The Hrushovski property for hypertournaments and profinite topologies

Marcin Sabok

BIRS, Banff, Nov 19, 2018

This is joint work with Jingyin Huang, Mike Pawliuk and Dani Wise

Theorem (Hrushovski)

If G is a finite graph and

$$\varphi_1, \ldots, \varphi_n : G \to G$$

are partial isomorphisms, then there exists a finite graph G' containing G as an induced subgraph such that all φ_i extend to automorphisms of G'.

Definition

A class K of finite structures has the **Hrushovski property** (a.k.a. the **EPPA**) if for any M in K and partial isomorphisms

$$\varphi_1, \dots, \varphi_n : G \to G$$

there exists M' in K extending M such that all φ_i extend to automorphisms of M'.

Herwig and Lascar gave an alternate proof of the Hrushovski theorem using profinite topology on the free group.

Herwig and Lascar gave an alternate proof of the Hrushovski theorem using profinite topology on the free group.

Question (Herwig and Lascar)

Does the class of finite tournaments have the Hrushovski property?

Definition

Let P be a set of prime numbers. The **pro-**P topology on the free group F_n is the one with neighborhoods of 1 consisting of those $N \triangleleft F_n$ such that

$$[F_n : N] \text{ is finite and divisible only by numbers in } P$$

Definition

Let P be a set of prime numbers. The **pro-**P topology on the free group F_n is the one with neighborhoods of 1 consisting of those $N \triangleleft F_n$ such that

$$[F_n : N] \text{ is finite and divisible only by numbers in } P$$

Examples

- if P is the set of all primes, then the above is called the profinite topology,
- if $P = \{p\}$, then the above is the pro-p topology,
- if $P = \{\text{all primes} > 2\}$, then this is called the pro-odd topology

Definition

A subgroup $G < F_n$ is **closed under q-th roots** if

$$g^q \in G \quad \text{implies} \quad g \in G.$$

When $q = 2$, we refer to this as **closed under square roots**.

Note

If G is pro-P closed, then it must be closed under q-th roots for every prime $q \notin P$.

Note

If G is pro-P closed, then it must be closed under q-th roots for every prime $q \notin P$.

proof

If $g^q \in G$ but $g \notin G$, then g cannot be separated from G in any finite quotient of size not divisible by q because in such finite quotient $\pi(g)$ is a power of $\pi(g^q)$.

Note

If G is pro-P closed, then it must be closed under q-th roots for every prime $q \notin P$.

proof

If $g^q \in G$ but $g \notin G$, then g cannot be separated from G in any finite quotient of size not divisible by q because in such finite quotient $\pi(g)$ is a power of $\pi(g^q)$.

In particular, a group which is pro-odd closed must be closed under square roots

Fact (Herwig and Lascar)

TFAE

Fact (Herwig and Lascar)

TFAE

1. The class of finite tournaments has the Hrushovski property,

Fact (Herwig and Lascar)

TFAE

1. The class of finite tournaments has the Hrushovski property,

2. For any n and any finitely generated $G < F_n$ TFAE:
   - G is closed in the pro-odd topology,
   - G is closed under square roots.

Question 2

Is it true that for any set P of primes, for any finitely generated $G < F_n$ TFAE:

- G is closed in the pro-P topology,
- G is closed under q-th roots for all prime $q \notin P$.

Definition

Let P be a set of primes. An P-**hypertournament** is a relational structure $(A, R_i : i \in P$ where R_i is an i-ary relation such that for every $i \in P$:

- for every $(x_1, \ldots, x_i) \in A^i$ there exists a permutation σ such that
$$A \models R_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)})$$
- there does not exist $(x_1, \ldots, x_i) \in A^i$ such that
$$A \models R_i(x_1, \ldots, x_i),$$
$$A \models R_i(x_2, \ldots, x_i, x_1),$$
$$\ldots,$$
$$A \models R_i(x_i, x_1, \ldots, x_{i-1})$$

Remark

In this terminology, a tournament is a $\{2\}$-hypertournament.

Remark

In this terminology, a tournament is a $\{2\}$-hypertournament.

Fact

For general set of primes P, Question 2 is equivalent to the Hrushovski property for the class of P-hypertournaments.

A positive answer (HPSW)

For any set of primes P and any subgroup $G < F_n$ of rank 1 (i.e. cyclic) TFAE:

- G is closed in the pro-P topology,
- G is closed under q-th roots for all primes $q \notin P$

Corollary

For any set of primes P, the class of finite P-hypertournaments has the Hrushovski property for families of partial isomorphisms with pariwise disjoint domains and pairwise disjoint ranges.

Corollary

For any set of primes P, the class of finite P-hypertournaments has the Hrushovski property for families of partial isomorphisms with pariwise disjoint domains and pairwise disjoint ranges.

In particular, this holds for tournaments.

A negative answer (HPSW)

There exists a finitely generated subgroup $G < F_2$ which is closed under q-th roots for all q and is not closed in any p-adic topology.

A negative answer (HPSW)

There exists a finitely generated subgroup $G < F_2$ which is closed under q-th roots for all q and is not closed in any p-adic topology.

The group has a nice presentation

$$G = \langle aba^{-1}b^{-1}a, b \rangle$$

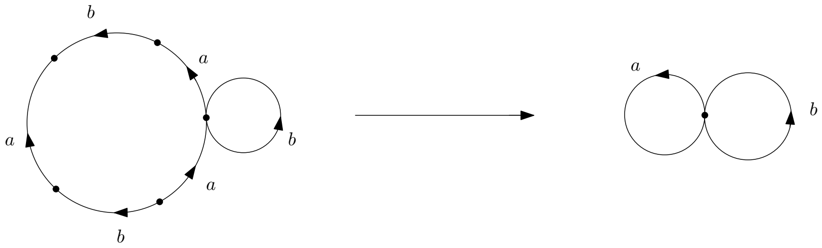
A negative answer (HPSW)

There exists a finitely generated subgroup $G < F_2$ which is closed under q-th roots for all q and is not closed in any p-adic topology.

The group has a nice presentation

$$G = \langle aba^{-1}b^{-1}a, b\rangle$$

The group corresponds to a 5-vertex hypertournament.


b
a
a
b
a
b
a
b
a
b

To see that this group is closed under q-th roots, one can either introduce the structure of an hypertournament on the set of vertices or argue more abstractly

To see that this group is closed under q-th roots, one can either introduce the structure of an hypertournament on the set of vertices or argue more abstractly

Definition

A subgroup $G < F_n$ is **malnormal** if for every $g \notin G$ we have $G \cap gGg^{-1} = \{1\}$

Fact

If $G < F_n$ is malnormal, then it is closed under q-th roots for all prime q

Fact

If $G < F_n$ is malnormal, then it is closed under q-th roots for all prime q

proof

Suppose $g \notin G$ but $g^q \in G$. Then both G and gGg^{-1} contain the infinite cyclic subgroup generated by g^q and so $G \cap gGg^{-1} \neq \{1\}$.

Fact

The group $G = \langle aba^{-1}b^{-1}a, b \rangle$ is malnormal in F_2.

Definition

Let $f : A \to X$ and $g : B \to X$ be maps between sets. Define

$$A \otimes_X B = \{(a, b) \in A \times B : f(a) = g(b).$$

If g is a covering map, then the above space is called the **pull-back**. It is also a covering space of X with the projection map.

Definition

Let $f : A \to X$ and $g : B \to X$ be maps between sets. Define

$$A \otimes_X B = \{(a, b) \in A \times B : f(a) = g(b).$$

If g is a covering map, then the above space is called the **pull-back**. It is also a covering space of X with the projection map.

Fact

Let $A \to X$ be an immersion between connected graphs. Then $\pi_1 A$ is malnormal in $\pi_1 X$ if and only if each non-diagonal component of $A \otimes_X A$ is a tree (i.e. simply-connected)

Theorem

The group $G = \langle aba^{-1}b^{-1}a, b\rangle < F_2$ is dense in the pro-p topology for every prime p.

Theorem

The group $G = \langle aba^{-1}b^{-1}a, b \rangle < F_2$ is dense in the pro-p topology for every prime p.

Fact

A subgroup $G < F_2$ is dense in the pro-P topology if and only if there is no proper subgroup $U < F_2$ such that U is pro-P open and $G < U$.

Sketch of a proof of the Theorem

In other words, we must show that there is no cover Y of the bouquet of two circles B such that $\pi_1(Y) = U$ is open in the pro-p topology and the immersion $f : X \to B$ defining

$$G = \langle aba^{-1}b^{-1}a, b\rangle = \pi_1(X)$$

lifts:

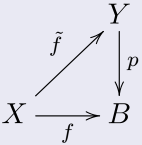
$$\begin{array}{ccc} & & Y \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ X & \xrightarrow[f]{} & B \end{array}$$

Sketch of a proof of the Theorem

In other words, we must show that there is no cover Y of the bouquet of two circles B such that $\pi_1(Y) = U$ is open in the pro-p topology and the immersion $f : X \to B$ defining

$$G = \langle aba^{-1}b^{-1}a, b \rangle = \pi_1(X)$$

lifts:



Note that in that case the pullback $X \otimes Y$ would be disconnected as X would be one of its connected components.

Sketch of a proof

For such a Y we could find a further cover $Y' \to Y$ such that $\pi_1(Y')$ is normal in F_2 and $F_2/\pi_1(Y')$ is a p-group.

$$\begin{array}{ccc} X \otimes Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow p' \\ X & \xrightarrow[f]{} & B \end{array}$$

Sketch of a proof

For such a Y we could find a further cover $Y' \to Y$ such that $\pi_1(Y')$ is normal in F_2 and $F_2/\pi_1(Y')$ is a p-group.

$$\begin{array}{ccc} X \otimes Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & B \end{array}$$

Now, the space $X \otimes Y'$ would be a cover of $X \otimes Y$ and hence disconnected.

Sketch of a proof

For such a Y we could find a further cover $Y' \to Y$ such that $\pi_1(Y')$ is normal in F_2 and $F_2/\pi_1(Y')$ is a p-group.

$$\begin{array}{ccc} X \otimes Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & B \end{array}$$

Now, the space $X \otimes Y'$ would be a cover of $X \otimes Y$ and hence disconnected.

However, all such pull-backs must be connected, as follows from the following lemma.

Theorem (Adams, Gersten)

Let p be a prime and $\hat{Y} \to Y$ be $\mathbb{Z}/p\mathbb{Z}$ cover. Suppose X, Y are connected, $f : X \to Y$ is a continuous map

$$\begin{array}{ccc} \hat{Y} \otimes X & \xrightarrow{\hat{f}} & \hat{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

If

$$f_* : H_1(X, \mathbb{Z}/p\mathbb{Z}) \to H_1(Y, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism, then

$$\hat{f}_* : H_1(\hat{Y} \otimes X, \mathbb{Z}/p\mathbb{Z}) \to H_1(\hat{Y}, \mathbb{Z}/p\mathbb{Z})$$

is an isomorphism and $\hat{Y} \otimes X$ is connected.

This is used to show that in a sequence of $Z/p\mathbb{Z}$ covers which lead to Y' all pull-backs are connected and the nduced maps on first $\mathbb{Z}/p\mathbb{Z}$-homology are always isomorphisms.

Thank you