Minimal k-partition for the p-norm of the eigenvalues

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Notation

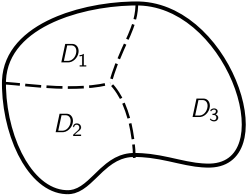
- $\Omega \subset \mathbf{R}^2$: bounded and connected domain

- $0 < \lambda_1(D) < \lambda_2(D) \leqslant \cdots$ eigenvalues of the Dirichlet-Laplacian on D

- $\mathcal{D} = (D_i)_{i=1,\ldots,k}$: k-partition of Ω

(i.e. D_i open, $D_i \cap D_j = \emptyset$, and $\cup D_i \subset \Omega$)

strong if $\mathrm{Int}\overline{D_i} \setminus \partial\Omega = D_i$ and $(\overline{\cup D_i}) \setminus \partial\Omega = \Omega$



- $\mathfrak{O}_k(\Omega) = \{$strong k-partitions of $\Omega\}$

Minimal k-partition

$$\mathfrak{L}_{k,\infty}(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k(\Omega)} \max_{1\leqslant i\leqslant k} \lambda_1(D_i)$$

[Conti–Terracini–Verzini, Helffer–Hoffmann-Ostenhof–Terracini, BN–Helffer–Vial, BN–Léna, ...]

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$$\mathfrak{L}_{k,1}(\Omega) = \inf_{\mathcal{D}\in\mathfrak{O}_k(\Omega)} \frac{1}{k}\sum_{i=1}^{k} \lambda_1(D_i)$$

[Bucur–Buttazzo-Henrot, Caffarelli–Lin, Bourdin–Bucur–Oudet, ...]

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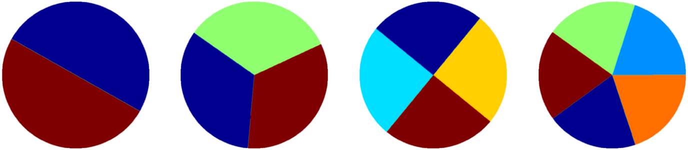

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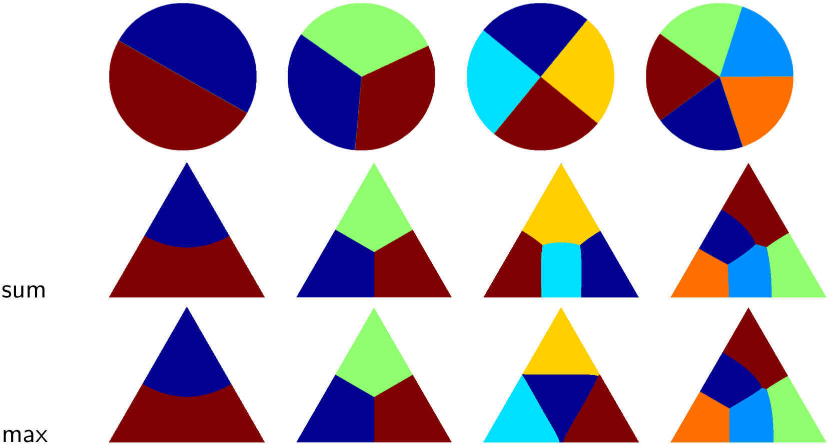

p-minimal k-partition

Definitions

- p-energy

$\mathcal{D} = (D_1, \ldots, D_k)$: k-partition of Ω

$$\boxed{\Lambda_{k,p}(\mathcal{D}) = \frac{1}{k^{1/p}} \Big\| \big(\lambda_1(D_1), \ldots, \lambda_1(D_k)\big) \Big\|_p}$$

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p	$p=1$	$1 \leq p < +\infty$	$p = +\infty$
$\Lambda_{k,p}(\mathcal{D})$	$\frac{1}{k}\sum_{i=1}^{k} \lambda_1(D_i)$	$\left(\frac{1}{k}\sum_{i=1}^{k} \lambda_1(D_i)^p\right)^{1/p}$	$\max_{1 \leqslant i \leqslant k} \lambda_1(D_i)$

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- $\mathcal{D}^*$ is called a p-minimal k-partition if $\Lambda_{k,p}(\mathcal{D}^*) = \mathfrak{L}_{k,p}(\Omega)$

Dimension 1

∞-minimal k-partition

Let $\Omega = (a, b)$, $p = \infty$

- $\lambda_1(\Omega) = \dfrac{\pi^2}{(b-a)^2} = \dfrac{\pi^2}{|\Omega|^2}$

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- Then

$$\mathfrak{L}_{k,\infty}(\Omega) = \frac{k^2\pi^2}{(b-a)^2}$$

and the equipartition $\mathcal{D}^* = (D_1, \ldots, D_k)$ is minimal

with $D_i = (a + (i-1)h, a + ih)$, $h = \frac{b-a}{k}$

Dimension 1

Nodal partition

Let $\Omega = (a, b)$

- k-th eigenvalue: $\lambda_k(\Omega) = \dfrac{k^2\pi^2}{(b-a)^2} = \dfrac{k^2\pi^2}{|\Omega|^2}$

$$\Longrightarrow \qquad \mathfrak{L}_{k,\infty}(\Omega) = \lambda_k(\Omega)$$

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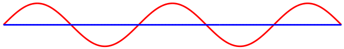
- k-th eigenfunctions

$$u_k(x) = \sin\left(k\pi\frac{x-a}{b-a}\right), \qquad \forall x \in \Omega$$

u_5 :



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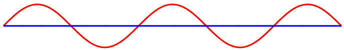
u_5 : 

- Any nodal partition associated with $\lambda_k(\Omega)$ gives a minimal k-partition

p-minimal k-partition

Existence of minimal partition

Theorem

For any $k \geq 1$ *and* $p \in [1, +\infty]$,

there exists a regular strong p-minimal k-partition

[*Bucur–Buttazzo-Henrot, Caffarelli–Lin, Conti–Terracini–Verzini, Helffer–Hoffmann-Ostenhof–Terracini*]

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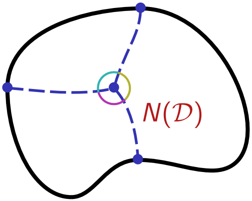

$$N(\mathcal{D}) = \overline{\cup(\partial D_i \cap \Omega)}$$

Regular : $N(\mathcal{D})$ is smooth curve except at finitely many points and

- $N(\mathcal{D}) \cap \partial\Omega$ is finite (boundary singular points)
- $N(\mathcal{D})$ satisfies the Equal Angle Property

Monotonicity

Let $k \geq 1$ and $1 \leq p \leq q < \infty$

- With respect to the domain

$$\Omega \subset \tilde{\Omega} \quad \Rightarrow \quad \mathfrak{L}_{k,p}(\tilde{\Omega}) \leq \mathfrak{L}_{k,p}(\Omega)$$

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- With respect to k

$$\mathfrak{L}_{k,p}(\Omega) < \mathfrak{L}_{k+1,p}(\Omega)$$

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- With respect to p

$$\frac{1}{k^{1/p}} \Lambda_{k,\infty}(\mathcal{D}) \leq \Lambda_{k,p}(\mathcal{D}) \leq \Lambda_{k,q}(\mathcal{D}) \leq \Lambda_{k,\infty}(\mathcal{D})$$

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$$\frac{1}{k^{1/p}}\Lambda_{k,\infty}(\mathcal{D}) \leq \Lambda_{k,p}(\mathcal{D}) \leq \Lambda_{k,q}(\mathcal{D}) \leq \Lambda_{k,\infty}(\mathcal{D})$$

$$\frac{1}{k^{1/p}}\mathfrak{L}_{k,\infty}(\Omega) \leq \mathfrak{L}_{k,p}(\Omega) \leq \mathfrak{L}_{k,q}(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega)$$

Equipartition

Proposition

- *If* $\mathcal{D}^* = (D_i)_{1\leq i\leq k}$ *is a* ∞*-minimal* k*-partition, then* $\mathcal{D}^*$ *is an equipartition*

$$\lambda_1(D_i) = \mathfrak{L}_{k,\infty}(\Omega)\,, \qquad \textit{for any} \quad 1 \leq i \leq k$$

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- *Let* $p\geq 1$ *and* $\mathcal{D}^*$ *a* p*-minimal* k*-partition*
  If $\mathcal{D}^*$ *is an equipartition, then*

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$$\mathfrak{L}_{k,q}(\Omega) = \mathfrak{L}_{k,p}(\Omega), \qquad \textit{for any} \quad q\geq p$$

We set

$$p_\infty(\Omega,k) = \inf\{p\geq 1, \mathfrak{L}_{k,p}(\Omega) = \mathfrak{L}_{k,\infty}(\Omega)\}$$

Lower bound

- Rayleigh quotient

$$\mathcal{Q}(u) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}, \qquad \forall u \in H^1_0(\Omega)$$

- Min-max principle

$$\lambda_k(\Omega) = \max_{u_1,\dots,u_{k-1}} \min \left\{ \mathcal{Q}(u), u \in H^1_0(\Omega), u \in [u_1, \dots, u_{k-1}]^\perp \right\}$$

$$\Longrightarrow \quad \lambda_k(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega)$$

Nodal partitions

Let u be an eigenfunction of $-\Delta$ on Ω

- The nodal sets of u are the components of

$$\Omega \setminus N(u) \qquad \text{with} \qquad N(u) = \overline{\{x \in \Omega |\, u(x) = 0\}}$$

- The partition composed by the nodal sets is called nodal partition

Regularity

$N(u)$ is a $\mathcal{C}^\infty$ curve except on some critical points $\{x\}$
If $x \in \Omega$, $N(u)$ is locally the union of an **even** number of half-curves ending at x with equal angle
If $x \in \partial\Omega$, $N(u)$ is locally the union of half-curves ending at x with equal angle

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Theorem

Any eigenfunction u associated with $\lambda_k(\Omega)$ has at most k nodal domains

[*Courant*]

u is said Courant-sharp if it has exactly k nodal domains

Nodal partitions

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u is said Courant-sharp if it has exactly k nodal domains For $k \geq 1$,

$L_k(\Omega)$ denotes the smallest eigenvalue (if any) for which there exists an eigenfunction with k nodal domains

We set $L_k(\Omega) = +\infty$ if there is no eigenfunction with k nodal domains

$$\Rightarrow \qquad \lambda_k(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega) \leq L_k(\Omega)$$

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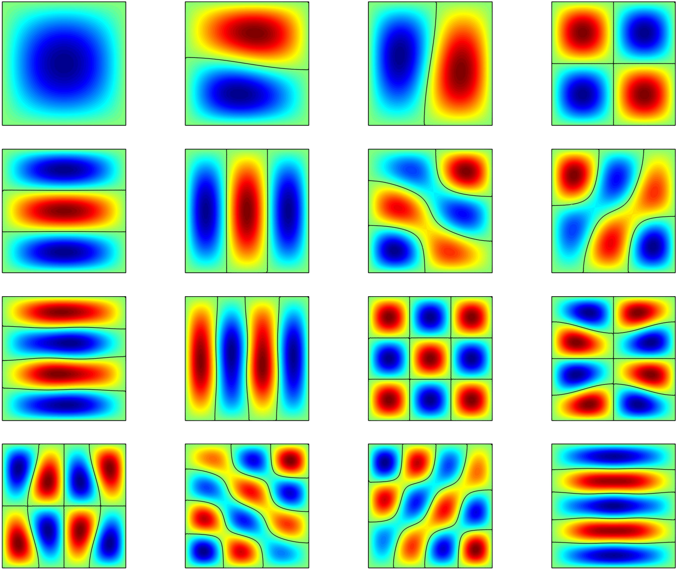
For $k \geq 1$, $\tilde{L}_k(\Omega)$ denotes the smallest eigenvalue for which there exists an eigenfunction with *at leat* k nodal domains

$$\Rightarrow \qquad \lambda_k(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega) \leq \tilde{L}_k(\Omega) \leq L_k(\Omega)$$

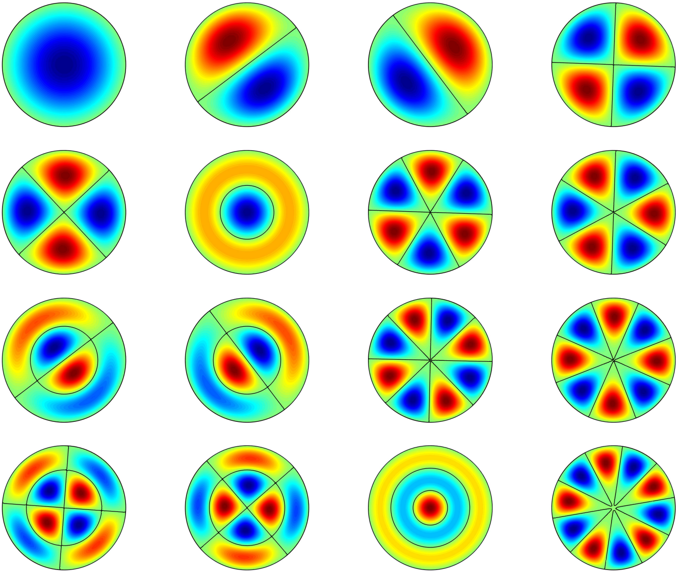
Nodal partitions

Square



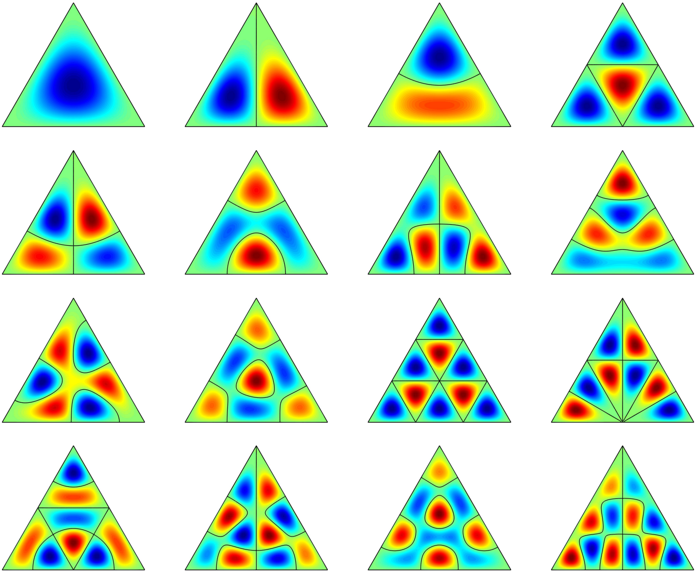
Nodal partitions

Disk



Nodal partitions

Equilateral triangle



Bounds

Theorem

$$\lambda_k(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega) \leq L_k(\Omega)$$

If $\mathfrak{L}_{k,\infty} = L_k$ *or* $\mathfrak{L}_{k,\infty} = \lambda_k$*, then* $\lambda_k(\Omega) = \mathfrak{L}_{k,\infty}(\Omega) = L_k(\Omega)$ *with a Courant sharp eigenfunction associated with* $\lambda_k(\Omega)$

[*Helffer–Hoffmann-Ostenhof–Terracini*]

Bounds

Theorem

$$\lambda_k(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega) \leq L_k(\Omega)$$

If $\mathfrak{L}_{k,\infty} = L_k$ or $\mathfrak{L}_{k,\infty} = \lambda_k$, then $\lambda_k(\Omega) = \mathfrak{L}_{k,\infty}(\Omega) = L_k(\Omega)$ with a Courant sharp eigenfunction associated with $\lambda_k(\Omega)$

[*Helffer–Hoffmann-Ostenhof–Terracini*]

Theorem

- *There exists k_0 such that $\lambda_k < L_k$ for $k \geq k_0$* [*Pleijel*]
- *Explicit upper-bound for k_0* [*Bérard-Helffer 16, van den Berg-Gittins 16*]

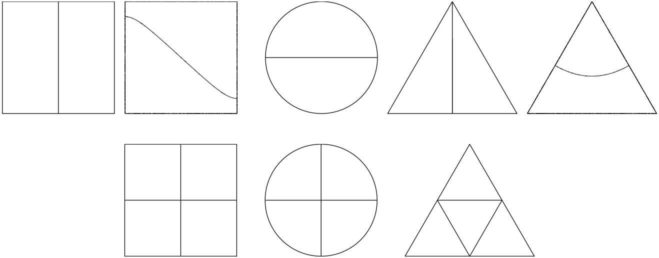
Examples

Minimal nodal partitions

- Let $\Omega = \square$, $\bigcirc$ or $\triangle$,

$$\lambda_k(\Omega) = \mathfrak{L}_{k,\infty}(\Omega) = L_k(\Omega) \qquad \text{iff} \qquad k = 1, 2, 4$$

Minimal nodal partitions



∞-minimal 2-partition

Theorem

$$\mathfrak{L}_{2,\infty}(\Omega) = \lambda_2(\Omega)$$

Any nodal partition associated with $\lambda_2(\Omega)$ is minimal

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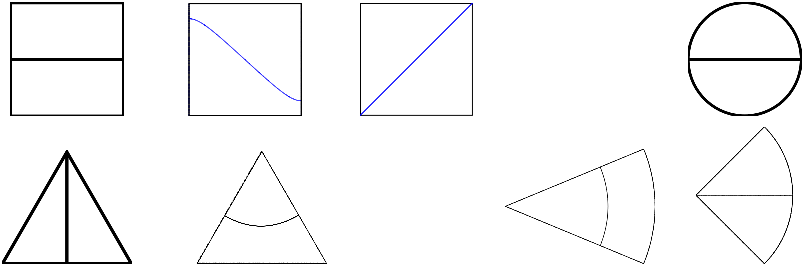


2-partition

$p = 1$ - $p = \infty$

Proposition

Let $\mathcal{D} = (D_1, D_2)$ be a ∞-minimal 2-partition of Ω
Suppose that there exists a second eigenfunction φ_2 of $-\Delta$ on Ω having D_1 and D_2 as nodal domains and such that

$$\int_{D_1} |\varphi_2|^2 \neq \int_{D_2} |\varphi_2|^2$$

Then

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

[*Helffer–Hoffman-Ostenhof*]

2-partition

$p = 1$ - $p = \infty$

Applications

Let $\mathcal{D} = (D_i)_{1 \le i \le k}$ be a ∞-minimal k-partition

Let $D_i \sim D_j$ be a pair of neighbors. We denote

$$D_{ij} = \mathrm{Int}\,\overline{D_i \cup D_i}$$

- $\lambda_2(D_{ij}) = \mathfrak{L}_{2,\infty}(D_{ij}) = \mathfrak{L}_{k,\infty}(\Omega)$
- Suppose that there exists a second eigenfunction φ_{ij} of $-\Delta$ on D_{ij} having D_i and D_j as nodal domains and such that

$$\int_{D_i} |\varphi_{ij}|^2 \neq \int_{D_j} |\varphi_{ij}|^2$$

Then

$$\mathfrak{L}_{k,1}(\Omega) < \Lambda_{k,\infty}(\mathcal{D})$$

[*Helffer–Hoffman-Ostenhof*]

2-partition

$p = 1$

- $\Omega = \square$?

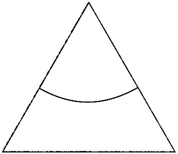
- $\Omega = \bigcirc$?

2-partition

$p = 1$

- $\Omega = \square, \ \bigcirc$?

- $\Omega = \triangle$



φ_2: symmetric eigenfunction associated with $\lambda_2(\Omega)$

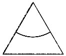
$$0.495 \simeq \int_{D_1} |\varphi_2|^2 < \int_{D_2} |\varphi_2|^2 \simeq 0.505$$

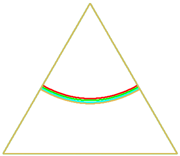
$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

2-partition

$p = 1$

- $\Omega = \square, \bigcirc$?

-  is a ∞-minimal 2-partition but not a 1-minimal 2-partition



2-partition

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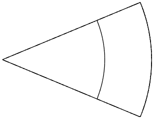
- $\Omega = \square, \ \bigcirc$?

-  is a ∞-minimal 2-partition but not a 1-minimal 2-partition

- Angular sector with opening $\pi/4$

φ_2: symmetric eigenfunction associated with $\lambda_2(\Omega)$



$$0.37 \simeq \int_{D_1} |\varphi_2|^2 < \int_{D_2} |\varphi_2|^2 \simeq 0.63$$

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

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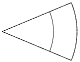
-  is a ∞-minimal 2-partition but not a 1-minimal 2-partition

- The inequality $\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$ is "generically" satisfied

[*Helffer–Hoffmann-Ostenhof*]

Lower bounds

Square, equilateral triangle, disk

$$\left(\frac{1}{k}\sum_{i=1}^{k}\lambda_i(\Omega)^p\right)^{1/p} \leq \mathfrak{L}_{k,p}(\Omega) \leq L_k(\Omega)$$

Explicit eigenvalues for $\square$, $\triangle$, $\bigcirc$

Ω	$\lambda_{m,n}(\Omega)$	m, n
$\square$	$\pi^2(m^2+n^2)$	$m, n \geq 1$
$\triangle$	$\frac{16}{9}\pi^2(m^2+mn+n^2)$	$m, n \geq 1$
$\bigcirc$	$j_{m,n}^2$	$m \geq 0,\ n \geq 1$ (multiplicity)

Properties

Dichotomy for the case $p = \infty$

Let $k > 2$
To determine a ∞-minimal k-partition,
we consider the eigenspace E_k associated with λ_k

Two cases:

- If there exists $u \in E_k$ with k nodal domains, then u produces a minimal k-partition and any minimal k-partition is nodal

$$\mathfrak{L}_{k,\infty}(\Omega) = \lambda_k(\Omega) = L_k(\Omega)$$

[Bipartite case]

- If $\mu(u) < k$ for any $u \in E_k$...

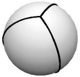
...we have to find another strategy
[Non bipartite case]

Known results in the non bipartite case, $p = \infty$

Sphere and fine flat torus

Theorem

The minimal 3-partition for the sphere is



[*Helffer–Hoffmann-Ostenhof–Terracini*]

Theorem

Let $0 < b \leq a$ *and* $T(a,b) = (\mathbf{R}/a\mathbf{Z}) \times (\mathbf{R}/b\mathbf{Z})$ *the flat torus*

$$\mathcal{D}_k(a,b) = \{]\tfrac{i-1}{k}a, \tfrac{i}{k}a[\times]0,b[, 1 \leq i \leq k \}$$

- k *even and* $\frac{b}{a} \leq \frac{2}{k} \Rightarrow \mathcal{D}_k(a,b)$ *is minimal*
- k *odd and* $\frac{b}{a} < \frac{1}{k} \Rightarrow \mathcal{D}_k(a,b)$ *is minimal* [*Helffer–Hoffmann-Ostenhof*]
- k *odd and* $\frac{1}{k} \leq \frac{b}{a} \leq \ell_* \Rightarrow \mathcal{D}_k(a,b)$ *is minimal* [*BN-Léna 16*]

The question is open for any other domain (in the non bipartite case)

Topological configurations

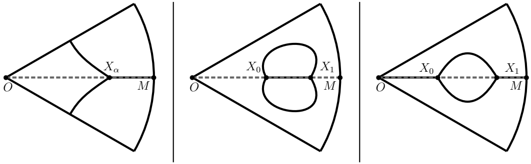
Euler formula

$$k = 1 + b_1 - b_0 + \sum_{\mathbf{x}_i \in X(\partial\mathcal{D})} \left(\frac{\nu(\mathbf{x}_i)}{2} - 1 \right) + \frac{1}{2} \sum_{\mathbf{y}_i \in Y(\partial\mathcal{D})} \rho(\mathbf{y}_i)$$

with

b_0	number of components of $\partial\Omega$
b_1	number of components of $\partial\mathcal{D} \cup \partial\Omega$
$\nu(\mathbf{x}_i)$	number of curves ending at $\mathbf{x}_i \in X(\partial\mathcal{D})$
$\rho(\mathbf{y}_i)$	number of curves ending at $\mathbf{y}_i \in Y(\partial\mathcal{D})$

$\Rightarrow$ 3 types of configurations

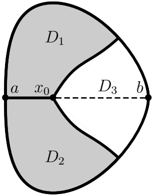


Question

If Ω is symmetric, does it exist a symmetric minimal 3-partition ?

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis



$\mathcal{D} = (D_1, D_2, D_3)$ minimal 3-partition

$\Rightarrow (D_1, D_3)$ minimal 2-partition for $\mathrm{Int}(\overline{D_1} \cup \overline{D_3})$

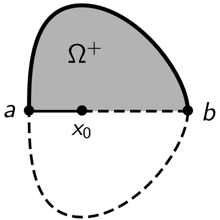
$\Rightarrow$ nodal partition on $\mathrm{Int}(\overline{D_1} \cup \overline{D_3})$

[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



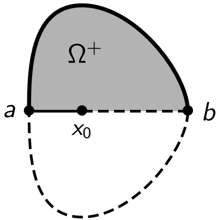
$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

[BN–Helffer–Vial 10]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

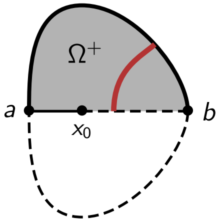
- $(\lambda_2(x_0), \varphi_{x_0})$ second eigenmode
- $x_0 \mapsto \lambda_2(x_0)$ is increasing
- the nodal line starts from (a, b) and reaches the boundary

[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



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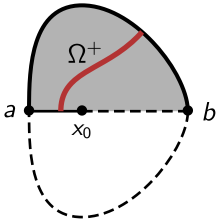
[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

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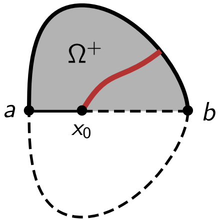
[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



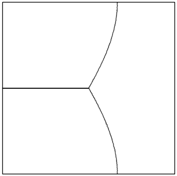
$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

- $(\lambda_2(x_0), \varphi_{x_0})$ second eigenmode
- $x_0 \mapsto \lambda_2(x_0)$ is increasing
- the nodal line starts from (a, b) and reaches the boundary

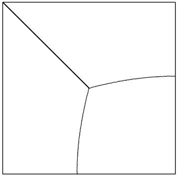
[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

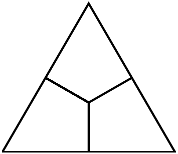
First configuration: examples



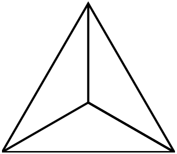
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 66.581$



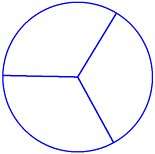
$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 66.581$

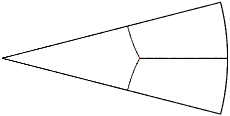


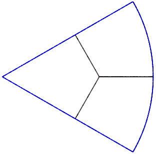
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 61.872$

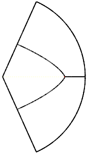


$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 93.156$



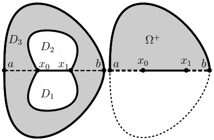






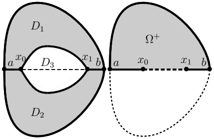
Non bipartite symmetric ∞-minimal 3-partition

Second and third configurations: Two critical points on the symmetry axis



Mixed Neumann-Dirichlet-Neumann problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [a, x_0] \cup [x_1, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$



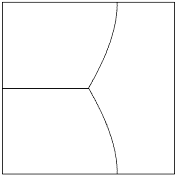
Mixed Dirichlet-Neumann-Dirichlet problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, x_1] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

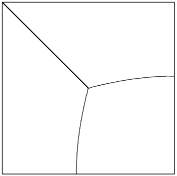
No candidate for the square, disk, angular sectors with two critical points!

∞-minimal 3-partition

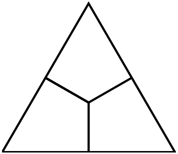
Candidates



$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 66.58$



$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 66.58$



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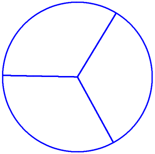
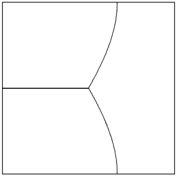


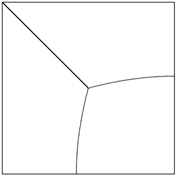
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 20.20$

∞-minimal 3-partition

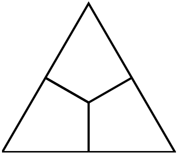
Candidates



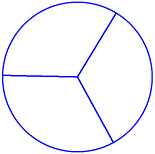
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 66.58$



$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 66.58$

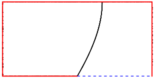


$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 61.872$

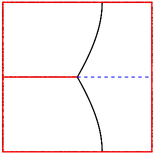


$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 20.20$

Applications



$$0.75 \simeq \int_{D_1} |\varphi_2|^2 > 2 \int_{D_2} |\varphi_2|^2 \simeq 0.51$$



$$\mathfrak{L}_{3,1}(\square) < \Lambda_{3,\infty}(\mathcal{D})$$

Numerical simulations

p-minimal 3-partition for the square

Since $\Lambda_3^{DN} \simeq 66.581$ and $L_3 = 10\pi^2 \simeq 98.696$

$$\lambda_3 < \mathfrak{L}_{3,\infty} < \Lambda_3^{DN}, \qquad \left(\frac{1}{3}\sum_{j=1}^{3}\lambda_j(\square)^p\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN}$$

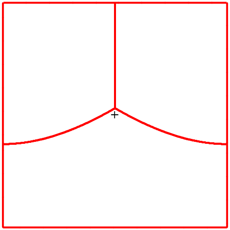
Numerical simulations

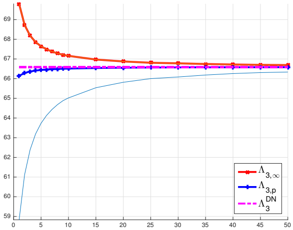
p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p+5^p+5^p}{3}\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \qquad \Rightarrow \qquad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 1$





[Bogosel-BN16]

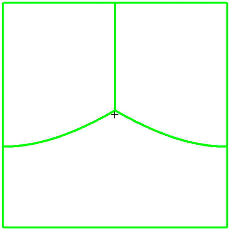
Numerical simulations

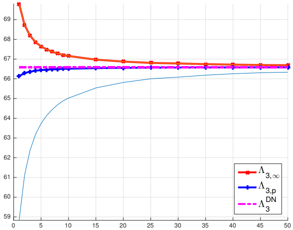
p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p + 5^p + 5^p}{3} \right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \qquad \Rightarrow \qquad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 2$





[*Bogosel-BN16*]

Numerical simulations

p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

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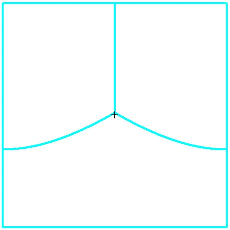
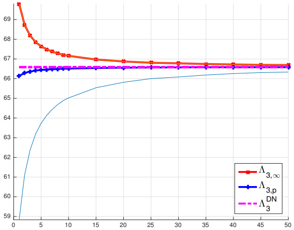
$p = 5$





[*Bogosel-BN16*]

Numerical simulations

p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

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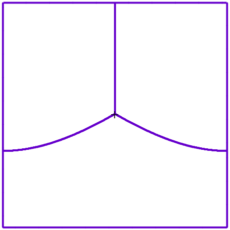
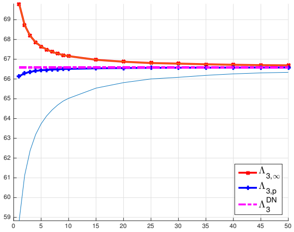
$p = 10$





[*Bogosel-BN16*]

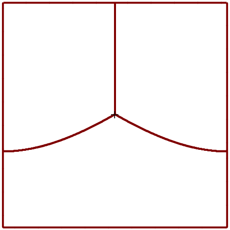
Numerical simulations

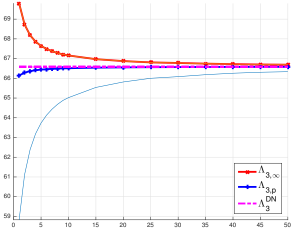
p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p+5^p+5^p}{3}\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \qquad \Rightarrow \qquad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 15$





[Bogosel-BN16]

Numerical simulations

p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

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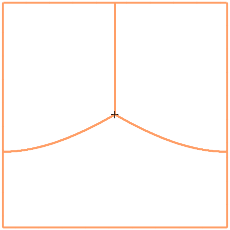
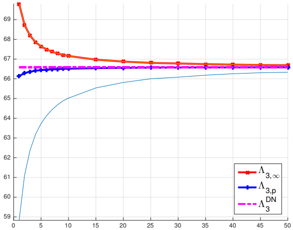
$p = 20$





[*Bogosel-BN16*]

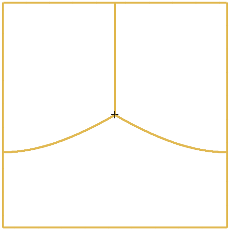
Numerical simulations

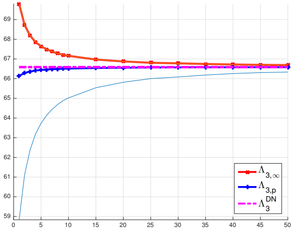
p-minimal 3-partition for the square

$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p + 5^p + 5^p}{3} \right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \qquad \Rightarrow \qquad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 50$





[*Bogosel-BN16*]

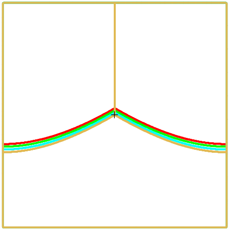
Numerical simulations

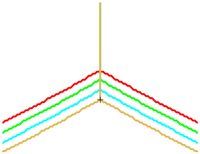
p-minimal 3-partition for the square

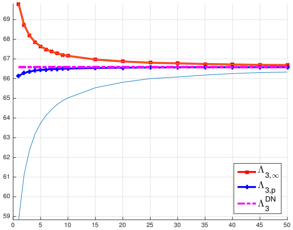
$$49.35 \simeq 5\pi^2 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2\left(\frac{2^p+5^p+5^p}{3}\right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \qquad \Rightarrow \qquad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 1, 2, 5, 50$







[*Bogosel-BN16*]

Numerical simulations

p-minimal 3-partition

Conjecture

For the square :

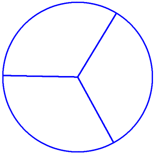
- $p \mapsto \mathfrak{L}_{3,p}(\square)$ is increasing
- $p_\infty(\square, 3) = +\infty$

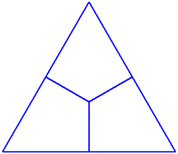
For the disk:

- $p_\infty(\bigcirc, 3) = 1$

For the equilateral triangle:

- $p_\infty(\triangle, 3) = 1$





is a p-minimal 3-partition for any $p \geq 1$

Iterative methods

Penalization

1. Instead of looking for k domains $(D_1, \ldots, D_K)$, we look for a k-upple of functions $(\varphi_1, \ldots, \varphi_k) \in M$ with

$$M = \left\{ (\varphi_1, \ldots, \varphi_k), \varphi_i : \Omega \to [0,1] \text{ measurable }, \sum_{i=1}^{k} \varphi_i = 1 \text{ a.e. } \Omega \right\}$$

[*Bourdin-Bucur-Oudet 09*]

Iterative methods

Penalization

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2. Penalized eigenvalue problem on Ω

$$-\Delta v_i + \frac{1}{\varepsilon}(1 - \varphi_i) v_i = \lambda(\varepsilon, \varphi_i) v_i \qquad \text{in } \Omega$$

Note that

$$\lim_{\varepsilon \to 0} \lambda(\varepsilon, \varphi_i) = \lambda_1(D_i)$$

[Bourdin-Bucur-Oudet 09]

Iterative methods

Penalization

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2. Penalized eigenvalue problem on Ω
$$-\Delta v_i + \frac{1}{\varepsilon}(1 - \varphi_i) v_i = \lambda(\varepsilon, \varphi_i) v_i \qquad \text{in } \Omega$$

3. Penalized optimization problem
$$\mathcal{M}(\varepsilon, k) = \inf \left\{ \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_1^p(\varepsilon, \varphi_i) \right)^{1/p}, (\varphi_1, \dots, \varphi_k) \in M \right\}$$
In some sense
$$\lim_{\varepsilon \to 0} \mathcal{M}(\varepsilon, k) = \mathfrak{L}_{k,p}(\Omega)$$

[Bourdin-Bucur-Oudet 09]

Iterative methods

Penalization

1. Instead of looking for k domains $(D_1, \dots, D_K)$, we look for a k-upple of functions $(\varphi_1, \dots, \varphi_k) \in M$ with

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4. Projected-gradient descent with adaptive step

[*Bourdin-Bucur-Oudet 09*]

Algorithm

Let $\rho > 0$, $\varepsilon > 0$

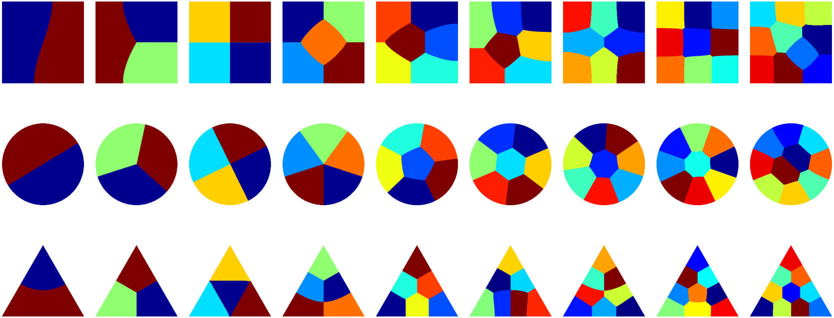
Initialisation k vectors Φ_ℓ^0 given randomly

Iteration Step p: for any $\ell = 1, \dots, k$:

1. Compute the first eigenmode $(\lambda(\Phi_\ell), U(\Phi_\ell))$ of $\mathbb{A}(\varepsilon, \Phi_\ell)$
2. Gradient descent : $\tilde{\Phi}_\ell^{p+1} = \Phi_\ell^{p+1} - \rho \nabla_{\Phi_\ell^p} \lambda(\Phi_\ell^p)$
3. Projection on $\mathcal{S}$: $\tilde{\Phi}_\ell^{p+1} = \Pi_{\mathcal{S}} \tilde{\Phi}_\ell^{p+1}$

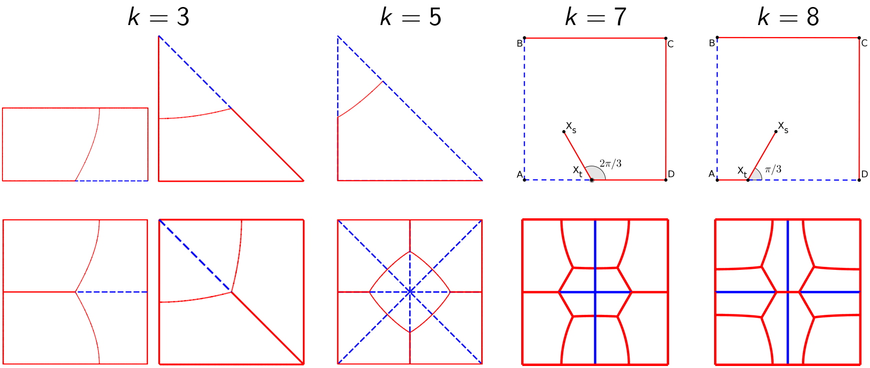
∞-minimal k-partition

Iterative method



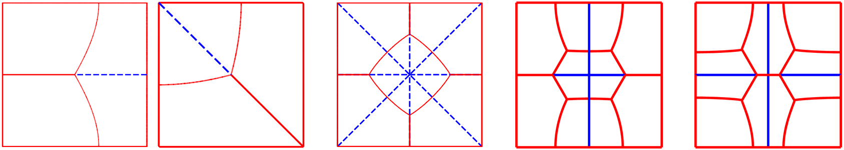
∞-minimal k-partition

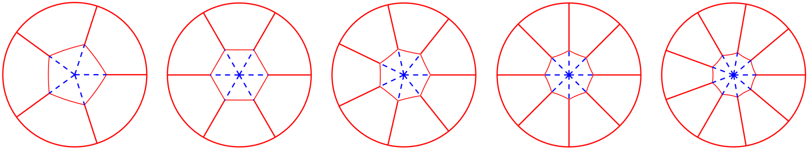
Dirichlet-Neumann approach



∞-minimal k-partition

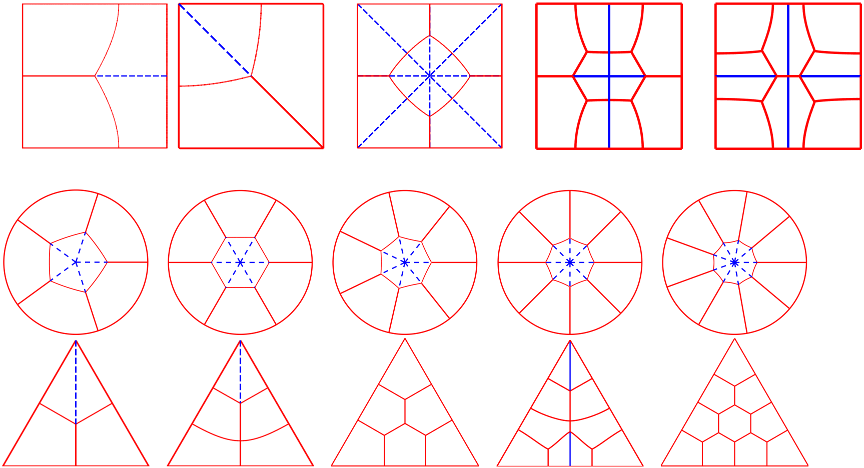
Dirichlet-Neumann approach





∞-minimal k-partition

Dirichlet-Neumann approach



Candidates for the max *vs.* the sum

Criteria of the L^2-norm

Let $\mathcal{D} = (D_i)_{1\leq i\leq k}$ be a ∞-minimal k-partition
Let $D_i \sim D_j$ be a pair of neighbors. We denote

$$D_{ij} = \mathrm{Int}\,\overline{D_i \cup D_i}$$

- $\lambda_2(D_{ij}) = \mathfrak{L}_{2,\infty}(D_{ij}) = \mathfrak{L}_{k,\infty}(\Omega)$

- Suppose that there exists a second eigenfunction φ_{ij} of $-\Delta$ on D_{ij} having D_i and D_j as nodal domains and such that

$$\int_{D_i} |\varphi_{ij}|^2 \neq \int_{D_j} |\varphi_{ij}|^2$$

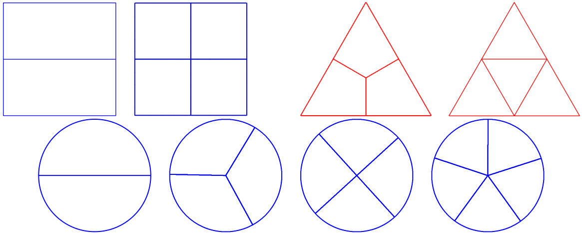
Then

$$\mathfrak{L}_{k,1}(\Omega) < \Lambda_{k,\infty}(\mathcal{D})$$

Candidates for the max *vs.* the sum

Criteria of the L^2-norm

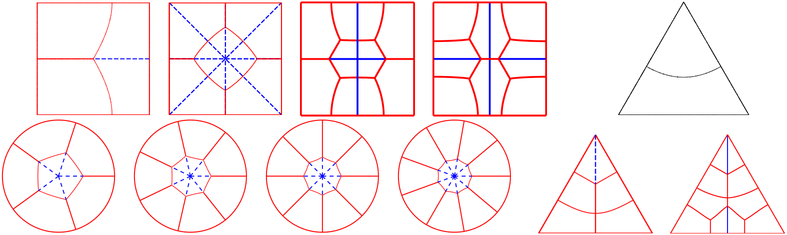
- Criteria not applicable when the subdomains are congruent



Candidates for the max *vs.* the sum

Criteria of the L^2-norm

- Criteria not applicable when the subdomains are congruent
- Cases where the criteria applies

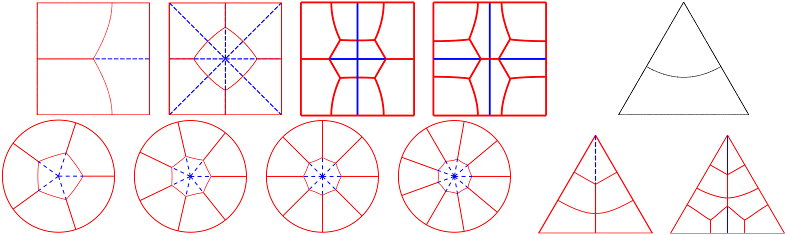


Non optimal partitions for the sum

Candidates for the max *vs.* the sum

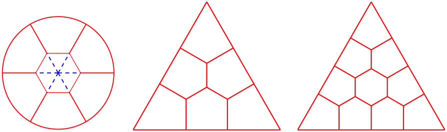
Criteria of the L^2-norm

- Criteria not applicable when the subdomains are congruent
- Cases where the criteria applies



Non optimal partitions for the sum

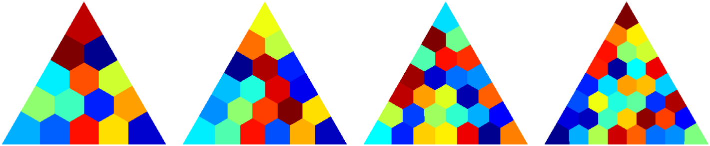
- No conclusion



∞-minimal k-partition

Triangular numbers

Let $k = \frac{n(n+1)}{2}$



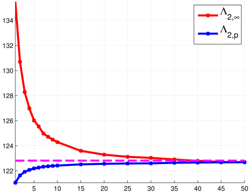
Numerical candidates for $k \in \{15, 21, 28, 36\}$

3 equal quadrilaterals, $3(n-2)$ pentagons, $\frac{(n-2)(n-3)}{2}$ regular hexagons

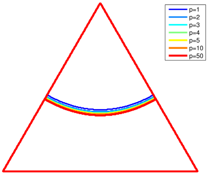
Numerical results for the p-norm

Equilateral triangle

$\Lambda_{2,p}(\mathcal{D}^{2,p})$, $\Lambda_{2,\infty}(\mathcal{D}^{2,p})$ and $\lambda_2(\triangle)$



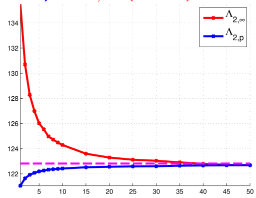
$\mathcal{D}^{2,p}$



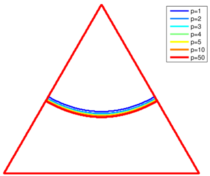
Numerical results for the p-norm

Equilateral triangle

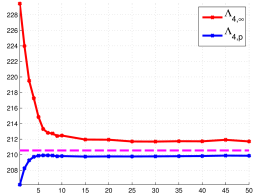
$\Lambda_{2,p}(\mathcal{D}^{2,p})$, $\Lambda_{2,\infty}(\mathcal{D}^{2,p})$ and $\lambda_2(\triangle)$



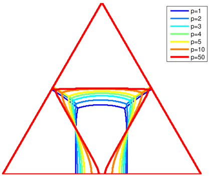
$\mathcal{D}^{2,p}$



$\Lambda_{4,p}(\mathcal{D}^{2,p})$, $\Lambda_{4,\infty}(\mathcal{D}^{4,p})$ and $\lambda_4(\triangle)$

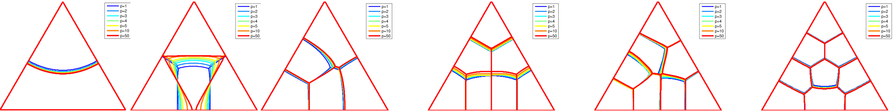


$\mathcal{D}^{4,p}$



Numerical results for the p-norm

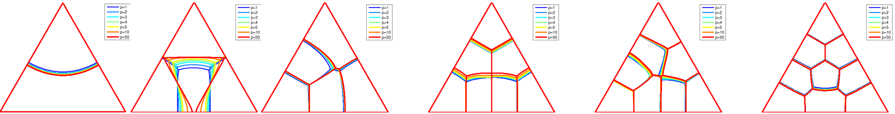
Equilateral triangle



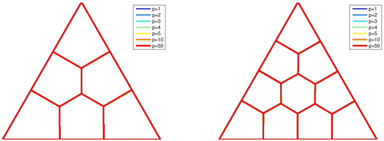
$$k \in \{2, 4, 5, 7, 8, 9\}$$

Numerical results for the p-norm

Equilateral triangle



$k \in \{2, 4, 5, 7, 8, 9\}$



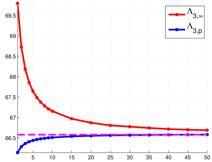
$k \in \{6,\ 10\}$

Numerical results for the p-norm

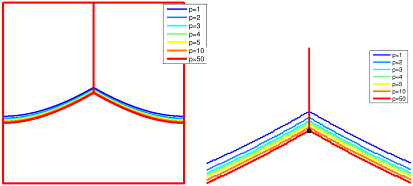
Square

- $k = 2, 4$: equipartitions
- $k \in \{3\}$

$\Lambda_{3,p}(\mathcal{D}^{3,p})$, $\Lambda_{3,\infty}(\mathcal{D}^{3,p})$ and $\Lambda_3^{DN}(\square)$ vs. p



$\mathcal{D}^{3,p}$ vs. p

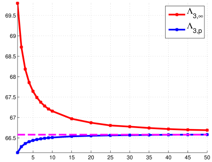


Numerical results for the p-norm

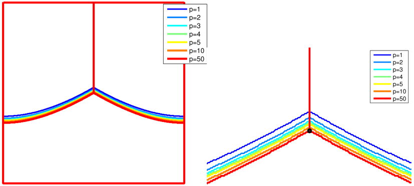
Square

- $k = 2, 4$: equipartitions
- $k \in \{3, 5\}$

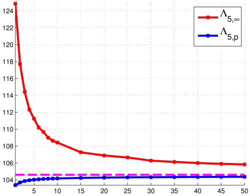
$\Lambda_{3,p}(\mathcal{D}^{3,p})$, $\Lambda_{3,\infty}(\mathcal{D}^{3,p})$ and $\Lambda_3^{DN}(\square)$ vs. p



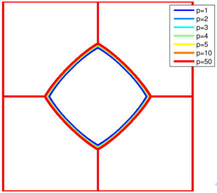
$\mathcal{D}^{3,p}$ vs. p



$\Lambda_{5,p}(\mathcal{D}^{5,p})$, $\Lambda_{5,\infty}(\mathcal{D}^{5,p})$ and $\Lambda_5^{DN}(\square)$



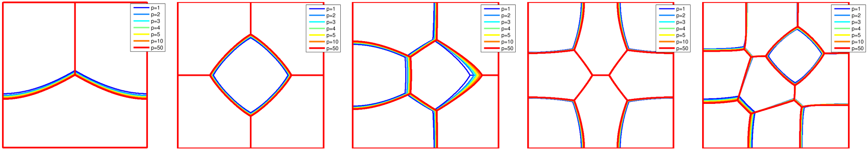
$\mathcal{D}^{5,p}$



Numerical results for the p-norm

Square

- $k = 2, 4$: equipartitions
- $k \in \{3, 5, 6, 8, 10\}$

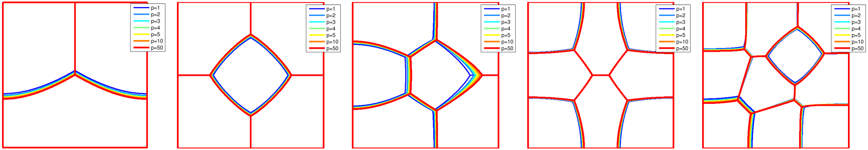


Numerical results for the p-norm

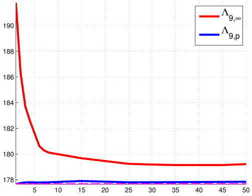
Square

- $k = 2, 4$: equipartitions

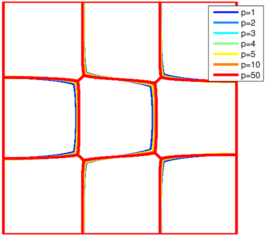
- $k \in \{3, 5, 6, 8, 10\}$



- $\Lambda_{9,p}(\mathcal{D}^{9,p})$, $\Lambda_{9,\infty}(\mathcal{D}^{9,p})$ and $L_9(\square)$



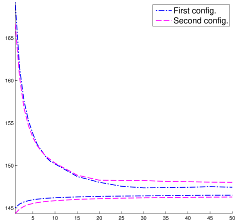
$\mathcal{D}^{9,p}$



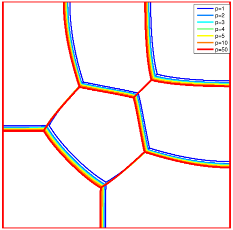
Numerical results for the p-norm

Square

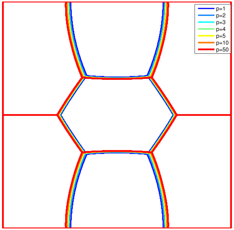
- $k = 2, 4$: equipartitions
- $\Lambda_{k,p/\infty}(\mathcal{D}_j^{7,p})$ for $j = 1,\ 2$



$\mathcal{D}_1^{7,p}(\Box)$



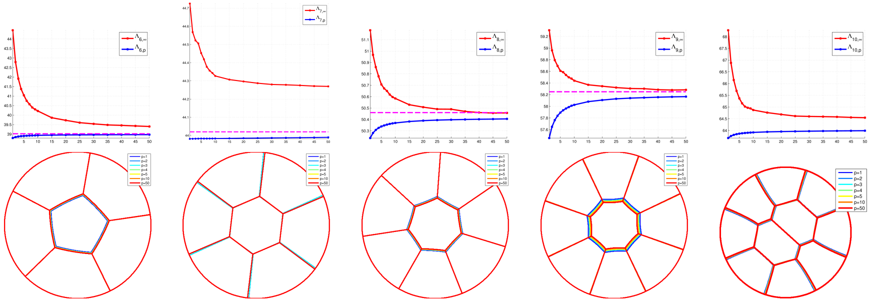
$\mathcal{D}_2^{7,p}(\Box)$



Numerical results for the p-norm

Disk

$k = 6$ $k = 7$ $k = 8$ $k = 9$ $k = 10$



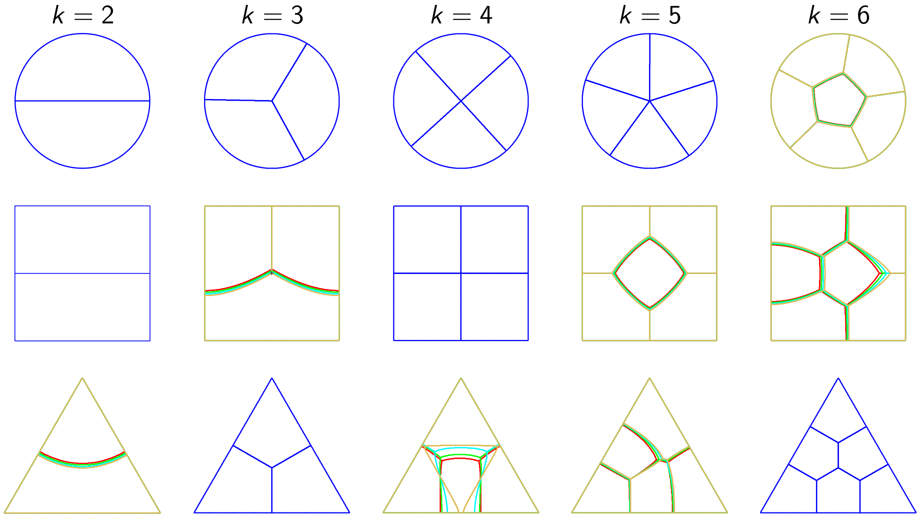
Conclusion

Conjectures

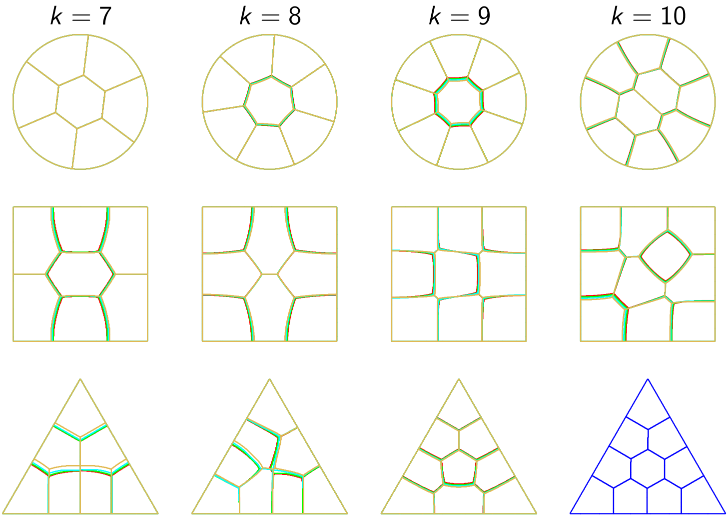
- $\mathfrak{L}_{k,p}(\bigcirc) = \lambda_1(\Sigma_{\frac{2\pi}{k}})$ for $k \in \{2,3,4,5\}$, $\forall p$
- $\mathfrak{L}_{k,p}(\square) = \lambda_k(\square)$ iff $k = 1, 2, 4$, $\forall p$
- $\mathfrak{L}_{k,p}(\triangle) = \lambda_k(\triangle)$ iff $k = 1, 2, 4$, $\forall p$

  $p \mapsto \mathfrak{L}_{k,p}(\triangle) = constant$ if $k = \frac{n(n+1)}{2}$

Conclusion



Conclusion



Asymptotics $k \to \infty$

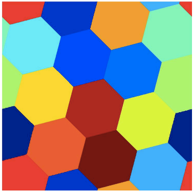
Hexagonal conjecture

- The limit of $\mathfrak{L}_k(\Omega)/k$ as $k \to +\infty$ exists and

$$\lim_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \frac{\lambda_1(\hexagon)}{|\Omega|}$$

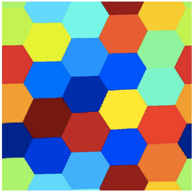
- The limit of $\mathfrak{L}_{k,1}(\Omega)/k$ as $k \to +\infty$ exists and

$$\lim_{k \to +\infty} \frac{\mathfrak{L}_{k,1}(\Omega)}{k} = \frac{\lambda_1(\hexagon)}{|\Omega|}$$

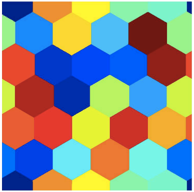


$k = 15$

$k = 20$



$k = 25$



$k = 30$

[Bourdin–Bucur–Oudet, BN–Léna]