Concentration, Relaxation and Mixing time for Restricted Lattices

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1 Overview of the Field

The study of the convergence of Markov chains/processes is a lively and active field in probability theory, with connections with different fields, including statistical mechanics – many models are coming from physics – and combinatorics, related to isoperimetry and functional analysis. Concentration of measure in various non-product spaces is also a fascinating topic with many potential applications, in spite of deep understanding of this phenomenon on product spaces (with product measures). During the Focused Research Group period, as per our submitted proposal, we investigated bounding relaxation and mixing times for two models/problems that have arisen in recent years: The first model was introduced by Jian Ding and Elchanan Mossel as an example of the study of mixing times of Markov chain under monotone censoring. The second model of interest for us is the non-crossing partition lattice which arises as a natural lattice/graph (or Catalan structure) in many topics, including enumerative combinatorics, free probability and statistical physics. Questions of concentration (of measure) and mixing time (of local dynamics) on the non-crossing partition lattice are also compelling and challenging to understand precisely on a host of other combinatorial structures whose enumeration is also given by the sequence of Catalan numbers.

Mixing in a monotone subset of the n-cube. Let $\Omega_n := \{0,1\}^n$ be the n-dimensional hypercube. We consider the standard partial order on $\Omega_n : x \leq y, x, y \in \Omega_n$ if $x_i \leq y_i$ for all $i = 1, \dots, n$. Now a subset $A \subset \Omega_n$ of the hypercube is said to be *monotone increasing* (or simply increasing) if $x \leq y, \quad x \in A \implies y \in A$. There has been recently some effort in understanding the following Poincaré inequality on any monotone increasing set. Namely, one is interested in estimating the best constant C_P such that, for any $f\colon A \to \mathbb{R}$, it holds

$$\mathrm{Var}_\pi(f) \leq C_P \mathcal{E}_\pi(f). \tag{1}$$

Here $\pi := \frac{1}{|A|}$ is the uniform measure on A (and $|\cdot|$ denotes the cardinal of the set), $\mathrm{Var}_\pi(f) = \pi(f^2) - \pi(f)^2$ is the variance of the function f with respect to the probability measure π (recall that $\pi(g) := \sum_{x \in A} g(x)\pi(x)$ is a short hand notation of the mean of g under π), and

$$\mathcal{E}_\pi(f) := \frac{1}{2} \sum_{x \in A} \sum_{\substack{y \in A: \\ y \sim x}} \pi(x)(f(y) - f(x))^2$$

denotes the Dirichlet form associated to the operator L acting on functions as follows $Lf(x) = \sum_{y\in A, y\sim x} f(y) - f(x)$. Finally $y \sim x$ means that x and y differ only on exactly one coordinate: $x \sim y$ if there exists $j \in \{1, \dots, n\}$ such that $x_i = y_i$ for all $i \neq j$ and $x_j = 1 - y_j$.

Tight estimate on C_P, that possibly depends on A and n, for all monotone set A is still an open question.

Random Catalan Structures. There are many realizations (at least 214, to be precise, according to a recent book by Richard Stanley) as combinatorial structures of the nth Catalan number, for $n \geq 1$. Some well-studied ones include the triangulations of a regular polygon of $n + 2$ sides, sets of expressions of balanced parentheses with n left and n right parentheses, non-crossing set partitions of an n-set, pattern 312-avoiding permutations of n distinct integers etc. Each of these realizations comes with its own set of natural local moves which can be used to define a Markov chain on the corresponding combinatorial structures whose equilibrium distribution is uniform over the nth Catalan number many of them. While this study is similar in spirit to understanding the mixing time properties, the cut-off phenomenon etc. of various card shuffles (that generate a permutation of n cards uniformly, or otherwise, at random), progress has been rather limited in analyzing various "Catalan shuffles". A classical example is the Markov chain on triangulations of an n-gon using diagonal flips, whose relaxation time was conjectured by Aldous (more two decades ago) to be of order $n^{3/2}$. The best known lower and upper bounds are of order $n^{3/2}$ (Molloy, Reed and Steiger, 1999), and n^4 (McShine and Tetali, 1999). Tighter bounds on this problem and several other Catalan shuffles are lacking despite much effort.

2 Recent Developments

Every participant contributed to Monday's session by reminding each other of the principal challenges (as well as related open problems) mentioned in the FRG proposal.

2.1 Presentation Highlights

Mixing in a monotone subset of the n-cube. Anna Ben-Hamou and Emma Cohen gave excellent presentations on the status of and recent progress made on the monotone censoring question. Emma described the general status of the problem and known bounds on the quantity C_P defined in the previous section, particularly her work with Piotr Nayar and Prasad Tetali. Anna's presentation is described a bit more in the next section. Paul-Marie Samson described analogous questions (and the convenience of working) in the smooth setting when one considers the real cube, as opposed to the discrete cube.

Random Catalan Structures. Alexandre Stauffer presented a recent result [1], obtained with Alessandra Caraceni, on the mixing time of the edge-flip Markov chain on random *quadrangulations* of the sphere. In this model, one considers the state space of the Markov chain as the set of all quadrangulations of the sphere with n faces. Such objects are in one-to-one correspondence with labelled plane trees, thus falling into the large class of Catalan structures. Caraceni and Stauffer showed that the *relaxation time* of this Markov chain is at most $n^{11/2}$ and at least $n^{5/4}$, up to absolute multiplicative constants. It is expected that the lower bound is the correct order for the relaxation. This improves upon a previous result by Budzinski, who showed a lower bound on the (total variation) *mixing time* of $n^{5/4}$. The best known upper bound on the mixing time is $n^{13/2}$, and is just a corollary of the result of Caraceni and Stauffer.

3 Scientific Progress Made and Open Problems

Mixing in a monotone subset of the n-cube. Anna Ben-Hamou described progress on the problem in an important case, by proving a Poincaré inequality in the case the monotone subset is covered by a set of minimal elements with disjoint support. During the meeting, we verified that the proof extended to provide the stronger logarithmic Sobolev inequality. The technical details are as follows.

Let $A \subset \{0,1\}^n$ be a monotone subset of the n-dimensional cube, defined by M minimal elements with disjoint support of lengths $L_1, \dots, L_M$. We denote by π the uniform measure on A, and by μ the uniform measure on $\{0,1\}^n$.

Let $f : A \to \mathbb{R}$ be a given real-valued function on A. For $k \in \{1, \dots, M\}$, let us define the function f_k on $\{0,1\}^n$ by

$$f_k(x) = \begin{cases} f(x) & \text{if } x \in A, \\ f(x^k) & \text{if } x \notin A, \end{cases}$$

where x^k is the element obtained from x by completing with ones the support of the k^{th} minimal element. The above allows us to use the classical logarithmic Sobolev inequality on the (whole) n-cube, along with much additional work, in obtaining the following theorem.

Theorem 1. *With $A \subset \{0,1\}^n$ and L_ℓ as above, for any $f : A \to \mathbb{R}$ we have:*

$$\mathrm{Ent}_\pi(f^2) \lesssim \frac{n}{\sum_{\ell=1}^M 2^{-L_\ell}} \left(\frac{\max_{1 \le \ell \le M} L_\ell}{\min_{1 \le k \le M} L_k} \right) \mathcal{E}_\pi(f)\,.$$

Note that this implies that in the case when L_ℓ are all within a constant multiple of each other, we obtain an optimal bound on the log-Sobolev inequality in its depends on n as well as the measure of A.

Random Catalan Structures. The proof of Caraceni and Stauffer goes by the introduction of two new Markov chains on plane trees, called the *leaf translation* and the *leaf replanting* chains. In the leaf translation (resp., leaf replanting) chain, one picks an edge of the tree uniformly at random and, if that edge is incident to a leaf of the tree, then the leaf is moved one step clockwise or counterclockwise (resp., is moved to a uniformly random position in the tree). For either chain, the relaxation is shown to be at most $n^{9/2}$, which yields the upper bound of $n^{11/2}$ on the relaxation time of quadrangulations via a standard comparison procedure (which adds a factor of n due to the maximum degree). In terms of lower bounds, the leaf replanting chain has relaxation at least n^2, whereas, as observed by Emma Cohen during the meeting, the leaf translation chain has relaxation at least n^3 as it is a subset of the peak-to-valley (a.k.a. adjacent transposition) chain on Dyck paths.

In an ongoing work, Caraceni and Stauffer are able to adapt the above analysis to derive a polynomial upper bound on the mixing time of p-angulations, for p even. An open problem is to tackle the problem for p odd. In particular, the best known upper bound on the mixing time for the edge-flip Markov chain on *triangulations* of the sphere (with n faces) is exponential in n.

3.1 Open Problems

The Poincaré Inequality on $\{0, \dots, n\}$. It is classical to derive inequalities on $\{0, \dots, n\}$ by projection of the hypercube. In this section, we go through this procedure, starting from (1). To that purpose, define $S_n(x) = \sum_{i=1}^n x_i$, $x = (x_1, \dots, x_n) \in \{0,1\}^n$. Then, given $f \colon \{0, \dots, n\} \to \mathbb{R}$, apply (1) to $f \circ S_n$ so that

$$\mathrm{Var}_\pi(f \circ S_n) \le \frac{1}{2} C_P \sum_{x \in A} \sum_{\substack{y \in A: \\ y \sim x}} \pi(x)(f(S_n(y)) - f(S_n(x)))^2.$$

Let us introduce some notations. Given $k \in \{0, \dots, n\}$, the k-th section of the hypercube is $C_k := \{x \in \{0,1\}^n : S_n(x) = k\}$. Then, we set $A_k := A \cap C_k$ and $\nu(k) := \pi(A_k) = |\{y \in A : S_n(y) = k\}|/|A|$. For any function g on $\{o, \dots, n\}$, we have

$$\mathbb{E}_\pi(g \circ S_n) = \sum_{k=0}^n \sum_{x \in A_k} g(S_n(x)) = \sum_{k=0}^n \nu(k) g(k) = \mathbb{E}_\nu(g)$$

so that $\mathrm{Var}_\pi(f \circ S_n) = \mathrm{Var}_\nu(f)$. Similarly the Dirichlet form can be rewritten as

$$\sum_{x \in A} \sum_{\substack{y \in A: \\ y \sim x}} \pi(x)(f(S_n(y)) - f(S_n(x)))^2$$
$$= \sum_{k=1}^{n} \nu(k)p(k,k-1)(f(k-1) - f(k))^2 + \sum_{k=0}^{n-1} \nu(k)p(k,k+1)(f(k+1) - f(k))^2 \,,$$

where we set $p(k,k-1) = |\{(x,y) \in A_k \times A_{k-1} : y \sim x\}|/|A_k|$ and $p(k,k+1) = |\{(x,y) \in A_k \times A_{k+1} : y \sim x\}|/|A_k|$. By construction, $\nu(k)p(k,k+1) = \nu(k+1)p(k+1,k)$ so that the latter can be interpreted as the Dirichlet form of a Markov process on $\{0,\dots,n\}$ with invariant measure ν and transition probabilities $p(k,k+1)$, $p(k,k-1)$. Using reversibility and additional computations, we infer that for any $f\colon \{0,\dots,n\} \to \mathbb{R}$, it holds

$$\mathrm{Var}_\nu(f) \leq C_P \sum_{k=1}^{n} \nu(k)p(k,k-1)(f(k-1) - f(k))^2$$

which is the Poincaré inequality associated to the Markov process described above on $\{0,\dots,n\}$.

Such an inequality could be analyzed for itself. This amounts to the following question. Given a monotone set A define ν and $p(k,k-1)$ as above: then what is the best constant C such that for all for any $f\colon \{0,\dots,n\} \to \mathbb{R}$, it holds

$$\mathrm{Var}_\nu(f) \leq C \sum_{k=1}^{n} \nu(k)p(k,k-1)(f(k-1) - f(k))^2? \tag{2}$$

Mixing times of Catalan Structures. As mentioned above, there are more than 200+ known classes of objects which are enumerated by the Catalan numbers, and each comes with its own "natural" Markov chain:

- Dyck paths support corner flips, ribbon additions, swaps of two (non-adjacent) steps, shifting of a step.
- Non-crossing partitions have merge/split operations and partial rotations.
- Triangulations have edge flips and partial rotations.
- Binary trees have Remy moves and several prune and splice chains.
- Plane trees have in addition leaf translation and leaf replanting.

Understanding the mixing times and spectral gaps of the corresponding Markov chains are open problems for almost all of these, with the ntable exceptions being corner flips on Dyck paths (n^3, Wilson 2004) and Remy moves on binary trees (n^2, Schweinsberg 2001).

These chains are related to each other through bijections between the relevant objects, and through the canonical paths method, so that polynomial bounds can be deduced from other polynomial bounds. However, the arguments here are lossy, and the true exponents are unknown in all other cases.

4 Outcome of the Meeting

It was an overall productive meeting. Several ideas from combinatorics, functional analysis and probability were exchanged. Partial progress was made on the main problems in the project. Besides the interesting versions mentioned above of the main challenges, additional specific problems such as the one below are identified as the next steps to pursue.

Returning to the various Markov chains on Catalan structures, there is a well-known bijection between triangulations of the n-gon and plane binary trees with n leaves (trees where each internal vertex has degree 3). During the working group meeting, a new Markov chain on plane binary trees was proposed, for which the triangulation of the p-gon corresponds to partial rotations. In this chain, one chooses a leaf and an edge uniformly at random, then breaks the edge into two, and attach the chosen leaf to the unique endpoint of the broken edge that would produce a labelled plane tree. Most of the test functions used to give a lower bound on the relaxation in previous case seem to yield only simple bounds of order n. Is n the correct order of the relaxation time?

Our focused research team agreed to continue collaborating on several of the open problems mentioned above. Relevant technical documents are shared online by the group to help facilitate future research discussions. The team is very grateful for the resources and the conducive atmosphere provided by BIRS and the Banff Center.

References

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[2] Cohen, Emma. Problems in Catalan Mixing and Matchings in Regular Hypergraphs. *Georgia Tech PhD thesis*, 2016.

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