On equality of two classes of homomorphism-homogeneous relational structures

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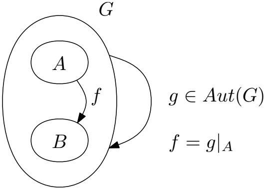
Banff 2018

Czech Academy of Sciences

Motivate

Definition (Ultrahomogeneity)

$\mathcal{A}$ is ultrahomogeneous if every isomorphism
$f : A \to B$ between finite substructures
$A, B \subset \mathcal{A}$ can be extended into automorphism



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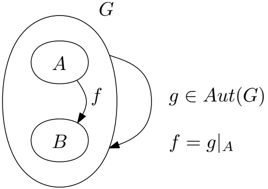

How far structures from homogeneity

- **Relational complexity** r (Cherlin, Martin, Saracino 1996) i.e. expand language with relations of arity $\leq r$ s.t. autmomorphism group remains and this lift is homogeneous

- For graphs (H., Hubička, Nešetřil, 2015) used result of (Hubička, Nešetřil, 2014) to homogenize graphs with forbidden homomorphism.

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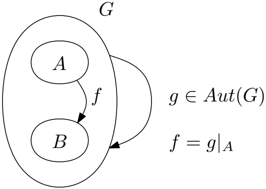

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- For graphs (H., Hubička, Nešetřil, 2015) used result of (Hubička, Nešetřil, 2014) to homogenize graphs with forbidden homomorphism.
- Relax definition of homogeneity
  Use homomorphism instead of isomorphism
  It also have Fraïssé type results

(Ultra) Homogeneity of structures

Considered structures a

- *relational structure* $\mathcal{A} = (A, \mathcal{R}_A)$ where $\mathcal{R}_A = (R^i_A; i \in I)$
- Usually interpreted as colored graphs
  (consider just binary relations - see later)

Classifications usually differs depending on

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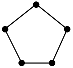
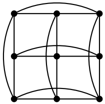
Classification for finite graphs (Gardiner 1976)

- Combinatorial argument utilizing finiteness of structures

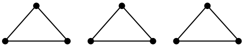
C_5:



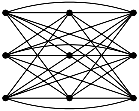
$L(K_{3,3})$:



$\dot{\bigcup}_{i=1}^{n} K_m$:



$K_{n,n,\ldots,n}$:



Ultrahomogeneity of countable graphs

Rado graph $\mathcal{R}$

- Countably infinite random graph

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- Useful property
  (∗) $\forall X, Y$ finite $\exists z$ s.t. $z \sim x \ \forall x \in X$ and $z \nsim y \ \forall y \in Y$

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Important notes

- It is, of course, a Fraïssé limit for class of all graphs
  - Showing this gives us all properties above
- Part of complete classification (Lachlan, Woodrow 1980)

Homomorphism-homogeneity

Variants of homogeneity (Cameron, Nešetřil 2006)

- homomorphism-homogeneity (**HH**)
  - local homomorphism $\rightarrow$ homomorphism
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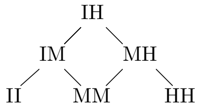

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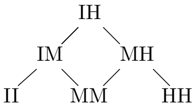


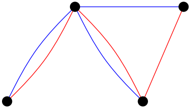
Problems

- Classification beyond finite graphs
  - Finite HH Graphs (Cameron, Nešetřil 2006)
- HH $\subseteq$ MH, HH $=$ MH ?
  - For countable graphs YES! (Rusinov, Schweitzer 2010)
  - For general structures NO!

HH = **MH** using pumping argument

Simple extension: **Bicolored graphs**

- graphs without loops having red/blue edges
- Are classes **HH** and **MH** equal?



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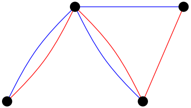

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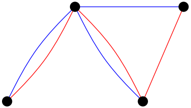

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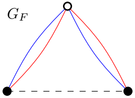

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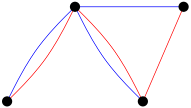

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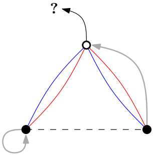

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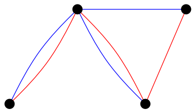

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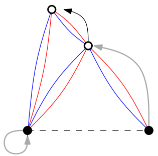

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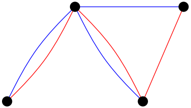

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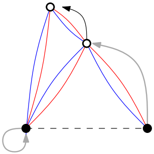

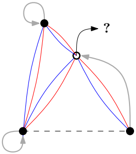

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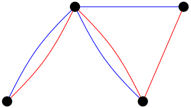

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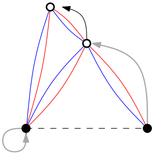

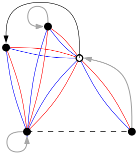

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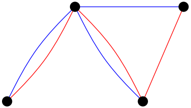

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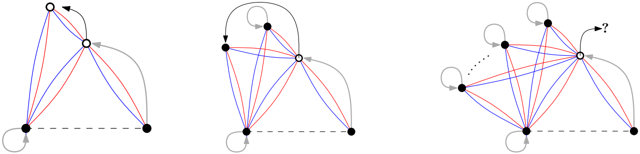

P, Q-colored graphs

Define P, Q-**colored graph** on vertex set V

- With two finite posets having minimal element 0 uses as
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$$\chi(v) \leq_P \chi(f(v)) \quad \text{and} \quad \xi(x, y) \leq_Q \xi(f(x), f(y))$$

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  - Corresponding classes $\mathbf{HH}_{P,Q}$ and $\mathbf{HH}_Q$

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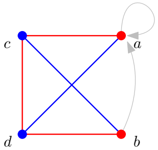

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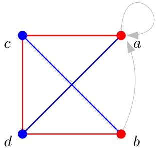

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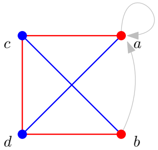

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Structure $\mathcal{R}_n$

- Let $\mathcal{C}_n$ be a class of finite graphs with edges colored by F_n (F_n is antichain extended by minimal element 0)
- $\mathcal{R}_n$ is universal for $\mathcal{C}_n$ and homogeneous

Properties of $\mathcal{R}_n$

Let G be F_n-colored graph

($\Diamond_n$) Let $G_1, G_2, \ldots, G_{n+1}$ be finite disjoint subsets of G then there exists $x \in G \setminus G_{n+1}$ s.t. each vertex of $G_i \sim_i x$ for $1 \leq i \leq n$ and for each $y \in G_{n+1}$ a pair xy is non-edge.

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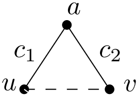
- not **HH**:
  for any non-edge uv there exists a s.t.



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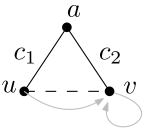

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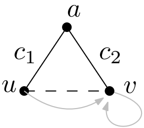

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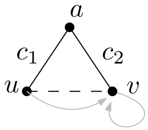

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Lesson learned: Vertex coloring is not needed!

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Show that $\mathbf{MH}_{P,Q} = \mathbf{HH}_{P,Q}$ implies

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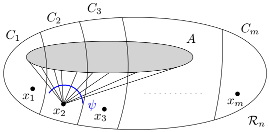

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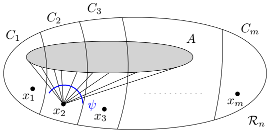

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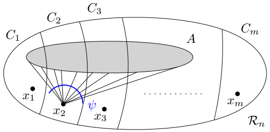

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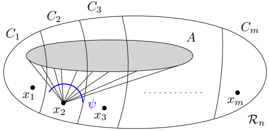

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- Idea is to impose structure given by conditions
- Enumerate
  - all finite subsets of X as $\{Y_i : i \in \omega\}$
  - all functions $t^i_j : Y_i \to F_n$ - there are $j \in \{2, \ldots, (n+1)^{|Y_i|}\}$

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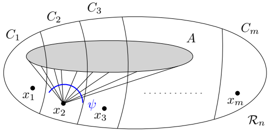

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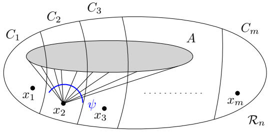

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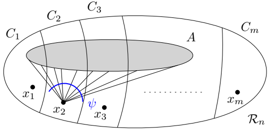

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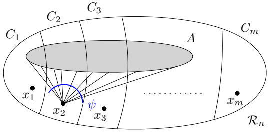

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Note that

- Each point of construction uses only finitely many elements - it's possible
- Condition 2. satisfied by construction and 1. follows

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What about sufficient condition for equality of classes?

Coloring using diamond

Construction of example structure M

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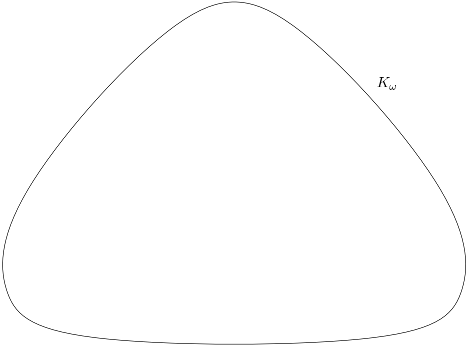

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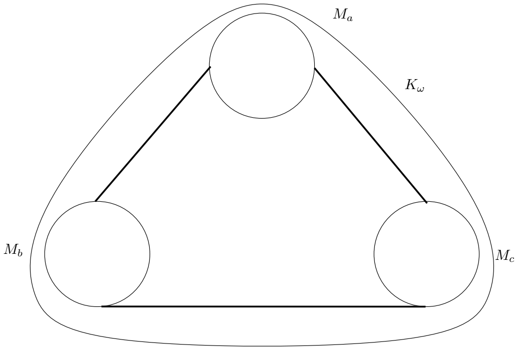

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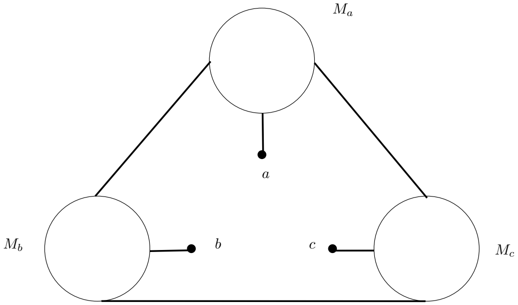

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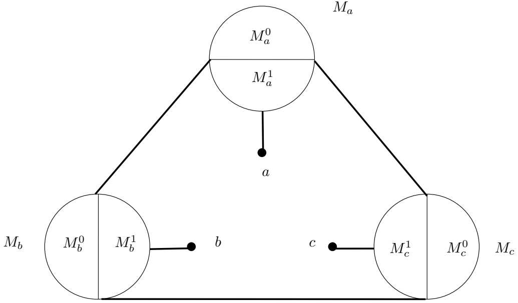

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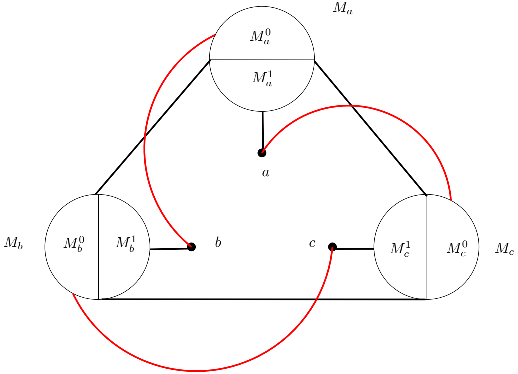

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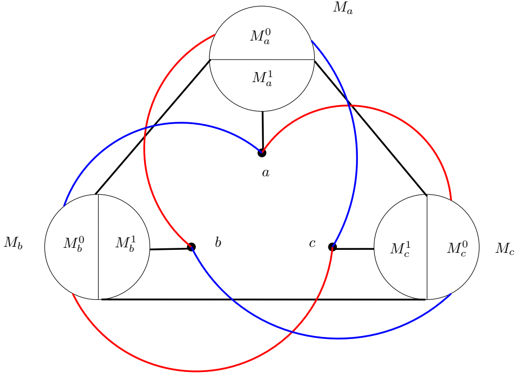

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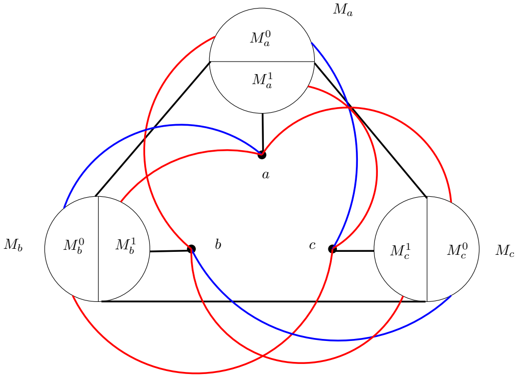

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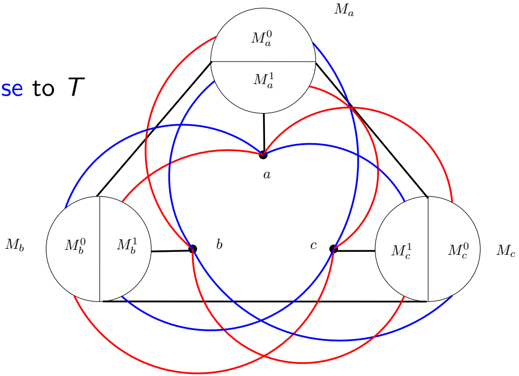

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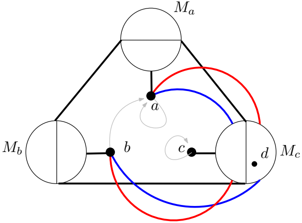

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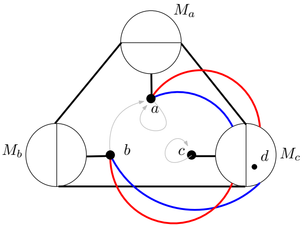

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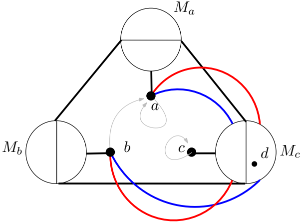

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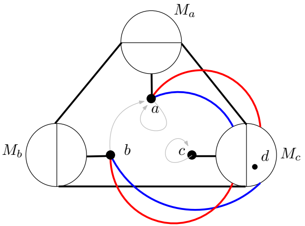

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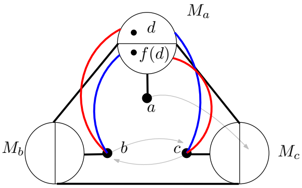
  Assume image of f to have exactly 2 elements
  - from $\{a, b, c\}$ - say b, c
  - study effect of f on $\{a, b, c\}$
    (case analysis)



Sufficient condition

Theorem (Aranda, H., 2018+)

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Let P and Q be finite partially ordered sets. $\mathbf{MH}_{P,Q} = \mathbf{HH}_{P,Q}$ iff Q is a linear order.

Forward idea: Q is a linear order:

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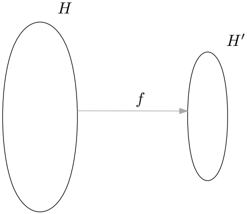

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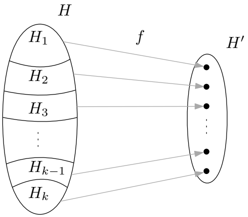

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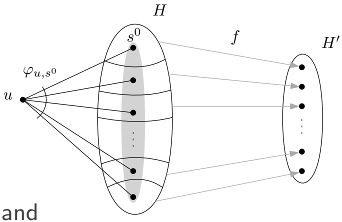

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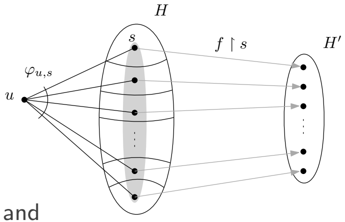

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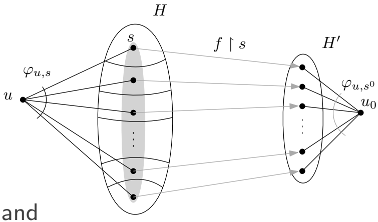

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  - use another realization of s^0 so $f \cup \{(u, (f \restriction s^0)(u_0))\}$

Extended example

Backward idea: If Q is not linear then $\exists M$ s.t. M is $\mathbf{MH}_{P,Q}$ but not $\mathbf{HH}_{P,Q}$
We know that Q is finite directed set

- Let top element be $\mathbb{1}$ and let $P_1, \ldots, P_n$ be top elements $Q \setminus \mathbb{1}$

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Construct M which is connected in $\mathbb{1}$ as well as P_i

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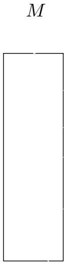

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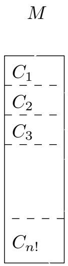

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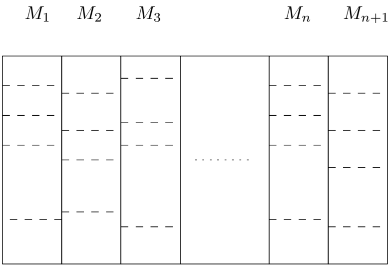

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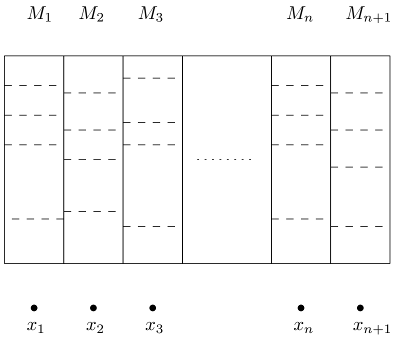

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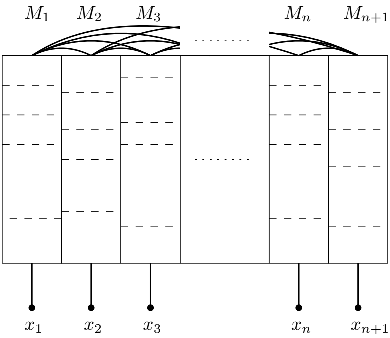

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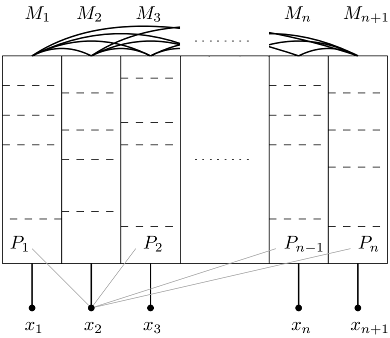
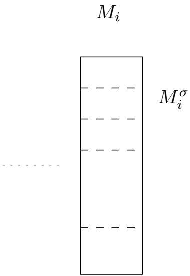


Connect x_j and M_i

Coloring edges between x_j and M_i

- Indices of parts of M_i corresponds to $\sigma \in S_n$, i.e.

$$M = \bigcup_{\sigma \in S_n} M_i^{\sigma}$$



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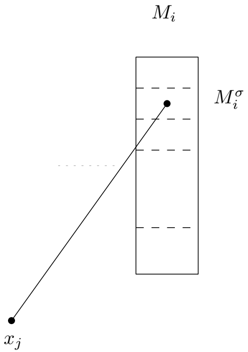

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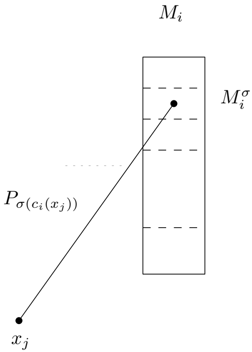

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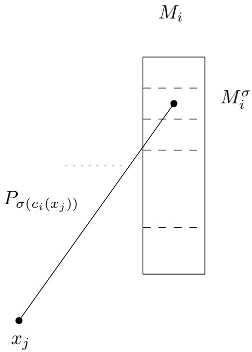
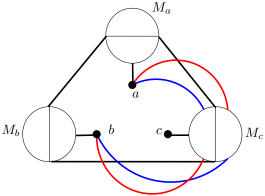


This construction generalize



Extended example - properties

Backward idea - homomorphism-homogeneity of M:

M is not $\mathbf{HH}_{P,Q}$

- x_i and x_j are in distance 3 in $\mathbb{1}$
- Take distinct $j, k, \ell \in \{1, \ldots, n+1\}$
- Local homomorphism
  $x_k \mapsto x_k, x_\ell \mapsto x_k, x_j \mapsto x_j$
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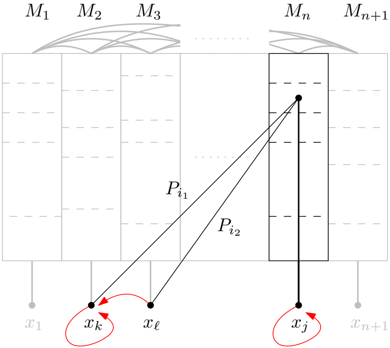

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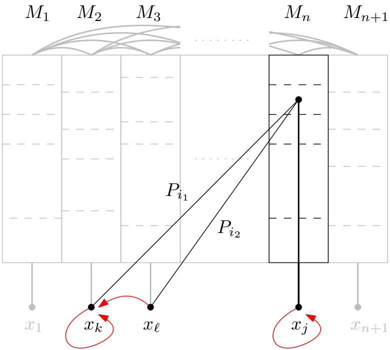


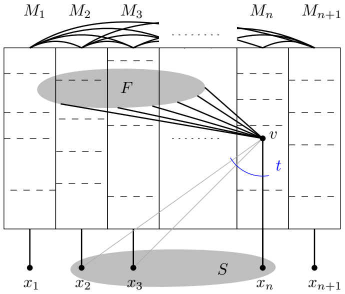
M is $\mathbf{MH}_{P,Q}$ Let $f : H \to K$ be surjective monomorphism

- If $|K \cap \{x_1, \ldots, x_{n+1}\}| \leq 1$ we have infinitely many cones
- Assume K contains at least two vertices from $\{x_1, \ldots, x_{n+1}\}$
  - Note that $x_i x_j$ is only non-edge in M, i.e.
  - its preimage is contained in $\{x_1, \ldots, x_{n+1}\}$

Final claim finishing the proof

Claim

Given any $F \subset M \setminus \{x_1, \ldots, x_{n+1}\}$, $S \subseteq \{x_1, \ldots, x_{n+1}\}$ and injective $t : S \to \{\mathbb{1}, P_1, \ldots, P_n\}$, there $\exists v \in M$ that is connected to all of F by edges of type $\mathbb{1}$ and satisfies t over S, i.e., $\varphi_{v,S} = t$.



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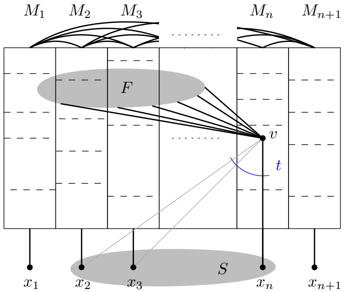

For $v_i \in M_i$ and any F

- $v_i \sim_{\mathbb{1}} u$ for $u \in M_j \cup F$ for $j \neq i$ by construction of M
- $v_i \sim_{\mathbb{1}} u$ for $u \in M_i \cup F$ by Rado-ness of M_i

For injective t

- using $\mathbb{1}$ for x_i choose v from M_i connected correctly (correct σ)
- others similarly

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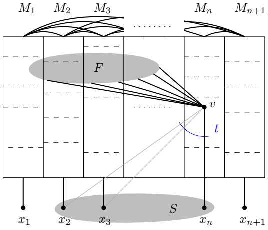

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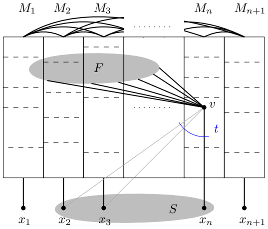

We use this claim to proof that M is $\mathbf{MH}_{P,Q}$

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- Suffice to show one vertex extension, i.e. For v being connected as $\varphi_{v,H}$ to w

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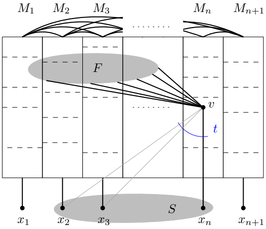

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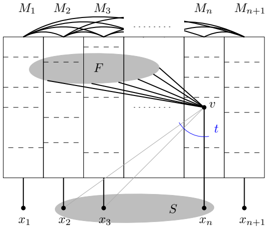

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Claim provides a vertex w satisfying t over S and connected by $\mathbb{1}$ to F

- Thus extending f

Thank you