}(U_\eta, U)
$$

$$
\Phi(U_\eta) < \Phi(V) + \eta \, \text{d}(U_\eta, V) \quad \text{for every } V \in X \setminus \{U_\eta\}. 
$$

In particular,

$$
|\partial \Phi|(U_\eta) \leq \mathcal{L}_0[\Phi](U_\eta) \leq \eta.
$$

Our idea is to apply Ekeland's variational principle to the functional

$$
V \mapsto \Phi(\tau, U_{\tau, \eta}^{n-1}, V) = \frac{1}{2\tau} \text{d}^2(U_{\tau, \eta}^{n-1}, V) + \phi(V).
$$

By letting $U \equiv U_{\tau, \eta}^{n-1}$ and choosing the above $\eta$ carefully, we can find $U_{\tau, \eta}^n$ satisfying

$$
\frac{1}{2\tau} \text{d}^2(U_{\tau, \eta}^{n-1}, U_{\tau, \eta}^n) + \phi(U_{\tau, \eta}^n) \leq \frac{1}{2\tau} \text{d}^2(U_{\tau, \eta}^{n-1}, V) + \phi(V) + \frac{\eta}{2} \text{d}(U_{\tau, \eta}^{n-1}, U_{\tau, \eta}^n) \, \text{d}(U_{\tau, \eta}^n, V)
$$

for every $V \in X$ and

$$
\frac{1}{2\tau} \text{d}^2(U_{\tau, \eta}^{n-1}, U_{\tau}^n) + \phi(U_{\tau, \eta}^n) \leq \phi(U_{\tau, \eta}^{n-1}).
$$
Two key inequalities satisfied by $\eta$-Ekeland movements

We denote by $\overline{U}_{\tau,\eta}$ the piecewise-constant interpolant of the $\eta$-Ekeland sequence $\{U_{\tau,\eta}^n\}_{n \in \mathbb{N}}$, which we call a discrete $\eta$-Ekeland movement.

In order to generate such a movement, we only need $\phi$ to be a l.s.c. functional quadratically bounded from below ($\tau$ small enough).

Let $\phi : X \to (-\infty, +\infty]$ be a l.s.c. $\lambda$-convex functional ($\lambda \leq 0$). Then, for any $\eta$-Ekeland sequence $\{U_{\tau,\eta}^n\}$ there hold

$$\tau \left(1 - \frac{\eta}{2} \tau\right)^2 |\partial \phi|^2(U_{\tau,\eta}^n) \leq \frac{d^2(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n)}{\tau}$$

and

$$\left(1 - \frac{\eta - \lambda}{2} \tau\right) \frac{d^2(U_{\tau,\eta}^{n-1}, U_{\tau,\eta}^n)}{\tau} \leq \phi(U_{\tau,\eta}^{n-1}) - \phi(U_{\tau,\eta}^n).$$

Such inequalities are closely related to the energy identity satisfied by solutions to EVI.
By exploiting the above inequalities plus the EVI properties, we can prove the following.

**Theorem**

Let $\phi : X \to (-\infty, +\infty]$ be a l.s.c. $\lambda$-convex functional ($\lambda \leq 0$), which admits a $\lambda$-Gradient Flow. Fix a time interval $[0, T]$ and $\tau \in (0, T)$. Then, if $U_{\tau, \eta}^0 = u_0 \in \text{Dom}(|\partial \phi|)$, there exists a constant $C = C(T, \lambda, \eta) > 0$ such that

$$d(u_t, \overline{U}_{\tau, \eta}(t)) \leq C |\partial \phi|(u_0) \sqrt{\tau} \quad \forall t \in [0, T],$$

whence $\overline{U}_{\tau, \eta}(t) \to u_t$ as $\tau \downarrow 0$ with rate $\sqrt{\tau}$ (at least).

Thus, the minimizing movement (limit of $\overline{U}_{\tau, \eta}(t)$ as $\tau \downarrow 0$) exists and coincides with $u_t$. 
We consider the delicate problem of stability w.r.t. $\phi$.

That is, let $\phi^h : X \to (-\infty, +\infty]$, $h \in \mathbb{N}$, be a family of l.s.c. functionals “converging” in a suitable sense as $h \to \infty$ to a l.s.c. functional $\phi : X \to (-\infty, +\infty]$.

We suppose that each $\phi^h$ admits a $\lambda$-Gradient Flow $S^h$ (except $\phi$).

The crucial questions

(Under which assumptions) Can we deduce that

also $\phi$ admits a $\lambda$-Gradient Flow $S$

and that

$S^h_t(u^h_0)$ converges to $S_t(u_0)$ as $h \to \infty$, if $u^h_0 \to u_0$?
Having in mind the Hilbert case, natural assumptions involve $\Gamma$-convergence [Dal Maso '93]. We recall the definitions of $\Gamma$-lim inf and $\Gamma$-lim sup of a sequence $\{\phi^h\}_{h \in \mathbb{N}}$:

\[
\Gamma\text{-lim inf } \phi^h(x) := \inf \left\{ \liminf_{h \to \infty} \phi^h(x^h) : x^h \to x \right\} = \lim_{r \downarrow 0} \liminf_{h \to \infty} \inf_{B_r(x)} \phi^h,
\]

\[
\Gamma\text{-lim sup } \phi^h(x) := \inf \left\{ \limsup_{h \to \infty} \phi^h(x^h) : x^h \to x \right\} = \lim_{r \downarrow 0} \limsup_{h \to \infty} \inf_{B_r(x)} \phi^h,
\]

for all $x \in X$. If the $\Gamma$-lim inf and the $\Gamma$-lim sup coincide, we set

\[
\phi = \Gamma\text{-lim } \phi^h = \Gamma\text{-lim inf } \phi^h = \Gamma\text{-lim sup } \phi^h,
\]

in which case we say that $\{\phi^h\}$ $\Gamma$-converges to $\phi$. This is equivalent to

\[
\forall x \in X, \ x^h \to x \ \Rightarrow \ \liminf_{h \to \infty} \phi^h(x^h) \geq \phi(x) \quad \text{(*)}
\]

\[
\forall x \in X \ \exists \{x^h\} : \ x^h \to x, \ \phi^h(x^h) \to \phi(x).
\]

If $X$ is Hilbert one also has weak topology. We say that $\{\phi^h\}$ Mosco-converges to $\phi$ if it $\Gamma$-converges w.r.t. both the strong and the weak topology, i.e. (*) holds for all $x^h \to x$. 
The stability result in the Hilbert case

Theorem (Crandall, Liggett, Bénilan, Pazy, Attouch – mostly during the 70’s)

Let $X$ be a Hilbert space and $\{\phi^h\}_{h \in \mathbb{N}} \cup \{\phi\}$ be a sequence of l.s.c. and convex functionals. Let $A^h := \partial \phi^h$ and $A := \partial \phi$. Then the following properties are equivalent:

- **Convergence of the flows:** if $u^h_0 \to u_0 \in \text{Dom}(\phi)$, with $u^h_0 \in \text{Dom}(\phi^h)$,
  \[ \lim_{h \to \infty} S^h_t(u^h_0) = S_t(u_0) \quad \forall t \geq 0. \]

- **Convergence of the resolvents:** for every $u \in X$ and $\tau > 0$
  \[ \lim_{h \to \infty} (I + \tau A^h)^{-1} u = (I + \tau A)^{-1} u. \]

- **Convergence of the Moreau-Yosida regularizations:** for every $u \in X$ and $\tau > 0$
  \[ \lim_{h \to \infty} \inf_{v \in X} \phi^h(v) + \frac{1}{2\tau} d^2(v, u) = \inf_{v \in X} \phi(v) + \frac{1}{2\tau} d^2(v, u). \]

- **Mosco-convergence of the functionals:** $\{\phi^h\}$ Mosco-converges to $\phi$.

- **G-convergence of the subgradients:** for every $v \in A(u)$ there exist $\{u^h\}, \{v^h\}$ s.t.
  \[ v^h \in A^h u^h, \quad u^h \to u, \quad v^h \to v. \]
Some related remarks

- Mosco-limits of convex functionals are convex: in particular, $S$ exists thanks e.g. to the Crandall-Liggett Theorem (without assuming \textit{a priori} the convexity of $\phi$).
- For every $v \in A(u)$ one can construct a recovery sequence $v^h \in A^h(u^h)$ s.t.
  $$u^h \to u, \quad v^h \to v, \quad \phi^h(u^h) \to \phi(u).$$
- If $\{\phi^h\}$ is strongly coercive (bdd sequences $\{x^h\}$ s.t. $\phi^h(x^h) \leq C$ are rel. compact), then Mosco convergence $\Leftrightarrow \Gamma$-convergence. Otherwise, limits of $\phi^h(x^h)$ along weakly convergent sequences are involved, whence the weak $\Gamma$-lim inf.
- The resolvent operator is strictly related to MM:
  $$U^{n,h}_t = (1 + \tau A^h)^{-1} U^{n-1,h}_t.$$
- In order to prove convergence of the flows, it is therefore convenient to exploit convergence of the minimizing movements along with uniform error estimates:
  $$d(u^h_t, u_t) \leq d(u^h_t, \bar{U}^h_t(t)) + d(\bar{U}^h_t(t), \bar{U}_\tau(t)) + d(\bar{U}_\tau(t), u_t),$$
  where $u^h_t := S^h_t(u^h_0)$ and $u_t := S_t(u_0)$. 
Additional difficulties due to the abstract metric setting

- We do not know a priori whether the limit $\lambda$-Gradient Flow $S$ exists.
- Resolvents are not well defined: one should use $\eta$-Ekeland movements instead.
- A natural weak topology is missing.
- We would like to study stability without strong-coercivity assumptions.
- On the other hand, if we lack coercivity, minimizing movements (a fortiori $\eta$-Ekeland movements) are not stable under $\Gamma$-convergence.

We point out that, at least in the strongly coercive case, it is possible to pass to the limit in the integral version of the EVI:

$$
\frac{e^{\lambda(t-s)}}{2} d^2(u^h_t, v^h) - \frac{1}{2} d^2(u^h_s, v^h) \leq E_\lambda(t - s) \left( \phi^h(v^h) - \phi(u^h_t) \right),
$$

for every $0 \leq s \leq t$ and $v^h \in \text{Dom}(\phi^h)$, which yields existence of $S$ “for free”.
The main stability result

Theorem

Let \( \{ \phi^h \}_{h \in \mathbb{N}} \cup \{ \phi \} \) be a sequence of l.s.c. functionals. Let each \( \phi^h \) admit a \( \lambda \)-Gradient Flow \( S^h \) and let \( \phi \) be \( \lambda \)-convex. The following claims are equivalent:

Convergence of the flows: also \( S \) exists and if \( u^h_0 \to u_0 \in \overline{\text{Dom}(\phi^\infty)} \), \( u^h_0 \in \overline{\text{Dom}(\phi^h)} \),

\[
\lim_{h \to \infty} S^h_t(u^h_0) = S_t(u_0) \quad \forall t \geq 0.
\]

Recovery sequence: for every \( u \in \text{Dom}(|\partial\phi|) \) there exists \( u^h \in \text{Dom}(|\partial\phi^h|) \) s.t.

\[
u^h \to u, \quad \phi^h(u^h) \to \phi(u), \quad \limsup_{h \to \infty} |\partial\phi^h|(u^h) \leq |\partial\phi|(u).
\]

\( \Gamma \)-convergence of \( \phi^h \) and \( |\partial\phi^h| \): \( \phi = \Gamma \)-lim \( \phi^h \) and \( |\partial\phi| = \Gamma \)-lim \( |\partial\phi^h| \) in \( \overline{\text{Dom}(\phi)} \).

Qualified \( \Gamma \)-convergence: \( \Gamma \)-lim sup \( \phi^h \leq \phi \) in \( \text{Dom}(|\partial\phi|) \) and for every \( u \in \text{Dom}(|\partial\phi|) \), \( \varepsilon > 0 \) and \( \bar{\tau} > 0 \), there exists \( \tau \in (0, \bar{\tau}) \) s.t.

\[
\liminf_{h \to \infty} \inf_{B_{\tau}(u)} \phi^h \geq \inf_{B_{\tau}(u)} \phi - \varepsilon \tau.
\]

Local Moreau-Yosida regularizations: \( \Gamma \)-lim sup \( \phi^h \leq \phi \) in \( \text{Dom}(|\partial\phi|) \) and for every \( u \in \text{Dom}(|\partial\phi|) \), \( \varepsilon > 0 \) and \( \bar{\tau} > 0 \), there exists \( \tau \in (0, \bar{\tau}) \) s.t.

\[
\liminf_{h \to \infty} \inf_{v \in X} \phi^h(v) + \frac{1}{2\tau} d^2(v, u) \geq \inf_{v \in X} \phi(v) + \frac{1}{2\tau} d^2(v, u) - \varepsilon \tau.
\]
Strategy of proof of the existence of the limit flow

- We generate a $\eta$-Ekeland sequence $\{U^n_{\tau,\eta}\}$ for $\phi$, which satisfies

$$
\tau \left(1 - \frac{\eta}{2}\tau\right)^2 |\partial\phi|^2(U^n_{\tau,\eta}) \leq \frac{d^2(U^{n-1}_{\tau,\eta}, U^n_{\tau,\eta})}{\tau} \leq \frac{\phi(U^{n-1}_{\tau,\eta}) - \phi(U^n_{\tau,\eta})}{1 - \frac{\eta - \lambda}{2}\tau}.
$$


- We exploit $\Gamma$-convergence of $\phi^h$ and $|\partial\phi^h|$ to approximate $U^n_{\tau,\eta}$ by sequences $U^{n,h}_{\tau,\eta}$ satisfying, for large $h$, the $\varepsilon$-version of ($\star$):

$$
\lim_{h \to \infty} \sup_{t \in [0, T]} d(\bar{U}_{\tau,\eta}(t), \bar{U}^h_{\tau,\eta}(t)) = 0.
$$

- We use the discrete-approximation error estimate, which yields

$$
d(u^h_t, \bar{U}^h_{\tau,\eta}(t)) \leq C \left(|\partial\phi^h|(u^h_0) \sqrt{\tau} + \sqrt{\varepsilon/\tau}\right) \quad \forall t \in [0, T].
$$

- By combining the two estimates and choosing $U^{0,h}_{\tau,\eta}$ appropriately, we deduce that

$$
\limsup_{h,k \to \infty} \sup_{t \in [0, T]} d(u^h_t, u^k_t) \leq C' \left(\sqrt{\tau} + \sqrt{\varepsilon/\tau}\right),
$$

which shows that $\{u^t_h\}_h$ is Cauchy, since $\tau > 0$ and $\varepsilon > 0$ are arbitrary.
An application to RCD spaces

Let $(X, d, m)$ be an RCD($\lambda$, $\infty$) metric measure space and let $\psi : X \to (-\infty, +\infty]$ be a continuous and geodesically $\lambda$-convex functional.

**Theorem (Sturm ’14)**

If $(X, d)$ is locally compact then $\psi$ admits a $\lambda$-Gradient Flow.

**Corollary of our results**

The local-compactness assumption can be removed.

Indeed, Sturm’s proof relies on the construction of the $\lambda$-GF for the functional

$$\phi(\mu) := \int_X \psi \, d\mu \quad \text{in} \ (\mathcal{P}_2(X), W_2)$$

by means of the approximations $\phi^h(\mu) := \phi(\mu) + \frac{1}{h} \text{Ent}(\mu|m)$. At least when $m \in \mathcal{P}(X)$, one can check that the assumptions of our main stability result are met.
Some extensions concerning the stability result

- Completeness of $X$ can be dropped: we only need $\phi$ to have complete sublevels.
- Convexity of $\phi$ can, to some extent, be relaxed: if $\text{Dom}(\phi)$ is geodesic, then it is just a consequence of the existence of the flows for $\phi^h$.
- Alternatively, it is enough to ask that $\phi$ is approximately $\lambda$-convex, namely that for every $x_0, x_1 \in \text{Dom}(\phi)$ and every $\vartheta, \varepsilon \in (0, 1)$ there exists $x_{\vartheta, \varepsilon} \in \text{Dom}(\phi)$ s.t.

$$\phi(x_{\vartheta, \varepsilon}) \leq (1 - \vartheta)\phi(x_0) + \vartheta \phi(x_1) - \frac{\lambda - \varepsilon}{2} \vartheta (1 - \vartheta) d^2(x_1, x_0)$$

and

$$d(x_{\vartheta, \varepsilon}, x_0) \leq \vartheta d(x_1, x_0) + \varepsilon, \quad d(x_{\vartheta, \varepsilon}, x_1) \leq (1 - \vartheta) d(x_1, x_0) + \varepsilon.$$
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- Alternatively, it is enough to ask that $\phi$ is approximately $\lambda$-convex, namely that for every $x_0, x_1 \in \text{Dom}(\phi)$ and every $\vartheta, \varepsilon \in (0, 1)$ there exists $x_{\vartheta, \varepsilon} \in \text{Dom}(\phi)$ s.t.

$$
\phi(x_{\vartheta, \varepsilon}) \leq (1 - \vartheta)\phi(x_0) + \vartheta\phi(x_1) - \frac{\lambda - \varepsilon}{2} \vartheta(1 - \vartheta)d^2(x_1, x_0)
$$

and

$$
d(x_{\vartheta, \varepsilon}, x_0) \leq \vartheta d(x_1, x_0) + \varepsilon, \hspace{1cm} d(x_{\vartheta, \varepsilon}, x_1) \leq (1 - \vartheta)d(x_1, x_0) + \varepsilon.
$$

THANK YOU FOR YOUR ATTENTION!