Approximate Ramsey properties of Banach spaces

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UNED (Madrid)

UNIFYING THEMES IN RAMSEY THEORY
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Introduction

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- Graham-Leeb Rothschild for the field $\mathbb{R}$;
- "multidimensional" Borsuk-Ulam Theorem;
- Extreme amenability.

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- Sketch proofs of known results.

Introduction

We are ... a family of finit...

- $\{\ell_\infty^n\}_n$;
- $\{\ell_2^n\}_n$, all f.d. Hilbert spaces;
- $\{\ell_1^n\}_n$, all $\{\ell_p^n\}$.

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- Relate (ARP) with well-known properties;
- Sketch proofs of known results.

This presentation is based on joint works with Dana Bartošová, J. LA, Martino Lupini and Brice Mbombo, and with Valentin Ferenczi, Brice Mbombo and Stevo Todorcevic.

Outline

1. What is the Approximate Ramsey Property
   - Grassmannians over the field $\mathbb{R}$
   - The definition
   - (ARP) and Extreme amenability
   - Examples
   - Borsuk-Ulam
2. Hints on proofs
   - $\{\ell_\infty^n\}_n$
   - $\{\ell_2^n\}_n$
   - $\{\ell_p^n\}_n$

Section 1

What is the Approximate Ramsey Property

Graham-Leeb-Rothschild

$\mathbb{F}$ denotes a finite field. Given $d, n \in \mathbb{N}$, let $\binom{\mathbb{F}^n}{\mathbb{F}^d}$ be the *d-Grassmannians* of the vector space $\mathbb{F}^n$.

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Theorem (Graham-Leeb-Rothschild)

For every $d, m \in \mathbb{N}$ and $r \in \mathbb{N}$ there exists $n \geq k$ such that every r-coloring of $\binom{\mathbb{F}^n}{\mathbb{F}^d}$ has a monochromatic set of the form $\binom{V}{\mathbb{F}^d}$ for some $V \in \binom{\mathbb{F}^n}{\mathbb{F}^m}$.

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Question

What if $\mathbb{F} = \mathbb{R}$?

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Besides other difficulties to understand colorings (signs, infinitely many values) there is the bad coloring "shape". Given a plane $\pi \in \binom{\mathbb{R}^3}{\mathbb{R}^2}$ we consider its section with the centered cube, and we record its shape

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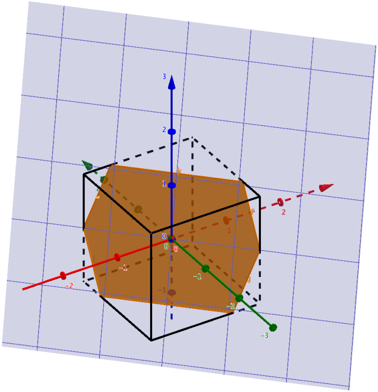

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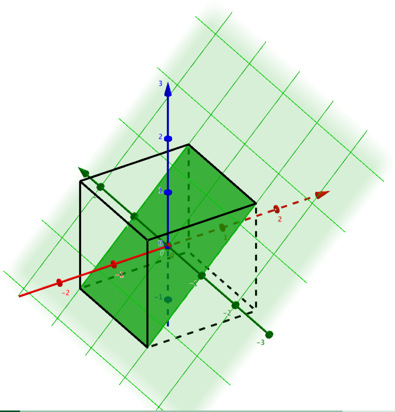


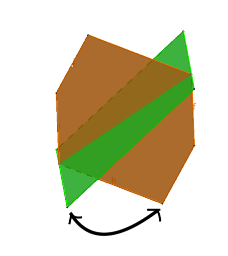
GLR for $\mathbb{F} = \mathbb{R}$

We write

$$\binom{\ell_p^n}{\mathbb{R}^d}$$

to denote the *metric* space of all d-dimensional subspaces of $\mathbb{R}^n$ endowed with the *p-opening (or gap)* Λ_p metric.

$\Lambda_p(V, W)$ is the p-Hausdorff metric between the p-unit ball of V, $B_V = B_p \cap V$ and that of W, $B_W = B_p \cap W$.



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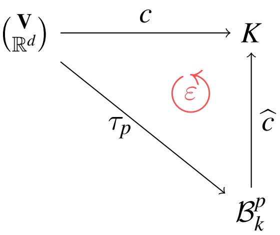
to denote the *metric* space of all d-dimensional subspaces of $\mathbb{R}^n$ endowed with the *p-opening (or gap)* Λ_p metric.
Similarly, for a f.d. normed space X of dimension d, we write $\binom{\ell_p^n}{X}$ to denote the set of all d-dimensional subspaces of ℓ_p^n that are isometric to X.

In the next $p \neq 4, 6, 8, \ldots$.

Theorem (GLR Theorem for $\mathbb{R}$, p-version)

For every d, m $\varepsilon > 0$ and every (K, d_K) compact metric there is n such that for every 1-Lipschitz coloring $c : \binom{\ell_p^n}{\mathbb{R}^d} \to (K, d_K)$ there is some $\mathbf{V} \in \binom{\ell_p^n}{\ell_p^m}$ and a 1-Lipschitz $\widehat{c} : (\mathcal{B}_k^p, \gamma_p) \to (K, d_K)$ such that



- $\mathcal{B}_k^p$ *=isometric types of subspaces of* $L_p[0,1]$;
- *this is a compactum, endowed with the Banach-Mazur metric;*
- *the metric* γ_p *is the Gromov-Hausdorff metric associated to* Λ_E, *that is uniformly equivalent to the Banach-Mazur metric*

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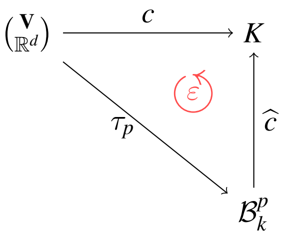

GLR Theorem for $\mathbb{R}$, Euclidean version

It follows from Dvoretzky theorem

GLR Theorem for $\mathbb{R}$, Euclidean version

n-dimensional normed spaces have almost Hilbertian subspaces of dimension uniformly proportional to $\log(n)$.

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GLR Theorem for $\mathbb{R}$, Euclidean version

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Theorem

For every d, m, $\varepsilon > 0$ and every $\mathcal{K}$ compact metric there is $n \geq k$ such that for every norm M on $\mathbb{R}^n$, every 1-Lipschitz coloring of $\binom{(\mathbb{R}^n, M)}{\mathbb{R}^d} \to \mathcal{K}$ ε-stabilizes in $\binom{\mathbf{V}}{\mathbb{R}^d}$ for some $\in \binom{(\mathbb{R}^n, M)}{\mathbb{R}^m}$, such that

$$\mathrm{diam}_K(c\binom{\mathbf{V}}{\mathbb{R}^d}) < \varepsilon$$

The Approximate Ramsey Property

Let $\mathcal{F}$ be a collection of finite dimensional normed spaces.

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Definition

- $\mathcal{F}$ has the Approximate Structural Ramsey Property when for every $F, G \in \mathcal{F}$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that every continuous coloring $c : \binom{H}{F} \to [0, 1]$ ε-stabilizes on $\binom{\mathbf{G}}{F}$ for some $\mathbf{G} \in \binom{H}{G}$.

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$\binom{H}{F}$ endowed with the H-induced Hausdorff distance between unit balls.

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- $\mathcal{F}$ has the Approximate Ramsey Property when for every $F, G \in \mathcal{F}$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that every continuous coloring $c : \mathrm{Emb}(F, H) \to [0,1]$ ε-stabilizes on $\varrho \circ \mathrm{Emb}(X, Y)$ for some $\varrho \in \mathrm{Emb}(G, H)$.

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- ... or every ... tinuous ... $\binom{H}{G}$.

  $\mathrm{Emb}(F,H)$ is the space of isometric linear embeddings from F into H endowed with the norm metric $d(\gamma,\eta) := \max_{\|x\|_F\le 1} \|\gamma x - \eta x\|_H$.

- ... $G \in \mathcal{F}$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that every continuous coloring $c : \mathrm{Emb}(F,H) \to [0,1]$ ε-stabilizes on $\varrho \circ \mathrm{Emb}(X,Y)$ for some $\varrho \in \mathrm{Emb}(G,H)$.

(ARP) and Extreme Amenability

Theorem (KPT correspondence)

For E approximately ultrahomogeneous the following are equivalent:

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Theorem (KPT correspondence)

For E approximately ultrahomogeneous the following are equivalent:

- $\mathrm{Iso}(E)$ *is extremely amenable.*
- $\mathrm{Age}(E)$ *has the approximate Ramsey property.*

Examples

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The unit ball has only finitely many extreme points

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- $\{\ell_p^n\}_n$ all $0 < p < \infty$.
- $\{\ell_\infty^n\}_n$.
- Polyhedral spaces.
- All f.d. normed spaces.

non-Examples

Theorem

For every $p \in 2\mathbb{N}$, $p > 2$, *the family* $\mathrm{Age}(L_p[0,1])$ *does not have the (ARP)*

The reason is that on those L_p's there are $X \equiv Y$ subspaces of L_p such that X is C-complemented

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There is a projection $P : L_p \to X$ of norm $\leq C$

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The reason is that on those L_p's there are $X \equiv Y$ subspaces of L_p such that X is C-complemented and Y is not $2C$ complemented. The coloring "being C-complemented or not" is a bad one.

(ARP) and Borsuk-Ulam

Recall that one of the several equivalent versions of the *Borsuk-Ulam* theorem states that

Theorem (Lusternik and Shnirel'man)

When the unit sphere $\mathbb{S}^n$ of ℓ_2^{n+1} is covered by $n+1$ many open sets, one of them contains a point x and its antipodal $-x$.

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Let (X, d) be a metric space, $\varepsilon > 0$. We say that an open covering $\mathcal{U}$ of X is ε*-fat* when $\{U_{-\varepsilon}\}_{U \in \mathcal{U}}$ is still a covering of X.

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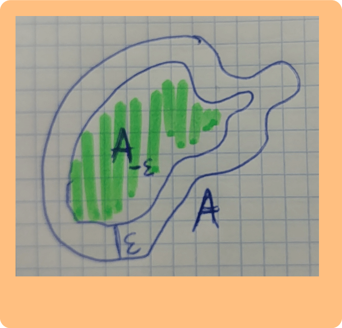

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It is not difficult to see that if X is compact, then every open covering is ε-fat for some $\varepsilon>0$.

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for every $1 \leq p \leq \infty$, every integers d, m and r and every $\varepsilon > 0$ there is some n such that for every ε-fat open covering $\mathcal{U}$ of $\mathrm{Emb}(\ell_p^d, \ell_p^n)$ with cardinality at most r there is an open set of $\mathcal{U}$ containing $\varrho \circ \mathrm{Emb}(\ell_p^d, \ell_p^n)$ for some $\varrho \in \mathrm{Emb}(\ell_p^m, \ell_p^n)$. Denote by $\mathbf{n}_p(d, m, r, \varepsilon)$ the minimal such n.

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because $\mathrm{Emb}(\ell_p^1, \ell_p^r) = \mathbb{S}_p^{r-1}$, and $\mathrm{Emb}(\ell_p^1, \ell_p^1) = \{\pm \mathrm{Id}_{\mathbb{R}}\}$

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Problem

Is $\mathbf{n}_p(d, m, r, \varepsilon)$ independent of ε?

Section 2

Hints on proofs

Matoušek-Rödl spreads

The case of $d = 1$ (i.e. coloring points of spheres) was proved independently by E. Odell, H. Rosenthal and Th. Schlumprecht

Matoušek-Rödl spreads

Using tools from Banach space theory (like unconditionality) to find many symmetries

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$$\mathrm{Spread}(a, s) := \sum_{j<m} a_j u_{k_j}.$$

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u_s is the vector with 1 in position s and 0 everywhere else

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Theorem

For every $p < \infty$, $m \in \mathbb{N}$ and $\varepsilon > 0$ there is some vector a and $n \in \mathbb{N}$ such that every Lipschitz coloring of the unit sphere $S_{\ell_p^n}$ ε-stabilizes on the unit sphere of the span of $\mathrm{Spread}(a, s_0), \dots, \mathrm{Spread}(a, s_{m-1})$ for some pairwise disjoint sequence $s_0, \dots, s_{m-1}$ of subsets of n.

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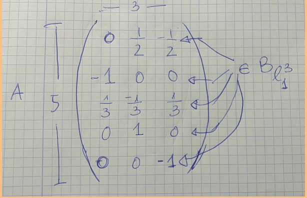
2. An linear $\gamma : \ell_\infty^d \to \ell_\infty^n$ represented in the unit basis by a matrix A is an isometry if and only if the rows of $\mathrm{rows}(A) \subseteq B_{\ell_1^d}$, and each $u_j \in \pm\mathrm{rows}(A)$.



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3. Given $D \subseteq B_{\ell_1^d}$ a (rigid) surjection $\sigma : n \to D$ we can define the $d \times n$-matrix A_σ whose j^{th}-row is $\sigma(j)$. When $\{u_j\}_{j<d} \subseteq \pm D$, A_σ^t represents an isometric embedding.

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After proving (ARP) of $\{\ell_\infty^n\}_n$, one proves the (ARP) of the f.d. polyhedral spaces, and then of all of f.d. normed spaces.

Hints of the Proof on Hilbertian

We want to prove that for every d, m and r, and every $\varepsilon > 0$ there is n such that r-colorings of $\mathrm{Emb}(\ell_2^d, \ell_2^n)$ have ε-monochromatic sets of the form $\varrho \circ \mathrm{Emb}(\ell_2^d, \ell_2^n)$.

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1. Since every Hilbert space is ultrahomogeneous, we may assume that $d = m$; Let D be a finite $\varepsilon/2$ dense subset of U_m such that $D = D^{-1}$ and containing the identity.

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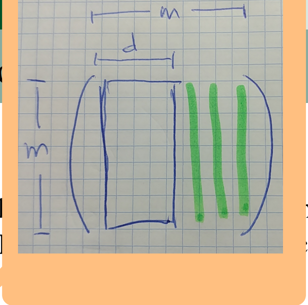

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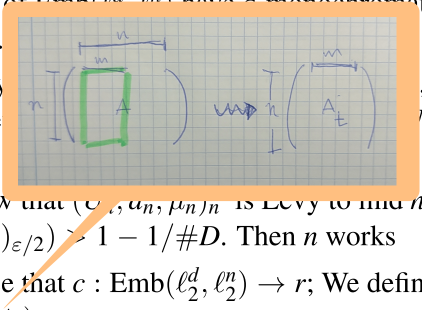

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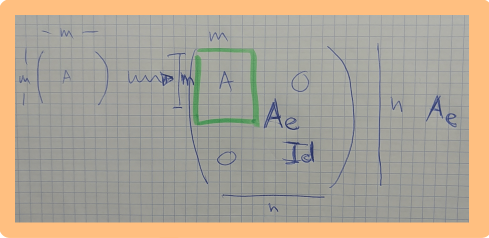

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6. A simple analysis shows that $A_t \cdot U_m \subseteq (c^{-1}(i))_{\varepsilon}$.

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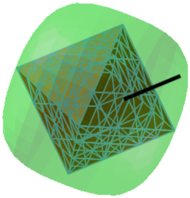

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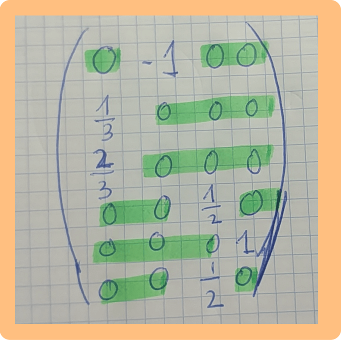

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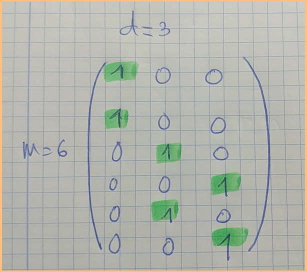

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3. The proof of the (ARP) of $\{\ell_1^n\}_n$ is a consequence of the Matoušek-Rödl, and the following

EquiDual Ramsey

Theorem

For every $d|m$ and every r there is n divided by m such that every r-coloring of $\mathcal{E}\mathcal{Q}_d(n)$ has a monochromatic set of the form $\langle\mathcal{R}\rangle_d^{\mathrm{eq}}$ for some $\mathcal{R} \in \mathcal{E}\mathcal{Q}_m(n)$

the collection of d-equipartitions of n

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Conjecture

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Theorem (ARP of equipartitions)

For every $d|m$ and every r and $\varepsilon > 0$ there is n divided by m such that every r-coloring of $\mathcal{EQ}_d(n)$ has an ε-monochromatic set of the form $\langle\mathcal{P}\rangle_d^{\mathrm{eq}}$ for some $\mathcal{R} \in \mathcal{EQ}_m(n)$

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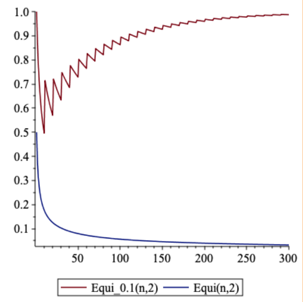

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- Arguing as we did in the Hilbert case, one proves the approximate result for ε-equipartitions.
- It is easily seen that when $d|n$, an ε-equipartition is $\varepsilon/2$-close to a equipartition. This and the (ARP) of ε-equipartitions gives the (ARP) of equipartitions.

Thank you!