Winter School 2019

January 26th–February 2nd 2019
Hejnice, Czech Republic

Invited speakers

- James Cummings
- Miroslav Hušek
- Wiesław Kubiś
- Jordi Lopez-Abad

www.winterschool.eu

Logic Colloquium 2019

August 11th–16th 2019, Prague, Czech Republic

www.lc2019.cz

Program Committee

- Andrew Arana
- Lev Beklemishev (chair)
- Agata Ciabattoni
- Russell Miller
- Martin Otto
- Pavel Pudlák
- Stevo Todorčević
- Alex Wilkie

The HL-property and indestructible reaping families

David Chodounský

Charles University in Prague

joint work with Osvaldo Guzmán and Michael Hrušák

HL families

Tree is a perfect initial subtree of $2^{<\omega}$ with no leaves.
The set of trees is denoted $\mathbf{S}$.
$(\mathbf{S}, \subset)$ ordered by inclusion forms the *Sacks forcing* $(\mathbf{S}, <)$
For $A \subset \omega$ and $p \in \mathbf{S}$ we denote $p \restriction A = \{\, t \in p \mid |t| \in A \,\}$.

Theorem (Halpern–Läuchli), weak version

Let $p \in \mathbf{S}$ and $c \colon p \to 2$. There exists $q \in \mathbf{S}$, $q \subseteq p$ and $A \in [\omega]^\omega$ such that $q \restriction A$ is c-monochromatic.

HL families

Tree is a perfect initial subtree of $2^{<\omega}$ with no leaves.
The set of trees is denoted $\mathbf{S}$.
$(\mathbf{S}, \subset)$ ordered by inclusion forms the *Sacks forcing* $(\mathbf{S}, <)$
For $A \subset \omega$ and $p \in \mathbf{S}$ we denote $p \restriction A = \{ t \in p \mid |t| \in A \}$.

Theorem (Halpern–Läuchli), weak version

Let $p \in \mathbf{S}$ and $c \colon p \to 2$. There exists $q \in \mathbf{S}$, $q \subseteq p$ and $A \in [\omega]^\omega$ such that $q \restriction A$ is c-monochromatic.

HL families

Tree is a perfect initial subtree of $2^{<\omega}$ with no leaves.
The set of trees is denoted $\mathbf{S}$.
$(\mathbf{S}, \subset)$ ordered by inclusion forms the *Sacks forcing* $(\mathbf{S}, <)$
For $A \subset \omega$ and $p \in \mathbf{S}$ we denote $p \restriction A = \{\, t \in p \mid |t| \in A \,\}$.

Theorem (Halpern–Läuchli), weak version

Let $p \in \mathbf{S}$ and $c \colon p \to 2$. There exists $q \in \mathbf{S}$, $q \subseteq p$ and $A \in [\omega]^{\omega}$ such that $q \restriction A$ is c-monochromatic.

Definition

$\mathcal{R} \subset \mathcal{P}(\omega)$ is *HL* if for every $c \colon 2^{<\omega} \to 2$ exists $q \in \mathbf{S}$ and $A \in \mathcal{R}$ such that $q \restriction A$ is c-monochromatic.

- $[\omega]^{\omega}$ is HL.

HL families

Tree is a perfect initial subtree of $2^{<\omega}$ with no leaves.
The set of trees is denoted $\mathbf{S}$.
$(\mathbf{S}, \subset)$ ordered by inclusion forms the *Sacks forcing* $(\mathbf{S}, <)$
For $A \subset \omega$ and $p \in \mathbf{S}$ we denote $p \restriction A = \{ t \in p : |t| \in A \}$.

Theorem (Halpern–Läuchli), weak version

Let $p \in \mathbf{S}$ and $c \colon p \to 2$. There exists $q \in \mathbf{S}$, $q \subseteq p$ and $A \in [\omega]^\omega$ such that $q \restriction A$ is c-monochromatic.

Definition

$\mathcal{R} \subset \mathcal{P}(\omega)$ is *HL* if for every $c \colon 2^{<\omega} \to 2$ exists $q \in \mathbf{S}$ and $A \in \mathcal{R}$ such that $q \restriction A$ is c-monochromatic.

- $[\omega]^\omega$ is HL.
- If $\mathcal{R}$ is HL, then $\mathcal{R}$ is reaping.

$\mathcal{R}$ is a *reaping* family if for each $A \subset \omega$ exist $R \in \mathcal{R}$ such that $R \subset A$ or $R \cap A = \emptyset$.

Sacks indestructibility

Theorem (Baumgartner–Laver, Miller, Yiparaki)

The following are equivalent for $\mathcal{R} \subset \mathcal{P}(\omega)$:

1. $\mathcal{R}$ is HL,
2. $\mathcal{R}$ is **S**-reaping indestructible,
3. $\mathcal{R}$ is reaping in a generic extension via forcing **S**,

Sacks indestructibility

Theorem (Baumgartner–Laver, Miller, Yiparaki)

The following are equivalent for $\mathcal{R} \subset \mathcal{P}(\omega)$:

1. $\mathcal{R}$ is HL,
2. $\mathcal{R}$ is $\mathbf{S}$-reaping indestructible,
3. $\mathcal{R}$ is reaping in a generic extension via forcing $\mathbf{S}$,
4. $\mathcal{R}$ is a reaping family in some extension containing a new real,

Sacks indestructibility

Theorem (Baumgartner–Laver, Miller, Yiparaki)

The following are equivalent for $\mathcal{R} \subset \mathcal{P}(\omega)$:

1. $\mathcal{R}$ is HL,
2. $\mathcal{R}$ is $\mathbf{S}$-reaping indestructible,
3. $\mathcal{R}$ is reaping in a generic extension via forcing $\mathbf{S}$,
4. $\mathcal{R}$ is a reaping family in some extension containing a new real,
5. for every $p \in \mathbf{S}$, $c\colon p \to 2$ there is $q \subseteq p$ and $A \in \mathcal{R}$ such that $q \restriction A$ is c-monochromatic,

Sacks indestructibility

Theorem (Baumgartner–Laver, Miller, Yiparaki)

The following are equivalent for $\mathcal{R} \subset \mathcal{P}(\omega)$:

1. $\mathcal{R}$ is HL,
2. $\mathcal{R}$ is $\mathbf{S}$-reaping indestructible,
3. $\mathcal{R}$ is reaping in a generic extension via forcing $\mathbf{S}$,
4. $\mathcal{R}$ is a reaping family in some extension containing a new real,
5. for every $p \in \mathbf{S}$, $c\colon p \to 2$ there is $q \subseteq p$ and $A \in \mathcal{R}$ such that $q \restriction A$ is c-monochromatic,
6. for every $p \in \mathbf{S}$ there is $q \subseteq p$ and $A \in \mathcal{R}$ such that $A \subset \bigcap[q]$ or $A \cap \bigcup[q] = \emptyset$.

Sacks indestructibility

Theorem (Baumgartner–Laver, Miller, Yiparaki)

The following are equivalent for $\mathcal{R} \subset \mathcal{P}(\omega)$:

1. $\mathcal{R}$ is HL,
2. $\mathcal{R}$ is $\mathbf{S}$-reaping indestructible,
3. $\mathcal{R}$ is reaping in a generic extension via forcing $\mathbf{S}$,
4. $\mathcal{R}$ is a reaping family in some extension containing a new real,
5. for every $p \in \mathbf{S}$, $c\colon p \to 2$ there is $q \subseteq p$ and $A \in \mathcal{R}$ such that $q \restriction A$ is c-monochromatic,
6. for every $p \in \mathbf{S}$ there is $q \subseteq p$ and $A \in \mathcal{R}$ such that $A \subset \bigcap[q]$ or $A \cap \bigcup[q] = \emptyset$.

Proposition

Let $\mathcal{R}$ be a reaping family. If $|\mathcal{R}| < \mathfrak{c}$, then $\mathcal{R}$ is HL.

Terminology

- $\mathcal{I} \subset \mathcal{P}(\omega)$ is an *ideal* if closed under finite unions and subsets.
- $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ – a *co-ideal.*
  Every co-ideal is a reaping family.
- $\mathcal{F} \subset \mathcal{P}(\omega)$ is a *filter* if closed under finite intersections and supersets.
- $\mathcal{U} \subset \mathcal{P}(\omega)$ is an *ultrafilter* if it is a reaping filter.
  Equivalently, a maximal filter.

Terminology

- $\mathcal{I} \subset \mathcal{P}(\omega)$ is an *ideal* if closed under finite unions and subsets.
- $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ – a *co-ideal*.
  Every co-ideal is a reaping family.
- $\mathcal{F} \subset \mathcal{P}(\omega)$ is a *filter* if closed under finite intersections and supersets.
- $\mathcal{U} \subset \mathcal{P}(\omega)$ is an *ultrafilter* if it is a reaping filter.
  Equivalently, a maximal filter.

Definition

Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal. $\mathcal{I}$ is an *HL-ideal* if $\mathcal{I}^+$ is HL.

Terminology

- $\mathcal{I} \subset \mathcal{P}(\omega)$ is an *ideal* if closed under finite unions and subsets.
- $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ – a *co-ideal.*
  Every co-ideal is a reaping family.
- $\mathcal{F} \subset \mathcal{P}(\omega)$ is a *filter* if closed under finite intersections and supersets.
- $\mathcal{U} \subset \mathcal{P}(\omega)$ is an *ultrafilter* if it is a reaping filter.
  Equivalently, a maximal filter.

Definition

Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal. $\mathcal{I}$ is an *HL-ideal* if $\mathcal{I}^+$ is HL.

- An ideal $\mathcal{I}$ is P^+ if for every sequence $\{ X_n \in \mathcal{I}^+ \mid n \in \omega \}$ there exists $Y = \{ y_n \in [X_n]^{<\omega} \mid n \in \omega \}$ such that $\bigcup Y \in \mathcal{I}^+$.

Proposition

Every P^+ ideal is an HL-ideal.

Definition

Let $\mathcal{I} \subset \mathcal{P}(\omega)$ be an ideal. $\mathcal{I}$ is an *HL-ideal* if $\mathcal{I}^+$ is HL.

- An ideal $\mathcal{I}$ is P^+ if for every sequence $\{ X_n \in \mathcal{I}^+ \mid n \in \omega \}$ there exists $Y = \{ y_n \in [X_n]^{<\omega} \mid n \in \omega \}$ such that $\bigcup Y \in \mathcal{I}^+$.

Proposition

Every P^+ ideal is an HL-ideal.

Example

Every Ramsey ultrafilter is an HL family.

- Ultrafilter $\mathcal{U}$ is *Ramsey* if $\mathcal{U} \to (\mathcal{U})^2_2$

Katětov order

Definition

For (ideals) $\mathcal{X}, \mathcal{Y} \subset \mathcal{P}(\omega)$ we define $\mathcal{X} \leq_{\mathrm{K}} \mathcal{Y}$ if there exists $f_{\mathrm{K}} \colon \omega \to \omega$ such that $f_{\mathrm{K}}^{-1}[X] \in \mathcal{Y}$ for every $X \in \mathcal{X}$.

Observation

Let $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(\omega)$ be ideals, $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$.
If $\mathcal{J}$ is an HL-ideal, then $\mathcal{I}$ is also an HL-ideal.

Katětov order

Definition

For (ideals) $\mathcal{X}, \mathcal{Y} \subset \mathcal{P}(\omega)$ we define $\mathcal{X} \leq_{\mathrm{K}} \mathcal{Y}$ if there exists $f_{\mathrm{K}} \colon \omega \to \omega$ such that $f_{\mathrm{K}}^{-1}[X] \in \mathcal{Y}$ for every $X \in \mathcal{X}$.

Observation

Let $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(\omega)$ be ideals, $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$.
If $\mathcal{J}$ is an HL-ideal, then $\mathcal{I}$ is also an HL-ideal.

For $c \colon 2^{<\omega} \to 2$ and $p \in \mathbf{S}$ let
$H_c(p) = \{ n \in \omega \mid p \restriction \{ n \} \text{ is } c\text{-monochromatic} \}$.
Let $\mathcal{I}_c$ be the ideal generated by $\{ H_c(p) \mid p \in \mathbf{S} \}$.

Observation

$\mathcal{J}$ is an HL-ideal iff $\mathcal{I}_c \not\leq_{\mathrm{K}} \mathcal{J}$ for each $c \colon 2^{<\omega} \to 2$.
Equivalently iff $\mathcal{I}_c \not\subseteq \mathcal{J}$ for each $c \colon 2^{<\omega} \to 2$.

Examples of HL-ideals

Theorem

The following are examples of HL-ideals.

- P^+ ideals, F_σ ideals, extendible to F_σ, …
- nwd; the ideal of nowhere dense subsets of $\mathbb{Q}$,
- $\mathcal{G}_{\mathrm{c}}$; an ideal on $[\omega]^2$, graphs which do not contain an infinite complete subgraph,
- $\mathcal{G}_{\mathrm{fc}}$; an ideal on $[\omega]^2$, graphs with finite chromatic number,
- $\mathcal{I}_{1/n}$, the ideal of summable sets on ω (is F_σ),
- $\mathcal{SC}$, the ideal generated by SC-sets
  $A = \{ a_n \mid n \in \omega \} \subset \omega$ is an SC-set if $\lim(a_{n+1} - a_n) = \infty$,
- $\mathrm{tr}(\mathrm{null}) =$
  $\{ A \subset 2^{<\omega} \mid \{ x \in 2^\omega \mid \exists^\infty n \in \omega : x \restriction n \in A \} \in \mathrm{null} \}$.

Examples of HL-ideals

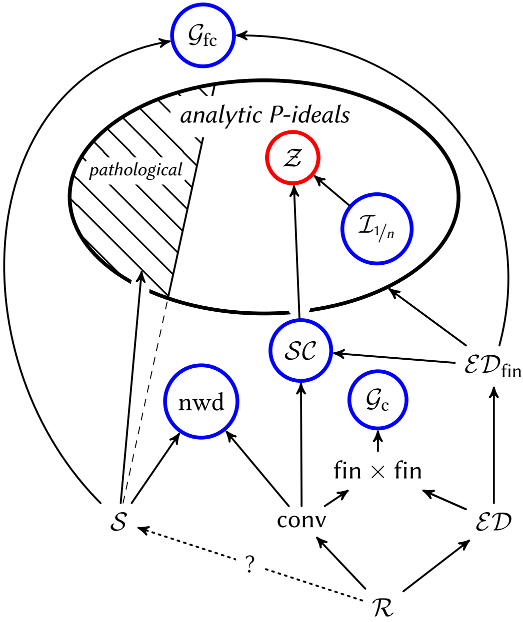
Theorem

The following are examples of HL-ideals.

- P^+ ideals, F_σ ideals, extendible to F_σ, …
- nwd; the ideal of nowhere dense subsets of $\mathbb{Q}$,
- $\mathcal{G}_{\mathrm{c}}$; an ideal on $[\omega]^2$, graphs which do not contain an infinite complete subgraph,
- $\mathcal{G}_{\mathrm{fc}}$; an ideal on $[\omega]^2$, graphs with finite chromatic number,
- $\mathcal{I}_{1/n}$, the ideal of summable sets on ω (is F_σ),
- $\mathcal{SC}$, the ideal generated by SC-sets
  $A = \{ a_n \mid n \in \omega \} \subset \omega$ is an SC-set if $\lim(a_{n+1} - a_n) = \infty$,
- $\mathrm{tr}(\mathrm{null}) =$
  $\{ A \subset 2^{<\omega} \mid \{ x \in 2^\omega \mid \exists^\infty n \in \omega : x \restriction n \in A \} \in \mathrm{null} \}$.

Theorem

$\mathcal{Z} = \left\{ A \subset \omega \mid \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}$ is not HL.



$\mathcal{G}_{fc}$
analytic P-ideals
pathological
$\mathcal{Z}$
$\mathcal{I}_{1/n}$
$\mathcal{SC}$
$\mathcal{ED}_{fin}$
nwd
$\mathcal{G}_c$
fin × fin
$\mathcal{S}$
conv
$\mathcal{ED}$
?
$\mathcal{R}$

Problems

Question

Is it consistent with ZFC that there are no HL-ultrafilters?
(I.e. no $\mathbf{S}$*-indestructible ultrafilters)?*
What about $\mathcal{Z}$*-ultrafilters? Property (s) ultrafilters?*

Problems

Question

Is it consistent with ZFC that there are no HL-ultrafilters?
(I.e. no $\mathbf{S}$*-indestructible ultrafilters)?*
What about $\mathcal{Z}$*-ultrafilters? Property (s) ultrafilters?*

Question

Let $\mathfrak{hl} = \min\{\operatorname{cof}(\mathcal{I}) \mid \mathcal{I}$ *is an ideal, not HL-ideal*$\}$
Is $\mathfrak{hl} = \mathfrak{d}$*?* *(We know that* $\mathfrak{d} \leq \mathfrak{hl} \leq \operatorname{cof} \mathcal{N}$*)*

Problems

Question

Is it consistent with ZFC that there are no HL-ultrafilters?
(I.e. no $\mathbf{S}$*-indestructible ultrafilters)?*
What about $\mathcal{Z}$*-ultrafilters? Property (s) ultrafilters?*

Question

Let $\mathfrak{hl} = \min\{\operatorname{cof}(\mathcal{I}) \mid \mathcal{I} \text{ is an ideal, not HL-ideal}\}$
Is $\mathfrak{hl} = \mathfrak{d}$*?* *(We know that* $\mathfrak{d} \leq \mathfrak{hl} \leq \operatorname{cof} \mathcal{N}$*)*

Question

What about products of trees?