Ramsey numbers of edge-ordered graphs

Martin Balko and Máté Vizer

Charles University,
Prague, Czech republic

November 21, 2018

Outline of the talk

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.
- We would like to know $\mathrm{R}(G)$ as precisely as possible.

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.
- We would like to know $\mathrm{R}(G)$ as precisely as possible.
- We discuss new variants of $\mathrm{R}(G)$ and their estimates.

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.
- We would like to know $\mathrm{R}(G)$ as precisely as possible.
- We discuss new variants of $\mathrm{R}(G)$ and their estimates.

- Outline:

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.
- We would like to know $\mathrm{R}(G)$ as precisely as possible.
- We discuss new variants of $\mathrm{R}(G)$ and their estimates.

- Outline:
  1. Ramsey numbers: graphs, some known (classical) bounds;

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.
- We would like to know $\mathrm{R}(G)$ as precisely as possible.
- We discuss new variants of $\mathrm{R}(G)$ and their estimates.
- Outline:
  1. Ramsey numbers: graphs, some known (classical) bounds;
  2. Ordered Ramsey numbers: ordered vertex sets, studied, but still relatively new;

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.
- We would like to know $\mathrm{R}(G)$ as precisely as possible.
- We discuss new variants of $\mathrm{R}(G)$ and their estimates.

- Outline:
  1. Ramsey numbers: graphs, some known (classical) bounds;
  2. Ordered Ramsey numbers: ordered vertex sets, studied, but still relatively new;
  3. Edge-ordered Ramsey numbers: ordered sets of edges, newly introduced variant, our work is still in progress.

Outline of the talk

- For a graph G, the Ramsey number $\mathrm{R}(G)$ of G is the smallest integer such that each 2-coloring of the edges of $K_{\mathrm{R}(G)}$ contains a monochromatic copy of G as a subgraph.
- We would like to know $\mathrm{R}(G)$ as precisely as possible.
- We discuss new variants of $\mathrm{R}(G)$ and their estimates.
- Outline:
  1. Ramsey numbers: graphs, some known (classical) bounds;
  2. Ordered Ramsey numbers: ordered vertex sets, studied, but still relatively new;
  3. Edge-ordered Ramsey numbers: ordered sets of edges, newly introduced variant, our work is still in progress.
- Unexplored area, a lot of interesting (and maybe difficult) problems.

Part 1

Ramsey numbers

Some classical bounds

Some classical bounds

- Obtaining good estimates on $\mathrm{R}(G)$ is notoriously hard.

Some classical bounds

- Obtaining good estimates on $\mathrm{R}(G)$ is notoriously hard.
- Except of very special cases, no exact formulas are known.

Some classical bounds

- Obtaining good estimates on $\mathrm{R}(G)$ is notoriously hard.
- Except of very special cases, no exact formulas are known.
- Classical bounds of Erdős and Szekeres: $2^{n/2} \leq \mathrm{R}(K_n) \leq 2^{2n}$.

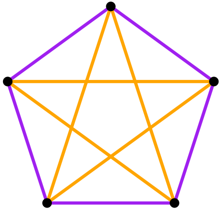
Some classical bounds

- Obtaining good estimates on $\mathrm{R}(G)$ is notoriously hard.
- Except of very special cases, no exact formulas are known.
- Classical bounds of Erdős and Szekeres: $2^{n/2} \leq \mathrm{R}(K_n) \leq 2^{2n}$.
- Smaller-term improvements obtained by Spencer and Conlon.

Some classical bounds

- Obtaining good estimates on $\mathrm{R}(G)$ is notoriously hard.
- Except of very special cases, no exact formulas are known.
- Classical bounds of Erdős and Szekeres: $2^{n/2} \leq \mathrm{R}(K_n) \leq 2^{2n}$.
- Smaller-term improvements obtained by Spencer and Conlon.

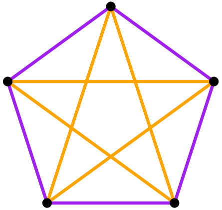
Example:



$\mathrm{R}(C_4) = 6$

Some classical bounds

- Obtaining good estimates on $\mathrm{R}(G)$ is notoriously hard.
- Except of very special cases, no exact formulas are known.
- Classical bounds of Erdős and Szekeres: $2^{n/2} \leq \mathrm{R}(K_n) \leq 2^{2n}$.
- Smaller-term improvements obtained by Spencer and Conlon.

Example:  $\mathrm{R}(C_4) = 6$

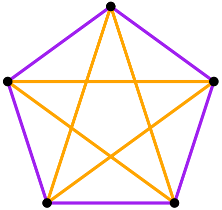
Theorem 1 (Chvátal, Rödl, Szemerédi, Trotter, 1983)

Every graph G on n vertices with bounded maximum degree satisfies

$$\mathrm{R}(G) \leq O(n).$$

Some classical bounds

- Obtaining good estimates on $\mathrm{R}(G)$ is notoriously hard.
- Except of very special cases, no exact formulas are known.
- Classical bounds of Erdős and Szekeres: $2^{n/2} \leq \mathrm{R}(K_n) \leq 2^{2n}$.
- Smaller-term improvements obtained by Spencer and Conlon.

Example:  $\mathrm{R}(C_4) = 6$

Theorem 1 (Chvátal, Rödl, Szemerédi, Trotter, 1983)

Every graph G on n vertices with bounded maximum degree satisfies

$$\mathrm{R}(G) \leq O(n).$$

- The linear upper bound holds even for graphs with bounded degeneracy (a solution of the Erdős–Burr conjecture by Lee, 2015).

Bounds for dense graphs

Bounds for dense graphs

- Thus very sparse graphs have small Ramsey numbers.

Bounds for dense graphs

- Thus very sparse graphs have small Ramsey numbers.
- How about dense graphs?

Bounds for dense graphs

- Thus very sparse graphs have small Ramsey numbers.
- How about dense graphs?
- If n-vertex G has density less than 1, is $\mathrm{R}(G) \leq (4-\varepsilon)^n$ for some $\varepsilon > 0$?

Bounds for dense graphs

- Thus very sparse graphs have small Ramsey numbers.
- How about dense graphs?
- If n-vertex G has density less than 1, is $\mathrm{R}(G) \leq (4-\varepsilon)^n$ for some $\varepsilon > 0$?

Theorem 2 (Conlon, 2012)

Every graph $G = (V, E)$ on n vertices with edge-density $\rho = |E|/n^2$ satisfies

$$\mathrm{R}(G) \leq 2^{O(\sqrt{\rho}\log(2/\rho)n)}.$$

Bounds for dense graphs

- Thus very sparse graphs have small Ramsey numbers.
- How about dense graphs?
- If n-vertex G has density less than 1, is $\mathrm{R}(G) \le (4-\varepsilon)^n$ for some $\varepsilon > 0$?

Theorem 2 (Conlon, 2012)

Every graph $G = (V, E)$ on n vertices with edge-density $\rho = |E|/n^2$ satisfies

$$\mathrm{R}(G) \le 2^{O(\sqrt{\rho}\log(2/\rho)n)}.$$

- Close to optimal, as a standard argument shows

$$\mathrm{R}(G) \ge 2^{\sqrt{\rho}n/4}$$

for some n-vertex graphs G with edge-density ρ.

Part 2

Ordered Ramsey numbers

Ordered Ramsey numbers

Ordered Ramsey numbers

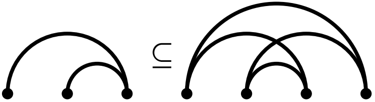
- An ordered graph $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertices.

Ordered Ramsey numbers

- An ordered graph $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertices.
- $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.

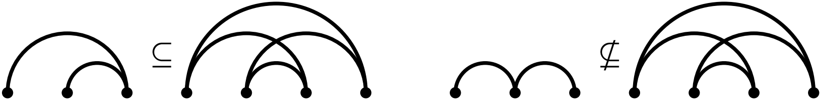
Ordered Ramsey numbers

- An ordered graph $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertices.
- $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



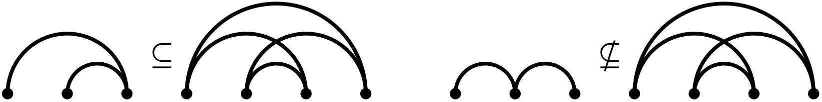
Ordered Ramsey numbers

- An ordered graph $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertices.
- $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



Ordered Ramsey numbers

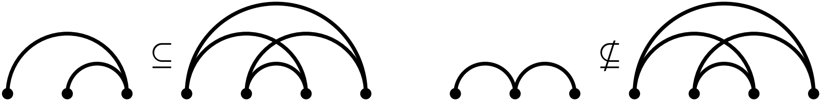
- An ordered graph $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertices.
- $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



- The ordered Ramsey number $\overline{\mathrm{R}}(\mathcal{G})$ of an ordered graph $\mathcal{G}$ is the minimum N such that every 2-coloring of the edges of $\mathcal{K}_N$ contains monochromatic copy of $\mathcal{G}$ as an ordered subgraph.

Ordered Ramsey numbers

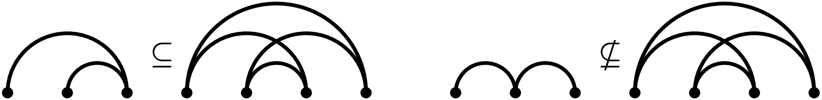
- An ordered graph $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertices.
- $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



- The ordered Ramsey number $\overline{\mathrm{R}}(\mathcal{G})$ of an ordered graph $\mathcal{G}$ is the minimum N such that every 2-coloring of the edges of $\mathcal{K}_N$ contains monochromatic copy of $\mathcal{G}$ as an ordered subgraph.
- Note that $\mathrm{R}(G) \leq \overline{\mathrm{R}}(\mathcal{G}) \leq \mathrm{R}(K_{|V(G)|})$ for every G and its ordering $\mathcal{G}$.

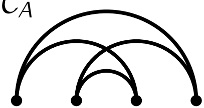
Ordered Ramsey numbers

- An ordered graph $\mathcal{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its vertices.
- $(H, \prec_1)$ is an ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



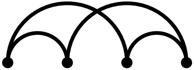
- The ordered Ramsey number $\overline{\mathrm{R}}(\mathcal{G})$ of an ordered graph $\mathcal{G}$ is the minimum N such that every 2-coloring of the edges of $\mathcal{K}_N$ contains monochromatic copy of $\mathcal{G}$ as an ordered subgraph.
- Note that $\mathrm{R}(G) \leq \overline{\mathrm{R}}(\mathcal{G}) \leq \mathrm{R}(K_{|V(G)|})$ for every G and its ordering $\mathcal{G}$.

$\mathcal{C}_A$



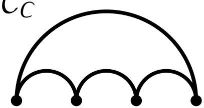
$\overline{\mathrm{R}}(\mathcal{C}_A) = 10$

$\mathcal{C}_B$



$\overline{\mathrm{R}}(\mathcal{C}_B) = 11$

$\mathcal{C}_C$



$\overline{\mathrm{R}}(\mathcal{C}_C) = 14$

Motivation

Motivation

- Motivation for studying ordered Ramsey numbers came from discrete geometry, especially due to its connections to the Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem.

Motivation

- Motivation for studying ordered Ramsey numbers came from discrete geometry, especially due to its connections to the Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem.

The Erdős–Szekeres Lemma (Erdős, Szekeres, 1935)

For every $n \in \mathbb{N}$, every sequence of $(n-1)^2+1$ distinct numbers contains an increasing or a decreasing subsequence of length n. This bound is tight.

Motivation

- Motivation for studying ordered Ramsey numbers came from discrete geometry, especially due to its connections to the Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem.

The Erdős–Szekeres Lemma (Erdős, Szekeres, 1935)

For every $n \in \mathbb{N}$, every sequence of $(n-1)^2 + 1$ distinct numbers contains an increasing or a decreasing subsequence of length n. This bound is tight.

- The monotone path $\mathcal{P}_n$ is an ordered graph with n vertices and edges formed by pairs of consecutive vertices.

Motivation

- Motivation for studying ordered Ramsey numbers came from discrete geometry, especially due to its connections to the Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem.

The Erdős–Szekeres Lemma (Erdős, Szekeres, 1935)

For every $n \in \mathbb{N}$, every sequence of $(n-1)^2 + 1$ distinct numbers contains an increasing or a decreasing subsequence of length n. This bound is tight.

- The monotone path $\mathcal{P}_n$ is an ordered graph with n vertices and edges formed by pairs of consecutive vertices.

Motivation

- Motivation for studying ordered Ramsey numbers came from discrete geometry, especially due to its connections to the Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem.

The Erdős–Szekeres Lemma (Erdős, Szekeres, 1935)

For every $n \in \mathbb{N}$, every sequence of $(n-1)^2+1$ distinct numbers contains an increasing or a decreasing subsequence of length n. This bound is tight.

- The monotone path $\mathcal{P}_n$ is an ordered graph with n vertices and edges formed by pairs of consecutive vertices.

- The Erdős–Szekeres Lemma is a corollary of the following result.

Motivation

- Motivation for studying ordered Ramsey numbers came from discrete geometry, especially due to its connections to the Erdős–Szekeres Lemma and the Erdős–Szekeres Theorem.

The Erdős–Szekeres Lemma (Erdős, Szekeres, 1935)

For every $n \in \mathbb{N}$, every sequence of $(n-1)^2+1$ distinct numbers contains an increasing or a decreasing subsequence of length n. This bound is tight.

- The monotone path $\mathcal{P}_n$ is an ordered graph with n vertices and edges formed by pairs of consecutive vertices.

- The Erdős–Szekeres Lemma is a corollary of the following result.

Proposition 1

For every $n \in \mathbb{N}$, we have

$$\overline{\mathrm{R}}(\mathcal{P}_n) = (n-1)^2 + 1.$$

Lower bounds

Lower bounds

- Ordered Ramsey numbers for graphs are relatively understood.

Lower bounds

- Ordered Ramsey numbers for graphs are relatively understood.
- They can behave very differently than classical Ramsey numbers.

Lower bounds

- Ordered Ramsey numbers for graphs are relatively understood.
- They can behave very differently than classical Ramsey numbers.

Theorem 3 (B., Cibulka, Kynčl, Král, 2015)

There are arbitrarily large ordered matchings $\mathcal{M}_n$ on n vertices such that

$$\overline{\mathrm{R}}(\mathcal{M}_n) \geq n^{\Omega\left(\frac{\log n}{\log \log n}\right)}.$$

Lower bounds

- Ordered Ramsey numbers for graphs are relatively understood.
- They can behave very differently than classical Ramsey numbers.

Theorem 3 (B., Cibulka, Kynčl, Král, 2015)

There are arbitrarily large ordered matchings $\mathcal{M}_n$ on n vertices such that

$$\overline{\mathrm{R}}(\mathcal{M}_n) \geq n^{\Omega\left(\frac{\log n}{\log \log n}\right)}.$$

- True for almost every ordered matching (Conlon, Fox, Lee, Sudakov).

Lower bounds

- Ordered Ramsey numbers for graphs are relatively understood.
- They can behave very differently than classical Ramsey numbers.

Theorem 3 (B., Cibulka, Kynčl, Král, 2015)

There are arbitrarily large ordered matchings $\mathcal{M}_n$ on n vertices such that

$$\overline{\mathrm{R}}(\mathcal{M}_n) \geq n^{\Omega\left(\frac{\log n}{\log \log n}\right)}.$$

- True for almost every ordered matching (Conlon, Fox, Lee, Sudakov).
- Contrasts with the result of Chvátal, Rödl, Szemerédi, Trotter.

Lower bounds

- Ordered Ramsey numbers for graphs are relatively understood.
- They can behave very differently than classical Ramsey numbers.

Theorem 3 (B., Cibulka, Kynčl, Král, 2015)

There are arbitrarily large ordered matchings $\mathcal{M}_n$ on n vertices such that

$$\overline{\mathrm{R}}(\mathcal{M}_n) \geq n^{\Omega\left(\frac{\log n}{\log \log n}\right)}.$$

- True for almost every ordered matching (Conlon, Fox, Lee, Sudakov).
- Contrasts with the result of Chvátal, Rödl, Szemerédi, Trotter.
- There are 3-regular graphs G such that no ordering $\mathcal{G}$ has linear $\overline{\mathrm{R}}(\mathcal{G})$.

Lower bounds

- Ordered Ramsey numbers for graphs are relatively understood.
- They can behave very differently than classical Ramsey numbers.

Theorem 3 (B., Cibulka, Kynčl, Král, 2015)

There are arbitrarily large ordered matchings $\mathcal{M}_n$ on n vertices such that

$$\overline{\mathrm{R}}(\mathcal{M}_n) \geq n^{\Omega\left(\frac{\log n}{\log \log n}\right)}.$$

- True for almost every ordered matching (Conlon, Fox, Lee, Sudakov).
- Contrasts with the result of Chvátal, Rödl, Szemerédi, Trotter.
- There are 3-regular graphs G such that no ordering $\mathcal{G}$ has linear $\overline{\mathrm{R}}(\mathcal{G})$.

Theorem 4 (B., Jelínek, Valtr, 2016)

For every $d \geq 3$, almost every d-regular graph G on n vertices satisfies $\overline{\mathrm{R}}(\mathcal{G}) \geq \frac{n^{3/2-1/d}}{4 \log n \log \log n}$ for every ordering $\mathcal{G}$ of G.

Upper bounds

Upper bounds

- Although $\overline{\mathrm{R}}(\mathcal{G})$ is almost always superpolynomial, there are classes of ordered graphs with polynomial ordered Ramsey numbers.

Upper bounds

- Although $\overline{\mathrm{R}}(\mathcal{G})$ is almost always superpolynomial, there are classes of ordered graphs with polynomial ordered Ramsey numbers.
- The interval chromatic number $\chi_{\prec}(G)$ of $(G, \prec)$ is the minimum number of intervals $V(G)$ can be partitioned into so that no two adjacent vertices are in the same interval.

Upper bounds

- Although $\overline{\mathrm{R}}(\mathcal{G})$ is almost always superpolynomial, there are classes of ordered graphs with polynomial ordered Ramsey numbers.
- The interval chromatic number $\chi_{\prec}(G)$ of $(G, \prec)$ is the minimum number of intervals $V(G)$ can be partitioned into so that no two adjacent vertices are in the same interval.

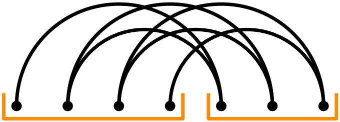
$$\chi_{\prec_{mon}}(P_n) = n$$

Upper bounds

- Although $\overline{\mathrm{R}}(\mathcal{G})$ is almost always superpolynomial, there are classes of ordered graphs with polynomial ordered Ramsey numbers.
- The interval chromatic number $\chi_{\prec}(G)$ of $(G, \prec)$ is the minimum number of intervals $V(G)$ can be partitioned into so that no two adjacent vertices are in the same interval.

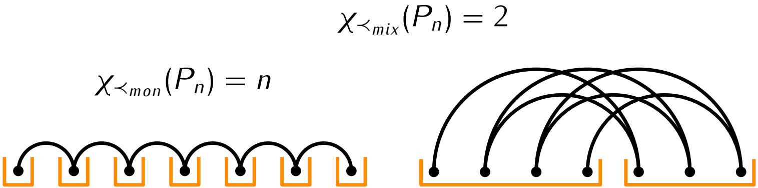
$$\chi_{\prec_{mon}}(P_n) = n$$

$$\chi_{\prec_{mix}}(P_n) = 2$$



Upper bounds

- Although $\overline{\mathrm{R}}(\mathcal{G})$ is almost always superpolynomial, there are classes of ordered graphs with polynomial ordered Ramsey numbers.
- The interval chromatic number $\chi_{\prec}(G)$ of $(G, \prec)$ is the minimum number of intervals $V(G)$ can be partitioned into so that no two adjacent vertices are in the same interval.



Theorem 5 (B., Cibulka, Kynčl, Král, 2015)

For all k and p every k-degenerate ordered graph $\mathcal{G} = (G, \prec)$ with n vertices and $\chi_{\prec}(G) = p$ satisfies

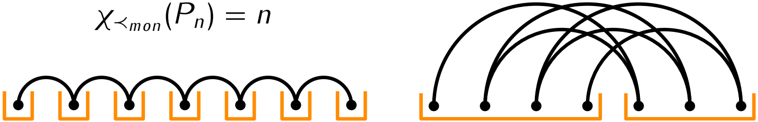
$$\overline{\mathrm{R}}(\mathcal{G}) \leq n^{O(k)^{\log p}}.$$

Upper bounds

- Although $\overline{\mathrm{R}}(\mathcal{G})$ is almost always superpolynomial, there are classes of ordered graphs with polynomial ordered Ramsey numbers.
- The interval chromatic number $\chi_{\prec}(G)$ of $(G, \prec)$ is the minimum number of intervals $V(G)$ can be partitioned into so that no two adjacent vertices are in the same interval.

$\chi_{\prec_{mix}}(P_n) = 2$

$\chi_{\prec_{mon}}(P_n) = n$



Theorem 5 (B., Cibulka, Kynčl, Král, 2015)

For all k and p every k-degenerate ordered graph $\mathcal{G} = (G, \prec)$ with n vertices and $\chi_{\prec}(G) = p$ satisfies

$$\overline{\mathrm{R}}(\mathcal{G}) \leq n^{O(k)^{\log p}}.$$

- Improved to $\overline{\mathrm{R}}(\mathcal{G}) \leq n^{O(k \log p)}$ (Conlon, Fox, Lee, and Sudakov).

Dense ordered graphs

Dense ordered graphs

- Conlon et al. derived a bound that works even for dense ordered graphs.

Dense ordered graphs

- Conlon et al. derived a bound that works even for dense ordered graphs.

Theorem 6 (Conlon, Fox, Lee, and Sudakov, 2015)

For every ordered graph $\mathcal{G}$ with n vertices and degeneracy d,

$$\overline{\mathrm{R}}(\mathcal{G}) \leq 2^{O(d \log^2 (2n/d))}.$$

Dense ordered graphs

- Conlon et al. derived a bound that works even for dense ordered graphs.

Theorem 6 (Conlon, Fox, Lee, and Sudakov, 2015)

For every ordered graph $\mathcal{G}$ with n vertices and degeneracy d,

$$\overline{\mathrm{R}}(\mathcal{G}) \leq 2^{O(d \log^2 (2n/d))}.$$

- To summarize:

Dense ordered graphs

- Conlon et al. derived a bound that works even for dense ordered graphs.

Theorem 6 (Conlon, Fox, Lee, and Sudakov, 2015)

For every ordered graph $\mathcal{G}$ with n vertices and degeneracy d,

$$\overline{\mathrm{R}}(\mathcal{G}) \leq 2^{O(d \log^2 (2n/d))}.$$

- To summarize:
  - Ordered Ramsey numbers are at most exponential.

Dense ordered graphs

- Conlon et al. derived a bound that works even for dense ordered graphs.

Theorem 6 (Conlon, Fox, Lee, and Sudakov, 2015)

For every ordered graph $\mathcal{G}$ with n vertices and degeneracy d,

$$\overline{\mathrm{R}}(\mathcal{G}) \leq 2^{O(d \log^2 (2n/d))}.$$

- To summarize:
  - Ordered Ramsey numbers are at most exponential.
  - They are at most quasi-polynomial for bounded-degree graphs.

Dense ordered graphs

- Conlon et al. derived a bound that works even for dense ordered graphs.

Theorem 6 (Conlon, Fox, Lee, and Sudakov, 2015)

For every ordered graph $\mathcal{G}$ with n vertices and degeneracy d,

$$\overline{\mathrm{R}}(\mathcal{G}) \leq 2^{O(d \log^2 (2n/d))}.$$

- To summarize:
  - Ordered Ramsey numbers are at most exponential.
  - They are at most quasi-polynomial for bounded-degree graphs.
  - For almost all ordered graphs $\mathcal{G}$, the numbers $\overline{\mathrm{R}}(\mathcal{G})$ are at least super-polynomial.

Dense ordered graphs

- Conlon et al. derived a bound that works even for dense ordered graphs.

Theorem 6 (Conlon, Fox, Lee, and Sudakov, 2015)

For every ordered graph $\mathcal{G}$ with n vertices and degeneracy d,

$$\overline{\mathrm{R}}(\mathcal{G}) \leq 2^{O(d \log^2 (2n/d))}.$$

- To summarize:
  - Ordered Ramsey numbers are at most exponential.
  - They are at most quasi-polynomial for bounded-degree graphs.
  - For almost all ordered graphs $\mathcal{G}$, the numbers $\overline{\mathrm{R}}(\mathcal{G})$ are at least super-polynomial.

- There is still a gap between the lower bound $n^{\Omega(\log n/ \log \log n)}$ and the upper bound $n^{O(\log n)}$ for ordered Ramsey numbers of ordered graphs on n vertices with bounded degeneracy.

Part 3

Edge-ordered Ramsey numbers

Edge-ordered graphs

Edge-ordered graphs

- Recently, there have been some new studies of Turán-type questions for edge-ordered graphs (done, for example, by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, Vizer).

Edge-ordered graphs

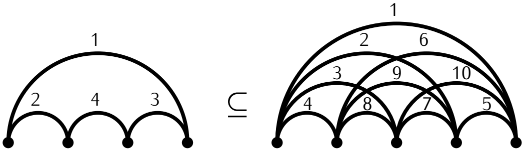
- Recently, there have been some new studies of Turán-type questions for edge-ordered graphs (done, for example, by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, Vizer).
- An edge-ordered graph $\mathfrak{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its edges.

Edge-ordered graphs

- Recently, there have been some new studies of Turán-type questions for edge-ordered graphs (done, for example, by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, Vizer).
- An edge-ordered graph $\mathfrak{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its edges.
- $(H, \prec_1)$ is an edge-ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.

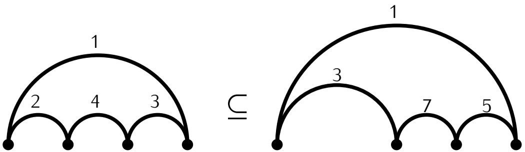
Edge-ordered graphs

- Recently, there have been some new studies of Turán-type questions for edge-ordered graphs (done, for example, by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, Vizer).
- An edge-ordered graph $\mathfrak{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its edges.
- $(H, \prec_1)$ is an edge-ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



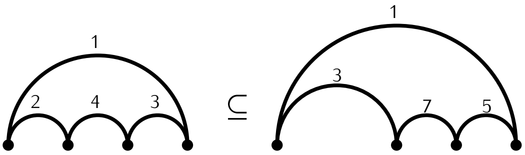
Edge-ordered graphs

- Recently, there have been some new studies of Turán-type questions for edge-ordered graphs (done, for example, by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, Vizer).
- An edge-ordered graph $\mathfrak{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its edges.
- $(H, \prec_1)$ is an edge-ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



Edge-ordered graphs

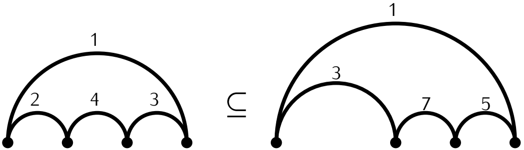
- Recently, there have been some new studies of Turán-type questions for edge-ordered graphs (done, for example, by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, Vizer).
- An edge-ordered graph $\mathfrak{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its edges.
- $(H, \prec_1)$ is an edge-ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



- What can we say about their Ramsey numbers?

Edge-ordered graphs

- Recently, there have been some new studies of Turán-type questions for edge-ordered graphs (done, for example, by Gerbner, Methuku, Nagy, Pálvölgyi, Tardos, Vizer).
- An edge-ordered graph $\mathfrak{G}$ is a pair $(G, \prec)$ where G is a graph and $\prec$ is a total ordering of its edges.
- $(H, \prec_1)$ is an edge-ordered subgraph of $(G, \prec_2)$ if $H \subseteq G$ and $\prec_1 \subseteq \prec_2$.



- What can we say about their Ramsey numbers?
- Not much yet, our work is still in progress.

Lexicographic edge-ordered Ramsey numbers

Lexicographic edge-ordered Ramsey numbers

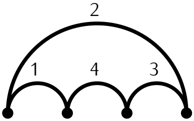
- How to define Ramsey numbers for edge-ordered graphs?

Lexicographic edge-ordered Ramsey numbers

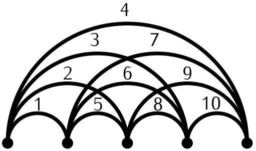
- How to define Ramsey numbers for edge-ordered graphs?
- For every $N \in \mathbb{N}$, there might be edge-ordered $\mathfrak{G}$ and $\mathfrak{K}_N$ such that $\mathfrak{G}$ is not an edge-ordered subgraph of $\mathfrak{K}_N$.

Lexicographic edge-ordered Ramsey numbers

- How to define Ramsey numbers for edge-ordered graphs?
- For every $N \in \mathbb{N}$, there might be edge-ordered $\mathfrak{G}$ and $\mathfrak{K}_N$ such that $\mathfrak{G}$ is not an edge-ordered subgraph of $\mathfrak{K}_N$.



$\not\subseteq$



Lexicographic edge-ordered Ramsey numbers

- How to define Ramsey numbers for edge-ordered graphs?
- For every $N \in \mathbb{N}$, there might be edge-ordered $\mathfrak{G}$ and $\mathfrak{K}_N$ such that $\mathfrak{G}$ is not an edge-ordered subgraph of $\mathfrak{K}_N$.

- An edge-ordered graph $(G, <_{lex})$ is lexicographically edge-ordered if there is a bijection $f\colon V \to [|V|]$ such that all edges uv and wt of G with $f(u) < f(v)$ and $f(w) < f(t)$ satisfy $uv <_{lex} wt$ if and only if $f(u) < f(w)$ or $(f(u) = f(w)\ \&\ f(v) < f(t))$.

Lexicographic edge-ordered Ramsey numbers

- How to define Ramsey numbers for edge-ordered graphs?
- For every $N \in \mathbb{N}$, there might be edge-ordered $\mathfrak{G}$ and $\mathfrak{K}_N$ such that $\mathfrak{G}$ is not an edge-ordered subgraph of $\mathfrak{K}_N$.

- An edge-ordered graph $(G, <_{lex})$ is lexicographically edge-ordered if there is a bijection $f\colon V \to [|V|]$ such that all edges uv and wt of G with $f(u) < f(v)$ and $f(w) < f(t)$ satisfy $uv <_{lex} wt$ if and only if $f(u) < f(w)$ or $(f(u) = f(w)\ \&\ f(v) < f(t))$.
- First idea: restrict ourselves to a special class of edge-ordered graphs.

Lexicographic edge-ordered Ramsey numbers

- How to define Ramsey numbers for edge-ordered graphs?
- For every $N \in \mathbb{N}$, there might be edge-ordered $\mathfrak{G}$ and $\mathfrak{K}_N$ such that $\mathfrak{G}$ is not an edge-ordered subgraph of $\mathfrak{K}_N$.

- An edge-ordered graph $(G, <_{lex})$ is lexicographically edge-ordered if there is a bijection $f\colon V \to [|V|]$ such that all edges uv and wt of G with $f(u) < f(v)$ and $f(w) < f(t)$ satisfy $uv <_{lex} wt$ if and only if $f(u) < f(w)$ or $(f(u) = f(w)\ \&\ f(v) < f(t))$.
- First idea: restrict ourselves to a special class of edge-ordered graphs.
- The lexicographic edge-ordered Ramsey number $\overline{\mathrm{R}}_{lex}(G)$ of G is the minimum N such that every 2-coloring of $(K_N, <_{lex})$ contains monochromatic copy of $(G, <_{lex})$ as an edge-ordered subgraph.

Bounds on lexicographic edge-ordered Ramsey numbers

Bounds on lexicographic edge-ordered Ramsey numbers

- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.

Bounds on lexicographic edge-ordered Ramsey numbers

- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.

Bounds on lexicographic edge-ordered Ramsey numbers

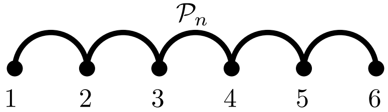
- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.
- So, for a bounded-degree G, $\overline{\mathrm{R}}_{lex}(G)$ is at most quasi-polynomial.

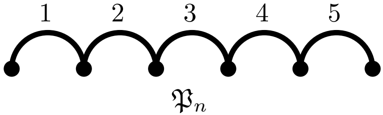
Bounds on lexicographic edge-ordered Ramsey numbers

- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.
- So, for a bounded-degree G, $\overline{\mathrm{R}}_{lex}(G)$ is at most quasi-polynomial.
- We can do better sometimes (edge-ordered stars, matchings).

Bounds on lexicographic edge-ordered Ramsey numbers

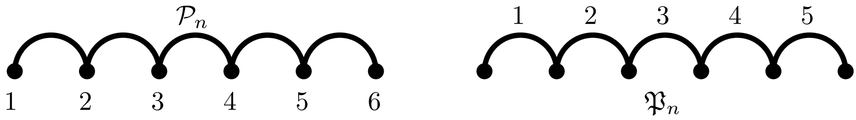
- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.
- So, for a bounded-degree G, $\overline{\mathrm{R}}_{lex}(G)$ is at most quasi-polynomial.
- We can do better sometimes (edge-ordered stars, matchings).
- Special case: edge-monotone path $\mathfrak{P}_n$.





Bounds on lexicographic edge-ordered Ramsey numbers

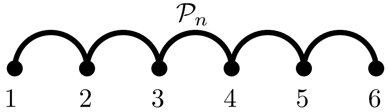
- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.
- So, for a bounded-degree G, $\overline{\mathrm{R}}_{lex}(G)$ is at most quasi-polynomial.
- We can do better sometimes (edge-ordered stars, matchings).
- Special case: edge-monotone path $\mathfrak{P}_n$.

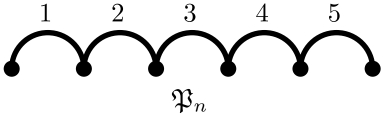


- Recall: $\overline{\mathrm{R}}(\mathcal{P}_n) = (n-1)^2 + 1$ (the Erdős–Szekeres Lemma).

Bounds on lexicographic edge-ordered Ramsey numbers

- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.
- So, for a bounded-degree G, $\overline{\mathrm{R}}_{lex}(G)$ is at most quasi-polynomial.
- We can do better sometimes (edge-ordered stars, matchings).
- Special case: edge-monotone path $\mathfrak{P}_n$.





- Recall: $\overline{\mathrm{R}}(\mathcal{P}_n) = (n-1)^2 + 1$ (the Erdős–Szekeres Lemma).

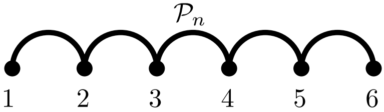
Proposition 2 (B., Vizer, 2018++)

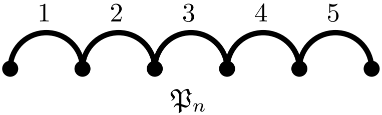
For every $n \in \mathbb{N}$,

$$\overline{\mathrm{R}}_{lex}(\mathfrak{P}_n) \leq O(n).$$

Bounds on lexicographic edge-ordered Ramsey numbers

- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.
- So, for a bounded-degree G, $\overline{\mathrm{R}}_{lex}(G)$ is at most quasi-polynomial.
- We can do better sometimes (edge-ordered stars, matchings).
- Special case: edge-monotone path $\mathfrak{P}_n$.





- Recall: $\overline{\mathrm{R}}(\mathcal{P}_n) = (n-1)^2 + 1$ (the Erdős–Szekeres Lemma).

Proposition 2 (B., Vizer, 2018++)

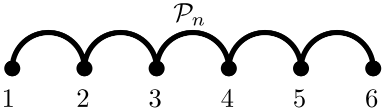
For every $n \in \mathbb{N}$,

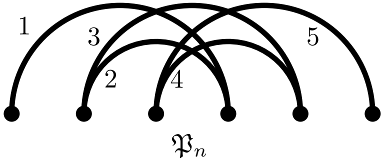
$$\overline{\mathrm{R}}_{lex}(\mathfrak{P}_n) \leq O(n).$$

- Proof is based on an estimate of $\overline{\mathrm{R}}((P_n, \prec_{mix}))$.

Bounds on lexicographic edge-ordered Ramsey numbers

- Clearly, $\mathrm{R}(G) \leq \overline{\mathrm{R}}_{lex}(G) \leq \mathrm{R}(K_{|V(G)|})$ for every graph G.
- In fact, $\overline{\mathrm{R}}_{lex}(G) \leq \overline{\mathrm{R}}(\mathcal{G})$ for each G and its ordering $\mathcal{G}$ induced by f.
- So, for a bounded-degree G, $\overline{\mathrm{R}}_{lex}(G)$ is at most quasi-polynomial.
- We can do better sometimes (edge-ordered stars, matchings).
- Special case: edge-monotone path $\mathfrak{P}_n$.





- Recall: $\overline{\mathrm{R}}(\mathcal{P}_n) = (n-1)^2 + 1$ (the Erdős–Szekeres Lemma).

Proposition 2 (B., Vizer, 2018++)

For every $n \in \mathbb{N}$,

$$\overline{\mathrm{R}}_{lex}(\mathfrak{P}_n) \leq O(n).$$

- Proof is based on an estimate of $\overline{\mathrm{R}}((P_n, \prec_{mix}))$.

Edge-ordered Ramsey numbers

Edge-ordered Ramsey numbers

- Using a “greedy embedding technique”, we obtain the following result.

Edge-ordered Ramsey numbers

- Using a "greedy embedding technique", we obtain the following result.

Theorem 7 (B., Vizer, 2018++)

Let $\mathfrak{H}$ be a d-degenerate edge-ordered graph on n vertices and let G be a bipartite graph on n vertices. Then there exists edge-ordered $\mathfrak{K}_{2n^{d+3}}$ such that every red-blue coloring of its edges contains either a blue $\mathfrak{H}$ or a red $(G, <_{lex})$ as an edge-ordered subgraph.

Edge-ordered Ramsey numbers

- Using a "greedy embedding technique", we obtain the following result.

Theorem 7 (B., Vizer, 2018++)

Let $\mathfrak{H}$ be a d-degenerate edge-ordered graph on n vertices and let G be a bipartite graph on n vertices. Then there exists edge-ordered $\mathfrak{K}_{2n^{d+3}}$ such that every red-blue coloring of its edges contains either a blue $\mathfrak{H}$ or a red $(G, <_{lex})$ as an edge-ordered subgraph.

- Should we consider a broader class of edge-ordered graphs?

Edge-ordered Ramsey numbers

- Using a "greedy embedding technique", we obtain the following result.

Theorem 7 (B., Vizer, 2018++)

Let $\mathfrak{H}$ be a d-degenerate edge-ordered graph on n vertices and let G be a bipartite graph on n vertices. Then there exists edge-ordered $\mathfrak{K}_{2n^{d+3}}$ such that every red-blue coloring of its edges contains either a blue $\mathfrak{H}$ or a red $(G, <_{lex})$ as an edge-ordered subgraph.

- Should we consider a broader class of edge-ordered graphs?
- The edge-ordered Ramsey number $\overline{\mathrm{R}}_e(\mathfrak{G})$ of an edge-ordered graph $\mathfrak{G}$ is the minimum N such that there exists edge-ordered $\mathfrak{K}_N$ such that for every 2-coloring of the edges of $\mathfrak{K}_N$ contains monochromatic copy of $\mathfrak{G}$ as an edge-ordered subgraph.

Edge-ordered Ramsey numbers

- Using a "greedy embedding technique", we obtain the following result.

Theorem 7 (B., Vizer, 2018++)

Let $\mathfrak{H}$ be a d-degenerate edge-ordered graph on n vertices and let G be a bipartite graph on n vertices. Then there exists edge-ordered $\mathfrak{K}_{2n^{d+3}}$ such that every red-blue coloring of its edges contains either a blue $\mathfrak{H}$ or a red $(G, <_{lex})$ as an edge-ordered subgraph.

- Should we consider a broader class of edge-ordered graphs?
- The edge-ordered Ramsey number $\overline{\mathrm{R}}_e(\mathfrak{G})$ of an edge-ordered graph $\mathfrak{G}$ is the minimum N such that there exists edge-ordered $\mathfrak{K}_N$ such that for every 2-coloring of the edges of $\mathfrak{K}_N$ contains monochromatic copy of $\mathfrak{G}$ as an edge-ordered subgraph.
- Clearly, $\overline{\mathrm{R}}_e((G, <_{lex})) \leq \overline{\mathrm{R}}_{lex}(G)$ for every G.

Edge-ordered Ramsey numbers

- Using a "greedy embedding technique", we obtain the following result.

Theorem 7 (B., Vizer, 2018++)

Let $\mathfrak{H}$ be a d-degenerate edge-ordered graph on n vertices and let G be a bipartite graph on n vertices. Then there exists edge-ordered $\mathfrak{K}_{2n^{d+3}}$ such that every red-blue coloring of its edges contains either a blue $\mathfrak{H}$ or a red $(G, <_{lex})$ as an edge-ordered subgraph.

- Should we consider a broader class of edge-ordered graphs?
- The edge-ordered Ramsey number $\overline{\mathrm{R}}_e(\mathfrak{G})$ of an edge-ordered graph $\mathfrak{G}$ is the minimum N such that there exists edge-ordered $\mathfrak{K}_N$ such that for every 2-coloring of the edges of $\mathfrak{K}_N$ contains monochromatic copy of $\mathfrak{G}$ as an edge-ordered subgraph.
- Clearly, $\overline{\mathrm{R}}_e((G, <_{lex})) \leq \overline{\mathrm{R}}_{lex}(G)$ for every G.
- Are these numbers finite for every edge-ordered $\mathfrak{G}$?

Bounds on edge-ordered Ramsey numbers

Bounds on edge-ordered Ramsey numbers

- It follows from the Graham–Rotschild theorem that $\overline{\mathrm{R}}_e(\mathfrak{G})$ is finite for every edge-ordered $\mathfrak{G}$.

Bounds on edge-ordered Ramsey numbers

- It follows from the Graham–Rotschild theorem that $\overline{R}_e(\mathfrak{G})$ is finite for every edge-ordered $\mathfrak{G}$.
- However, the obtained bound is enormous.

Bounds on edge-ordered Ramsey numbers

- It follows from the Graham–Rotschild theorem that $\overline{\mathrm{R}}_e(\mathfrak{G})$ is finite for every edge-ordered $\mathfrak{G}$.
- However, the obtained bound is enormous.
- By a result of Shelah (1988), the bound is primitive recursive.

Bounds on edge-ordered Ramsey numbers

- It follows from the Graham–Rotschild theorem that $\overline{\mathrm{R}}_e(\mathfrak{G})$ is finite for every edge-ordered $\mathfrak{G}$.
- However, the obtained bound is enormous.
- By a result of Shelah (1988), the bound is primitive recursive.
- Can we get better bounds?

Bounds on edge-ordered Ramsey numbers

- It follows from the Graham–Rotschild theorem that $\overline{\mathrm{R}}_e(\mathfrak{G})$ is finite for every edge-ordered $\mathfrak{G}$.
- However, the obtained bound is enormous.
- By a result of Shelah (1988), the bound is primitive recursive.
- Can we get better bounds?

Theorem 8 (B., Vizer, 2018++)

Let $\mathfrak{H}$ be an edge-ordered graph on n vertices and let $\mathfrak{G}$ be a bipartite edge-ordered graph with n vertices and m edges. Then there exists edge-ordered $\mathfrak{K}_N$ with $N \leq 2^{O(nm \log m)}$ such that every red-blue coloring of its edges contains either a blue $\mathfrak{H}$ or a red $\mathfrak{G}$ as an edge-ordered subgraph.

Bounds on edge-ordered Ramsey numbers

- It follows from the Graham–Rotschild theorem that $\overline{\mathrm{R}}_e(\mathfrak{G})$ is finite for every edge-ordered $\mathfrak{G}$.
- However, the obtained bound is enormous.
- By a result of Shelah (1988), the bound is primitive recursive.
- Can we get better bounds?

Theorem 8 (B., Vizer, 2018++)

Let $\mathfrak{H}$ be an edge-ordered graph on n vertices and let $\mathfrak{G}$ be a bipartite edge-ordered graph with n vertices and m edges. Then there exists edge-ordered $\mathfrak{K}_N$ with $N \leq 2^{O(nm \log m)}$ such that every red-blue coloring of its edges contains either a blue $\mathfrak{H}$ or a red $\mathfrak{G}$ as an edge-ordered subgraph.

- No nontrivial lower bounds.

Open problems

Open problems

- We have more questions than answers.

Open problems

- We have more questions than answers.
- Some open problems:

Open problems

- We have more questions than answers.
- Some open problems:
  - Can we improve upper bounds on edge-ordered Ramsey numbers? (At least for some classes of edge-ordered graphs.)

Open problems

- We have more questions than answers.
- Some open problems:
  - Can we improve upper bounds on edge-ordered Ramsey numbers? (At least for some classes of edge-ordered graphs.)
  - Are there asymptotically better lower bounds on edge-ordered Ramsey numbers of complete graphs than the ones obtained from lower bounds on Ramsey numbers?

Open problems

- We have more questions than answers.
- Some open problems:
  - Can we improve upper bounds on edge-ordered Ramsey numbers? (At least for some classes of edge-ordered graphs.)
  - Are there asymptotically better lower bounds on edge-ordered Ramsey numbers of complete graphs than the ones obtained from lower bounds on Ramsey numbers?
  - Do edge-ordered Ramsey numbers grow significantly faster than ordered Ramsey numbers for some edge-ordered graphs?

Open problems

- We have more questions than answers.
- Some open problems:
  - Can we improve upper bounds on edge-ordered Ramsey numbers? (At least for some classes of edge-ordered graphs.)
  - Are there asymptotically better lower bounds on edge-ordered Ramsey numbers of complete graphs than the ones obtained from lower bounds on Ramsey numbers?
  - Do edge-ordered Ramsey numbers grow significantly faster than ordered Ramsey numbers for some edge-ordered graphs?

Thank you.