Counting V-tangencies and nodal domains

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1 Overview of the Field

Given a "nice" function $f : \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$, or $f : M \to \mathbb{R}$ with M a compact Riemannian d-manifold (equipped with some Riemannian metric), the nodal set of f is its zero set $f^{-1}(0)$; if f is Morse, then its nodal set is a smooth hypersurface. The *nodal components* of f are the connected components of the nodal set. The most basic question one is interested is in the *nodal count* of f, i.e. the total number of the nodal components of f, which we denote by $\mathscr{N}(f)$. Classical and celebrated results in $\mathbb{R}^d$ go back to Sturm and Courant.

Understanding the "typical" nature of the nodal structures of *Gaussian random fields*, rather than individual functions, is an actively pursued subject especially in the last few years. Let us introduce the specific Gaussian random fields of our focus. The space $L^2(M)$ of square-summable functions on M has an orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ consisting of Laplace eigenfunctions, i.e.

$$\Delta\phi_j + t_j^2\phi_j = 0, \tag{1}$$

where Δ is the Laplace-Beltrami operator on M acting on L^2, and $\{t_j\}_{j\geq 0}$ is its purely discrete spectrum

$$0 = t_0 \leq t_2 \leq \dots,$$

satisfying $t_j \to \infty$. For a "band" $\alpha \in [0, 1)$ and spectral parameter $T > 0$ (with the intention of taking the limit $T \to \infty$), we define the random band-limited functions to be

$$f_T(x) = f_{\alpha;T}(x) = \frac{1}{|\{j : \alpha \cdot T < t_j < T\}|^{1/2}} \sum_{\alpha \cdot T < t_j < T} c_j\varphi_j(x), \tag{2}$$

where the c_j are independent and identically distributed standard Gaussians. For $\alpha = 1$, we interpret the summation as

$$f_{1;T}(x) = \frac{1}{|\{j : T - \eta(T) < t_j < T\}|^{1/2}} \sum_{T-\eta(T) < t_j < T} c_j\varphi_j(x), \tag{3}$$

with the convention $\eta(T) = o_{T\to\infty}(T)$ but $\eta(T) \to \infty$; we will drop the subscript α when the context is clear. The breakthrough works of Nazarov-Sodin [NS09, NS15] demonstrate that as $T \to \infty$,

$$\mathbb{E}\left[\left|\frac{\mathscr{N}(f_T)}{Vol(M) \cdot T^2} - c(d, \alpha)\right|\right] \to 0 \tag{4}$$

with $c(d, \alpha) > 0$ being a universal constant, itself a quantity that has generated tremendous interest.

2 Focus of the Workshop

For the purposes of the workshop, we first let $\mathtt{V}(M)$ be the class of all C^∞-smooth vector fields on M with finitely many zeros; the class $\mathtt{V}(M)$ is non-empty for every smooth M by the existence of Morse functions (that is, we can simply take the gradient field of a given Morse function). For a nodal component $\gamma \subseteq f_T^{-1}(0)$ and $V \in \mathtt{V}(M)$ fixed, define

$$\mathscr{N}_V(f_T, k) := \#\{\gamma \subseteq f_T^{-1}(0) : \ \gamma \text{ has precisely } k \text{ tangencies w.r.t. } V\}. \tag{5}$$

Note that we have

$$\mathscr{N}(f_T) = \sum_{k=0}^{\infty} \mathscr{N}_V(f_T, k), \tag{6}$$

hence turning the nodal count for f_T into a series of counts involving tangencies. Our aim is to understand various asymptotics, in T, surrounding the distribution of the random variable $\mathscr{N}_V(f_T, k)$.

3 Recent Developments

As inferred in the previous section, one might refine the study of $\mathscr{N}(f_T)$ by separately counting the nodal components of f_T belonging to a given topology class $\mathscr{T}$ or more generally to a class of some prescribed geometric type. For the expected Betti number, or the expected number of components of a certain diffeomorphism type $\mathscr{T}$, Gayet-Welschinger [GW14] were able to obtain upper and lower bounds for the corresponding expected values for the Kostlan ensemble, which is different in many aspects from those considered in (2) or (3). A local refinement of their lower bound was very recently obtained by Wigman [Wig19].

The work of Sarnak-Wigman [SW18] provides some finer results surrounding the count for the components of a particular topology $\mathscr{T}$, particularly in the case of ensembles given in (2) and (3). To be more specific and for the sake of concreteness, take a 3-dimensional (M, g). The work [SW18] shows the existence of a probability measure $\mu^{H(2)}$, with support on every element n of $\mathbb{Z}_{\geq 0}$, where $\mu^{H(2)}(n)$ gives the expected limiting fraction of nodal components of $f_T^{-1}(0)$ of genus n (in this case, smooth and compact surfaces of genus n). Moreover, this measure is shown to be universal in the sense of its independence of the geometry or topology of M but still dependent on the spectral measure for the corresponding scaling limit $\mathfrak{g}_\alpha$ (attached to the ensemble $\{f_{\alpha,T}\}_{T>0}$) and the dimension d.

Building upon the techniques of Nazarov-Sodin and Sarnak-Wigman, Beliaev-Wigman [BW17] addressed the following question: what is the asymptotic volume distribution of the nodal domains of f_T, the sets where f_T is either positive or negative? A similar deterministic universal law to that established in [SW18] was obtained; some basic qualitative properties of the cumulative probability distribution were also proven.

The interaction between tangent/normal spaces to nodal sets and various other geometric quantities of these submanifolds is another natural topic of study. The work of Dang-Rivière (who themselves were motivated by the work [GW14]) give asymptotics pertaining to the equidistribution (in T^*M) of conormal cycles for f_T on general compact manifolds [DR17]. In the setting of the base space M, the study of the distribution of tangencies to a fixed vector field $V \in \mathtt{V}(M)$ was initiated in the work of Rudnick-Wigman [RW18], who considered a count in the arithmetic setting of the flat torus $M = \mathbb{T}^d$ related to that described in (5), specifically

$$N_V(f_T) = \#\{x : \ V(x) \neq 0, \ f_T(x) = V f_T(x) = 0\} \tag{7}$$

where the f_T is taken to be a Gaussian toral eigenfunction and $V = \zeta \in S^{d-1}$ a fixed direction. The authors obtained asymptotics for $\mathbb{E}\left[N_V(f_T)\right]$, along with some deterministic results, while the subsequent work of Eswarathasan [E18] gave asymptotics for Gaussian spherical harmonics on S^2 for fixed vector fields $V \in \mathtt{V}(M)$.

4 Scientific Progress Made

Let us encapsulate all the individual counts $\mathscr{N}_V(f_k)$ into a single (random) probability measure, the "direction distribution measure", as follows:

$$\mu_f(V) = \frac{1}{\mathscr{N}(f)} \sum_{k=0}^{\infty} \mathscr{N}_V(f, k) \cdot \delta_k, \tag{8}$$

on $\mathbb{Z}_{\geq 0}$. Given two probability measures μ_1, μ_2 on $\mathbb{Z}$ we will use the total variation distance function

$$\mathscr{D}(\mu_1, \mu_2) = \sup_{F \subseteq \mathbb{Z}_{\geq 0}} |\mu_1(F) - \mu_2(F)|. \tag{9}$$

During our week-long workshop, we were able to establish all the necessary details behind the following:
Theorem: Let (M, g) be 2 dimensional. Given $\alpha \in [0, 1)$, there exists a (deterministic) probability measure μ_α on $\mathbb{Z}_{\geq 0}$, supported on the positive even integers $2\mathbb{Z}_{>0}$, so that for all $V \in \mathtt{V}(M)$ and every $\epsilon > 0$,

$$\lim_{T \to \infty} \mathbb{P}\left(\mathscr{D}(\mu_{f_{\alpha;T}}(V), \mu_\alpha) > \epsilon\right) = 0, \tag{10}$$

where $\mathscr{D}(\cdot, \cdot)$ is the total variation distance.

This theorem can be seen as the natural next step after those established in [SW18, BW17] with the added property that the support of our limiting direction distribution measure μ_α is "essentially half" that of the connectivity measure introduced by Sarnak-Wigman and is actually independent of V. The remaining case that we would like to include before publication is $\alpha = 1$. The approach we took followed that of [SW18] which itself closely follows that of [NS15]. However, a significant juncture in our methods occurred at the stage of establishing the stability between the nodal counts involving tangencies for f_T and the corresponding counts for the universal scaling limit $\mathfrak{g}_\alpha$. Thanks to some elementary differential geometry in the plane, we were able to reduce the question of counting components whose number of tangencies with respect to V is precisely k to a question of counting components with precisely k points of (quantitative) transversal intersection to another set of curves, namely $\{V f_T = 0\}$.

A critical obstacle that we faced was in showing such components with precisely k transversal intersections, but that also have at least one intersection that is quantitatively near-degenerate, are few; this was a crucial part to establishing our desired stability property. A Kac-Rice calculation allowing us to bound an expected volume however, hinging upon some technical work of Cammarota-Marinnucci-Wigman [CMW16], allowed us to overcome this roadblock.

As mentioned before, the only missing piece before submitting the paper is to show that band-limited ensembles for $\alpha = 1$ give the same measure result as recorded above. The main hurdle en route to this goal is to apply the barrier method of Nazarov-Sodin [NS09, NS15] with added regularity assumptions and allowing our target eigenfunctions to have singular nodal sets. We will work out the details of this approach in the near future.

5 Outcome of the Meeting

We expect to complete the case of $\alpha = 1$ with relatively few obstructions and hope to subsequently submit our article for publication. We warmly thank BIRS for its hospitality and an environment which is in many ways second to none!

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