The classification of homogeneous finite-dimensional permutation structures

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BIRS, Unifying Themes in Ramsey Theory
November 21, 2018

Overview

PERMUTATIONS

- A *permutation* of size n is a rearrangement of the numbers $\{1, \dots, n\}$.
- These may be viewed as structures in a language of 2 linear orders.

$$231 \preceq 4,5,2,1,6,8,7,3$$

Homogeneous Permutations

Theorem (Cameron, 2002)

Up to interdefinability, there are 3 homogeneous permutations.

1. $\mathbb{Q}$ *equipped with its standard order twice.*
2. *The fully generic permutation, i.e. the Fraïssé limit of the class of all finite permutations*
3. $\mathbb{Q}^2$ *equipped with the equivalence relation of agreement in the first coordinate, and the lexicographic order*

- The first two structures are primitive, i.e. have no $\emptyset$-definable equivalence relation.
- The last is imprimitive, equipped with a convex linear order.

Problem (Cameron)

Classify, for each n, the homogeneous n-dimensional permutation structures.

Replacing Equivalence Relations

- Consider a structure M equipped with an equivalence relation E and an E-convex linear order $<$.
- E can be interdefinably replaced with the linear order $<^*$ defined as follows.

Definition

- If xEy, then $x <^* y \iff x < y$.
- Otherwise, $x <^* y \iff y < x$.

- There are more efficient ways of doing this for several equivalence relations.

Cameron's Problem

- The first step in such a classification problem is the production of a catalog of examples, which will later serve to guide the classification procedure.
- Cameron's classification yields a satisfactory catalog of the primitive finite-dimensional permutation structures.

Theorem (Primitivity Theorem, Simon)

Let M be a primitive homogeneous n-dimensional permutation structure, in which no orders are equal up to reversal.
Then M is the fully generic n-dimensional permutation structure.

- What about imprimitive structures?
  1. Start with a homogeneous skeleton of equivalence relations.
  2. Expand by convex linear orders.
  3. Replace the equivalence relations with more linear orders.

Λ-Ultrametric Spaces

- In an ω-categorical structure, the $\emptyset$-definable equivalence relations form a lattice.
- In a structure equipped with a lattice Λ of equivalence relations, passing to a substructure, e.g. a single point, may collapse Λ.
- In order to keep Λ fixed as we pass to substructures, we change the language to Λ-*ultrametric spaces*.

Definition

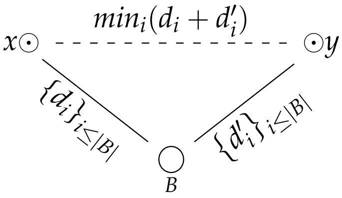
Let Λ be a lattice. Then (M, d) is a Λ-*ultrametric space* if d is a metric taking values in Λ, with the triangle inequality

$$d(x, y) \leq d(x, z) \vee d(z, y)$$

- To go from a structure equipped with a lattice Λ of equivalence relations to a Λ-ultrametric space, take $d(x, y)$ to be the finest equivalence relation between x and y.

Homogeneous Λ-Ultrametric Spaces

- Shortest-path completion provides a standard amalgamation strategy for metric spaces.



- We adapt this to Λ-ultrametric spaces in the obvious way to prove the following.

Theorem

Let Λ be a distributive lattice. Then the class of all finite Λ-ultrametric spaces is an amalgamation class.

Generating a Catalog (First Try)

1. Let Λ be a finite distributive lattice, and M_Λ the fully generic Λ-ultrametric space.
2. Expand M_Λ by enough linear orders, generic modulo convexity conditions, so that every meet-irreducible equivalence relation is convex with respect to at least one order.
3. Interdefinably replace the equivalence relations with linear orders as before.

- One problem: "generic modulo convexity conditions" is too restrictive a requirement to capture all examples.

SUBQUOTIENT ORDERS

- An E-convex linear order may be split into pieces; one within E-classes, and one on the quotient.
- Pieces of distinct linear orders may be interdefinable, even if the orders are not, killing genericity.
- Our solution: if $<$ is an E-convex linear order, then break it into two partial orders; one giving encoding the order within E-classes, and one between E-classes.

Definition

Let X be a structure, and $E \leq F$ equivalence relations on X. A *subquotient order from E to F* is a partial order on X/E in which two E-classes are comparable iff they lie in the same F-class.

GENERATING A CATALOG (FINAL VERSION)

1. Let Λ be a finite distributive lattice, and M_Λ the fully generic Λ-ultrametric space.
2. For each meet-irreducible $E \in \Lambda$, expand M_Λ by at least one generic subquotient order between E-classes.

- The resulting structure is interdefinable with a finite-dimensional permutation structure.

Minimal linear orders

- The following definitions and results come from Simon's work on NIP ω-categorical structures.
- Idea: Certain linear orders can interact in only a few prescribed ways.
- Let M be ω-categorical.
- Let $(V, \leq, \dots)$ be $\emptyset$-definable, with $\leq$ distinguished.

Definition

$(V, \leq, \dots)$ has *topological rank 1* if it has no parameter-definable convex equivalence relation with infinitely many infinite classes.

Definition

$(V, \leq, \dots)$ is *minimal* if it has topological rank 1, and further conditions implied by transitivity.

A TRICHOTOMY

- Given $(V, \leq_V, \dots)$, $(W, \leq_W, \dots)$ both minimal, one of the following holds.

1. $\leq_V$ and $\leq_W$ in monotone bijection.
2. $\leq_V$ and $\leq_W$ are *intertwined*, i.e. there is a monotone $\emptyset$-definable function from $\leq_V$ to the parameter-definable cuts of $\leq_W$.
3. $\leq_V$ and $\leq_W$ are *independent*, i.e. none of the above.

Theorem (Product Theorem)

A closed $\emptyset$-definable set in a product of independent orders is a finite union of products of closed $\emptyset$-definable sets.

Applying the trichotomy

- Let M be a primitive homogeneous n-dimensional permutation structure.

1. Prove every order has topological rank 1.
2. Rule out the possibility of intertwined orders.
3. Consider the diagonal in $(M, \leq_1) \times \cdots \times (M, \leq_n)$.
   a. By the Product Theorem, its closure is M^n.
   b. So the diagonal is dense, i.e. the structure is fully generic.

- We have thus proven the Primitivity Theorem.
- For $n = 2, 3$ this was done by increasingly lengthy amalgamation arguments.

THE IMPRIMITIVE CASE

- Let M be a homogeneous finite-dimensional permutation structure, with lattice of $\emptyset$-definable equivalence relations Λ.

1. Proceed by induction. The restriction to any E-class is understood, and we wish to understand the quotients.
2. Let E_i be the maximal $\leq_i$-convex equivalence relation. Then the $(M/E_i, \leq_i)$ are minimal and independent.
3. The maximal equivalence relations are cross-cutting.
4. Λ is distributive, and thus the reduct to the language of equivalence relations is fully generic.

Ramsey theory

- Let Λ be a finite distributive lattice, and M_Λ be the generic Λ-ultrametric space.

1. The catalog essentially adds linear orders to M_Λ making every $\emptyset$-definable equivalence relation convex. This looks suspiciously like a Ramsey expansion.
2. The minimal Ramsey expansion adds a generic subquotient order to each meet-irreducible in Λ.
3. Thus every homogeneous finite-dimensional permutation structure is Ramsey.
4. The proof uses the Hubička-Nešetřil local finiteness machinery.
5. A special case of the semigroup-valued metric spaces from Konečný's talk.

The number of orders needed

- The catalog is not given in the language of linear orders.

Problem

Given a structure in the catalog, what is the minimum number of orders needed to represent it?

Problem

Given a finite distributive lattice Λ, what is the minimum number of orders (d_Λ) needed to represent it?

- We may encode chains with maximal efficiency.

Proposition

Let Λ be a finite distributive lattice, Λ_0 the poset of meet-irreducibles of $\Lambda \setminus \{\mathbb{0}, \mathbb{1}\}$, $\mathcal{L}$ a set of chains covering Λ_0, and ℓ the minimum size of any such $\mathcal{L}$. Then $2\ell \leq d_\Lambda \leq |\mathcal{L}| + \sum_{L \in \mathcal{L}} \lceil \log_2(|L| + 1) \rceil$.

Homogeneous ordered structures

- The homogeneous ordered graphs are expansions of homogeneous proper reducts (graphs, tournaments, or partial orders) by a linear order.
- In the primitive case, this linear order is generic.

Question

Is this true in general for homogeneous ordered structures?

- Homogeneous finite-dimensional permutation structures are a natural test case. Here it is true.
- False for intertwined orders (with an intertwining relation).

Problem

Find an appropriate modification of the question above.
I suppose resolve it as well.

- One option is to also allow expansion by intertwined orders.

References

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