Unique ergodicity, the semigeneric directed graph and short exact sequences

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joint work with Andy Zucker

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Definitions

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We call a continuous G-action on a compact space a G-flow.

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Examples:

- Compact groups.
- No known locally compact example.
- $G = \mathrm{Aut}(\mathbb{F})$ where $\mathbb{F}$ is a Fraissé limit ?

Unique ergodicity and automorphism groups

When $G = \mathrm{Aut}(\mathbb{F})$ where $\mathbb{F}$ is a Fraissé limit: Angel, Kechris and Lyons proved that if $\mathbb{F}$ is the random graph or the Fraissé limit of uniform hypergraphs or graphs with forbidden subgraphs (or a variety of other structures) then G is uniquely ergodic.

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Theorem (J.)

The automorphism group of the semigeneric directed graph is uniquely ergodic.

The semigeneric directed graph

Let $\mathcal{S}$ be the class of finite directed graphs such that:

i) The absence of edge is an equivalence relation $\sim$.

ii) For any two pairs $x \sim y$ and $x' \sim y'$ the number of (directed) edges from $\{x, y\}$ to $\{x', y'\}$ is even.

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Remark

Given a $\sim$-equivalence class $x^\sim$ and a point $y \notin x^\sim$, we get a partition of $x^\sim$ in two classes $x^\sim_{y+}$ and $x^\sim_{y-}$. This partition only depends on the class of y.

$M(\mathrm{Aut}(\mathbb{S}))$

(Kechris, Pestov, Todorcevic '05 - Nguyen Van Thé '13) To find the UMF of the automorphism group of a Fraïssé structure, it suffices to find a good Ramsey expansion for the structure.

Jasiński, Laflamme, Nguyen Van Thé and Woodrow found a suitable Ramsey expansion of the semigeneric directed graph.

A Ramsey expansion of the semigeneric directed graph

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- Given two equivalence classes $x^\sim < y^\sim$, we choose one of the classes $x^\sim_{y+}$ and $x^\sim_{y-}$, call it $x^\sim_{y^\sim}$ and say $R(y', x') = 1$ iff $x' \in x^\sim_{y^\sim}$.

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Remark

There are $n!2^{\binom{n}{2}}$ ways to define the order and define R on n equivalence classes.

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We call $\mathcal{S}^*$ the Fraïssé class of structures obtained this way.

We write $\mathbb{S}^* = Flim(\mathcal{S}^*) = (\mathbb{S}, <^*, R^*)$.

Theorem (Jasiński, Laflamme, Nguyen Van Thé, Woodrow '14)

$\mathrm{M}\left(\mathrm{Aut}(\mathbb{S})\right) = \mathrm{Aut}(\mathbb{S}) \curvearrowright \overline{\mathrm{Aut}(\mathbb{S}) \cdot (<^*, R^*)}$, *where the closure is taken in the compact space* $\{0,1\}^{\mathbb{S}^2} \times \{0,1\}^{\mathbb{S}^2}$.

Borel sets of $\mathrm{M}(\mathrm{Aut}(\mathbb{S}))$

The Borel sets of $\mathrm{M}(\mathrm{Aut}(\mathbb{S}))$ are generated by clopen sets of the form:

$$U_{x_1,\ldots,x_n,(\varepsilon_1^2,\ldots,\varepsilon_{n-1}^n)} \cap V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)}.$$

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where:

$x_1,\ldots,x_n$ are in different classes in $\mathbb{S}$,
$\varepsilon_i^j \in \{0,1\}$ with $i < j \leq n$,

$$U = \{(<', R') \in \mathrm{M}(\mathrm{Aut}(\mathbb{S})) \mid x_1^{\sim} <' \cdots <' x_n^{\sim} \text{ and } R(x_j, x_i) \Leftrightarrow \varepsilon_i^j = 1\}$$

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and,
$a_i^k \sim a_j^k \ \forall i,j,k$,

$$V = \{(<',R') \in \mathrm{M}(\mathrm{Aut}(\mathbb{S})) \mid (a_1^1 <' \cdots <' a_{i_1}^1),\ldots,(a_1^k <' \ldots <' a_{i_k}^k)\}.$$

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Theorem (Pawliuk, Sokic '15)

There is an $\mathrm{Aut}(\mathbb{S})$*-invariant measure* μ_0 *such that:*

$$\mu_0\left(U_{x_1,\ldots,x_n,(\varepsilon_1^2,\ldots,\varepsilon_{n-1}^n)} \cap V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)}\right) = \frac{1}{n!2^{\binom{n}{2}}}\frac{1}{\prod_{j=1}^{k} i_j!}.$$

Sketch of Proof

Let μ be an $\mathrm{Aut}(\mathbb{S})$-invariant measure.

Proposition

$$\mu(U_{x_1,\ldots,x_n,(\varepsilon_1^2,\ldots,\varepsilon_{n-1}^n)}) = \frac{1}{n!2^{\binom{n}{2}}}$$

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Proposition

$$\mu\left(U \cap V\right) = \mu\left(U\right)\mu\left(V\right)$$

for all $U = U_{x_1,\ldots,x_n,(\varepsilon_i^j)}$ *and* $V = V_{(a_1^1,\ldots,a_{i_1}^1),\ldots,(a_1^k,\ldots,a_{i_k}^k)}$.

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Proof.

Take $x_1, \ldots, x_n$ and $U_1, \ldots, U_m$ the clopen sets corresponding to all the way to order their columns and add R.

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Proof.

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Define $\mu_U(\cdot) = \frac{\mu(\cdot \cap U)}{\mu(U)}$, as a measure on LO_p, the space of orderings inside columns.

$$\mu = \sum \mu_{U_i} \mu(U_i)$$

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There exists H a subgroup of $\mathrm{Aut}(\mathbb{S})$ such that:
i) $H \cdot U_i = U_i$ for all $i \leq m$.
ii) μ is H-ergodic.
Ergodic measures being extreme points, $\mu = \mu_{U_i}$ $\forall i \leq m$.

□

Other instances where this method works:

- Relational quotients (as defined by Sokic in ′13).
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All those groups are extensions and "carry" that extension in their UMFs.

Stability under extension

Let G be a Polish group, H a closed normal subgroup and K such that

$$1 \to H \to G \to K \to 1$$

is an exact sequence.

Theorem (J., Zucker)

If $\mathrm{M}(H)$ and $\mathrm{M}(K)$ are metrizable then $\mathrm{M}(G)$ is metrizable. Moreover, under these hypotheses, if H and K are uniquely ergodic, then G is uniquely ergodic.

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