Guaranteed Lower Bounds for Eigenvalues

Joscha Gedicke
(joint work with Carsten Carstensen)

joscha.gedicke@univie.ac.at

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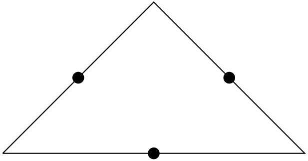
Der Wissenschaftsfonds.

Crouzeix-Raviart Nonconforming FEM

$$
\begin{aligned}
CR_0^1(\mathcal{T}) := \{v \in P_1(\mathcal{T}) \mid\ & v \text{ is continuous at } \mathrm{mid}(\mathcal{E}) \text{ and} \\
& v = 0 \text{ at } \mathrm{mid}(\mathcal{E}(\partial\Omega))\}
\end{aligned}
$$

$$
P_1(\mathcal{T}) := \left\{v \in L^2(\Omega) \mid \forall T \in \mathcal{T},\ v|_T \text{ is affine}\right\}
$$



Nonconforming interpolant $\mathcal{I}_{NC} : H_0^1(\Omega) \to CR_0^1(\mathcal{T})$

$$
\mathcal{I}_{NC} v(\mathrm{mid}(E)) := \frac{1}{|E|} \int_E v \, ds \quad \text{for all } E \in \mathcal{E}
$$

Symmetric Eigenvalue Problem

Seek eigenpair $(u, \lambda) \in V \times \mathbb{R}^+ := H_0^1(\Omega) \times \mathbb{R}^+$ with $\|u\|_{L^2(\Omega)} = 1$ s.t.

$$-\Delta u = \lambda u \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$u = 0 \qquad \text{on } \partial\Omega$$

Variational formulation: Seek $(\lambda_{CR}, u_{CR}) \in \mathbb{R} \times CR_0^1(\mathcal{T})$ with $b(u_{CR}, u_{CR}) = 1$ and

$$a_{NC}(u_{CR}, v_{CR}) = \lambda_{CR} b(u_{CR}, v_{CR}) \quad \text{for all } v_{CR} \in CR_0^1(\mathcal{T})$$

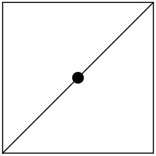
with nonconforming bilinear form

$$a_{NC}(u_{CR}, v_{CR}) := \sum_{T \in \mathcal{T}} \int_T \nabla u_{CR} \cdot \nabla v_{CR} \, dx \quad \text{for all } u_{CR}, v_{CR} \in CR_0^1(\mathcal{T})$$

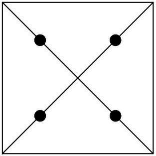
Guaranteed Lower Bounds for Eigenvalues

$$\frac{\lambda_{CR,1}}{1 + 0.1931\lambda_{CR,1}H^2} \leq \lambda_1$$

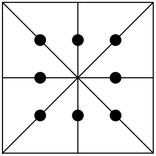
Example: $\lambda_1 = 2\pi^2 = 19.7392$



$2.3371 \leq \lambda_1 < \lambda_{CR,1} = 24$



$4.2594 \leq \lambda_1 < \lambda_{CR,1} = 24$



$6.6182 \leq \lambda_1$
$\lambda_{CR,1} = 18.3344$

Lower Bound for Exact Solve

$$\frac{\lambda_{CR,1}}{1+\kappa^2\lambda_{CR,1}H^2} \leq \lambda_1$$

- Orthogonality

$$\lambda_1 = \|\!| u_1 - \mathcal{I}_{NC} u_1 \|\!|_{NC}^2 + \|\!| \mathcal{I}_{NC} u_1 \|\!|_{NC}^2$$

- Rayleigh-Ritz principle

$$\|\!| u_1 - \mathcal{I}_{NC} u_1 \|\!|_{NC}^2 + \lambda_{CR,1} \| \mathcal{I}_{NC} u_1 \|^2 \leq \lambda_1$$

- binomial expansion & adding zero

$$1 - s^2 - 2(1-s)\| u_1 - \mathcal{I}_{NC} u_1 \| \leq \| \mathcal{I}_{NC} u_1 \|^2$$

- Interpolation estimate

$$\| u_1 - \mathcal{I}_{NC} u_1 \| \leq \sqrt{1/8 + j_{1,1}^{-2}}\, H \|\!| u_1 - \mathcal{I}_{NC} u_1 \|\!|_{NC}$$

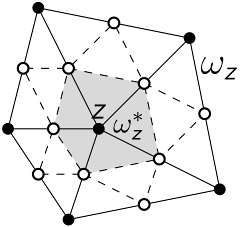
Interpolation for Upper Bound

$$\mathcal{I}_{CM} v_{CR}(z) := \begin{cases} 0 & \text{if } z \text{ lies on the boundary } \partial\Omega \\ v_{CR}(z) & \text{if } z \text{ is the midpoint of an edge } E \in \mathcal{E}(\Omega) \\ v_{min}(z) & \text{if } z \in \mathcal{N}(\Omega) \end{cases}$$

$$W_z := \{w \in P_1(\mathcal{T}^*(z)) \cap C(\overline{\omega}_z^*) \mid w = v_{CR} \text{ on } \partial\omega_z^*\}$$

$v_{min} \in W_z$ is the unique minimizer of

$$\min_{w \in W_z} \sum_{T \in \mathcal{T}^*(z)} \|\nabla(v_{CR} - w)\|^2_{L^2(T)}$$



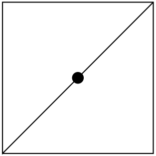
Guaranteed Bounds for Eigenvalues

Theorem

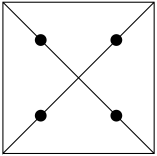
Let $(\tilde{\lambda}_{CR,1}, \tilde{u}_{CR,1}) \in \mathbb{R} \times CR_0^1(\mathcal{T})$ *be an approximation of the eigenpair* (λ_1, u_1) *of the smallest eigenvalue with* $\|\tilde{u}_{CR,1}\|_{L^2(\Omega)} = 1$ *and with algebraic residual* $\mathbf{r} := \mathbf{A}\tilde{\mathbf{u}}_{CR,1} - \tilde{\lambda}_{CR,1}\mathbf{B}\tilde{\mathbf{u}}_{CR,1}$ *and let* $\mathcal{I}_{CM}\tilde{u}_{CR,1}$ *be the quasi-interpolant of* $\tilde{u}_{CR,1}$*. Suppose separation of* $\tilde{\lambda}_{CR,1}$ *from the remaining discrete spectrum in the sense that* $\tilde{\lambda}_{CR,1}$ *is closer to the smallest discrete eigenvalue* $\lambda_{CR,1}$ *than to any other discrete eigenvalue and suppose that* $\|\mathbf{r}\|_{\mathbf{B}^{-1}} < \tilde{\lambda}_{CR,1}$*. Then it holds that*

$$\frac{\tilde{\lambda}_{CR,1} - \|\mathbf{r}\|_{\mathbf{B}^{-1}}}{1 + \kappa^2(\tilde{\lambda}_{CR,1} - \|\mathbf{r}\|_{\mathbf{B}^{-1}})H^2} \leq \lambda_1 \leq R(\mathcal{I}_{CM}\tilde{u}_{CR,1})$$

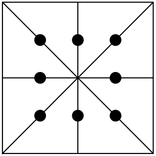
Example: $\lambda_1 = 2\pi^2 = 19.7392$



$2.3371 \leq \lambda_1 \leq 32$



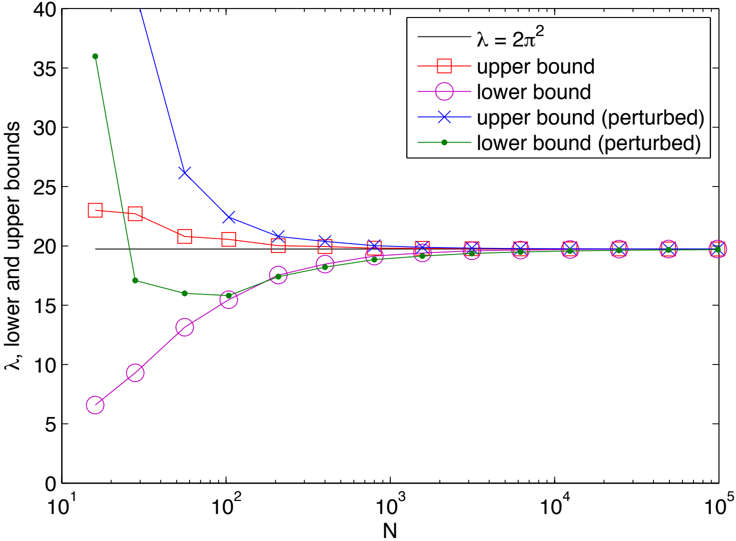
$4.2594 \leq \lambda_1 \leq 24$



$6.6182 \leq \lambda_1 \leq 22.0397$

$\lambda_{C,1} = 24$

Guaranteed Bounds for Eigenvalues



Efficiency for the Class of Graded Meshes

$$\eta := R(\mathcal{I}_{CM}\tilde{u}_{CR,1}) - \frac{\tilde{\lambda}_{CR,1} - \|\mathbf{r}\|_{\mathbf{B}^{-1}}}{1 + \kappa^2(\tilde{\lambda}_{CR,1} - \|\mathbf{r}\|_{\mathbf{B}^{-1}})H^2}$$

Theorem

For all graded meshes the eigenvalue bounds are efficient in the sense that the difference η of the upper and lower bounds satisfies

$$\begin{aligned}\eta \lesssim\ & (1 + H^2\tilde{\lambda}_{CR,1})\|\!|u_1 - \tilde{u}_{CR,1}|\!\|_{NC}^2 \\ & + H^2\left((\lambda_1 - \lambda_{CR,1})^2 + \lambda_1\lambda_{CR,1}\|u_1 - u_{CR,1}\|^2\right) \\ & + |\lambda_1 - \tilde{\lambda}_{CR,1}| + \|\mathbf{A}(\mathbf{u}_{CR,1} - \tilde{\mathbf{u}}_{CR,1})\|_{\mathbf{B}^{-1}} \\ & + \lambda_{CR,1}\|u_{CR,1} - \tilde{u}_{CR,1}\| + |\lambda_{CR,1} - \tilde{\lambda}_{CR,1}|\end{aligned}$$

Adaptive Finite Element Method & Error Balancing

$$|\lambda_1 - \tilde{\lambda}_{1,\ell}| \leq \eta_1 + \eta_2 + \eta_3$$

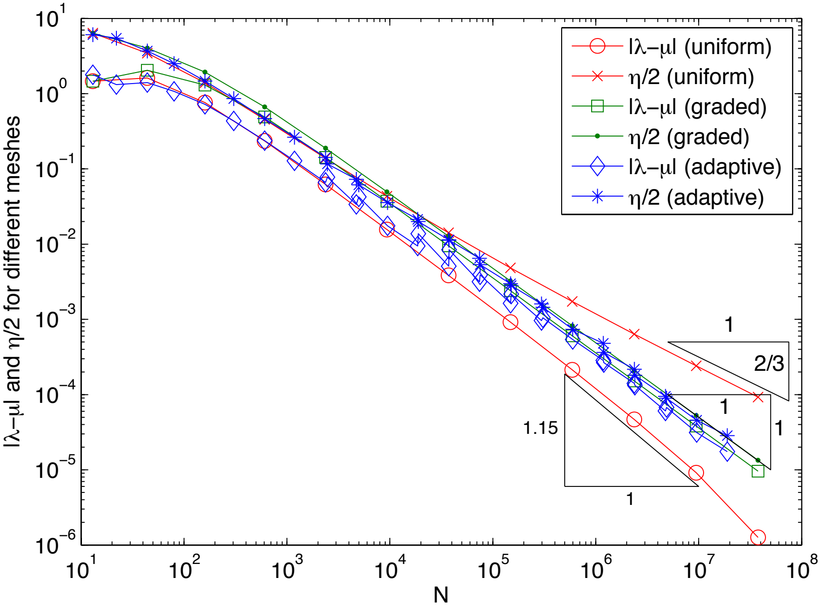
with

$$\eta_1 := \frac{\tilde{\lambda}_{1,\ell}\kappa^2(\tilde{\lambda}_{1,\ell} - \|\mathbf{r}_\ell\|_{\mathbf{B}_\ell^{-1}})H_\ell^2}{1 - \kappa^4(\tilde{\lambda}_{1,\ell} - \|\mathbf{r}_\ell\|_{\mathbf{B}_\ell^{-1}})^2 H_\ell^4}$$

$$\eta_2 := \frac{\|\mathbf{r}_\ell\|_{\mathbf{B}_\ell^{-1}}}{1 + \kappa^2(\tilde{\lambda}_{1,\ell} - \|\mathbf{r}_\ell\|_{\mathbf{B}_\ell^{-1}})H_\ell^2}$$

$$\eta_3 := R(\mathcal{I}_{CM}\tilde{u}_{1,\ell}) - \frac{\tilde{\lambda}_{1,\ell}}{1 - \kappa^4(\tilde{\lambda}_{1,\ell} - \|\mathbf{r}_\ell\|_{\mathbf{B}_\ell^{-1}})^2 H_\ell^4}$$

L-shaped Domain Example



Higher Eigenvalues

Theorem

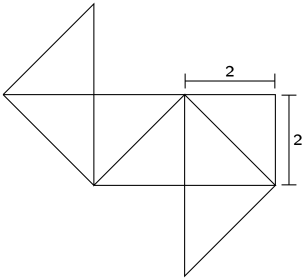
Suppose that the separation condition

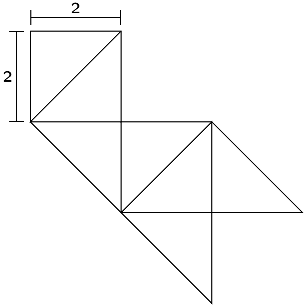
$$H < \left(\sqrt{1+1/J}-1\right)/(\kappa\lambda_J^{1/2})$$

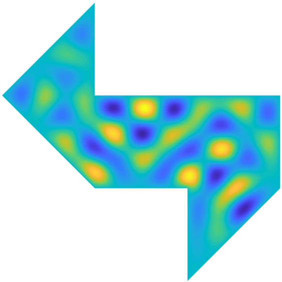
holds for the J-th exact eigenvalue λ_J. *Let* $(\tilde{\lambda}_{CR,J}, \tilde{u}_{CR,J}) \in \mathbb{R} \times CR_0^1(\mathcal{T})$ *with* $\|\tilde{u}_{CR,J}\|_{L^2(\Omega)} = 1$ *and algebraic residual* $\mathbf{r} := \mathbf{A}\tilde{\mathbf{u}}_{CR,J} - \tilde{\lambda}_{CR,J}\mathbf{B}\tilde{\mathbf{u}}_{CR,J}$ *approximate the J-th eigenpair* (λ_J, u_J). *Suppose separation of* $\tilde{\lambda}_{CR,J}$ *from the remaining discrete spectrum in the sense that* $\tilde{\lambda}_{CR,J}$ *is closer to the discrete eigenvalue* $\lambda_{CR,J}$ *than to any other discrete eigenvalues and that* $\|\mathbf{r}\|_{\mathbf{B}^{-1}} < \tilde{\lambda}_{CR,J}$. *Then it holds that*

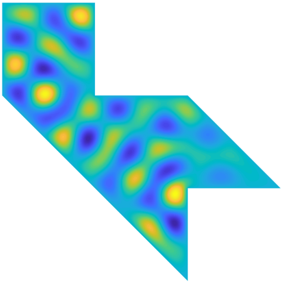
$$\frac{\tilde{\lambda}_{CR,J} - \|\mathbf{r}\|_{\mathbf{B}^{-1}}}{1+\kappa^2(\tilde{\lambda}_{CR,J} - \|\mathbf{r}\|_{\mathbf{B}^{-1}})H^2} \le \lambda_J \le \max_{\xi\in\mathbb{R}^J\setminus\{0\}} R\left(\sum_{j=1}^{J} \xi_j \mathcal{I}_{CM}\tilde{u}_{CR,j}\right)$$

"Can one hear the shape of a drum?" (Kac 1966)









Shape of a Drum Eigenvalue Bounds

$\lambda_{50} = 54.1879356$	lower bounds	
N	left domain	right domain
2760	40.13930504264308	40.13930504264337
10896	49.82373624915233	49.82373624915240
43296	53.022275017108896	53.022275017108903
172608	53.88987045942154 5	53.88987045942153 7
689280	54.112360562895724	54.112360562895560
2754816	54.16872379682151 0	54.16872379682153 8
11014656	54.183012990240513	54.183012990240186

$\lambda_{50} = 54.1879356$	upper bounds	
N	left domain	right domain
2760	56.619351329573185	56.619351329573249
10896	54.81842468456033 4	54.81842468456030 6
43296	54.352753736838082	54.352753736838132
172608	54.231273697990432	54.231273697990602
689280	54.199573365120656	54.199573365121147
2754816	54.191162363149061	54.191162363147861
11014656	54.188868310930701	54.188868310929948

Lehmann-Goerisch, Weinstein and Kato Bounds

- Based on flux approximations
- Allow for higher order FEM
- Need *a priori* information on λ_{J-1} and λ_{J+1}
  $\Rightarrow$ use guaranteed bounds based on $CR_0^1(\mathcal{T})$ FEM on coarse meshes

Left domain with continuous $P_8(\mathcal{T})$ FEM and 'exact' solve:

N	lower	upper
14701	53.89423022818913	54.18796257468186
15961	54.01893045692087	54.18795201892610
17769	54.10749616313925	54.18794349206255
27669	54.18376165215992	54.18793604159895
33905	54.18706280054064	54.18793573134040
50505	54.18792572767124	54.18793562650560
59893	54.18793452877433	54.18793562557860
73201	54.18793551973868	54.18793562538509

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