On hypergraph Ramsey numbers

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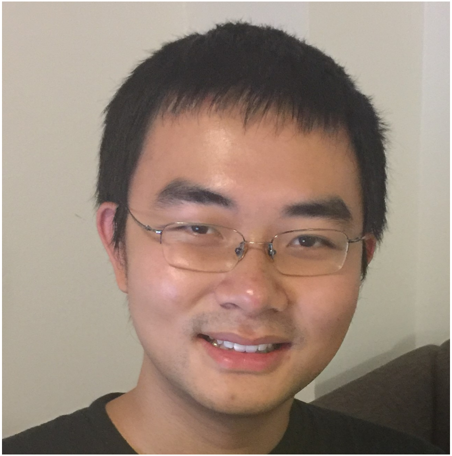

Xiaoyu He

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For k-graphs H and F, the Ramsey number $r(H, F)$ is the minimum N such that every k-graph on N vertices contains a copy of H or its complement contains a copy of F.

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Let $r_k(s, n) = r(K_s^{(k)}, K_n^{(k)})$.

Theorem: (Erdős-Szekeres 1935, Erdős 1947)

$$2^{n/2} \leq r_2(n, n) \leq 2^{2n}.$$

Hypergraphs ($k \geq 3$)

Definition:

The tower function $t_i(x)$ is given by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$.

Theorem: (Erdős-Rado 1952, Erdős-Hajnal 1960s)

$$2^{cn^2} \leq r_3(n,n) \leq 2^{2^{c'n}}.$$

$$t_{k-1}(cn^2) \leq r_k(n,n) \leq t_k(c'n).$$

Remarks:

- $k = 3$ case is central because of the stepping up lemma.
- For 4 colors, $r_3(n,n,n,n) \geq 2^{2^{cn}}$.

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Theorem: (Conlon-F.-Rödl 2017, F.-Li 2019)

There is a 3-graph H on n vertices with
$r(H,H) = O(n \log n)$ but $r(H,H,H,H) = 2^{\Theta(\sqrt{n})}$.

Off-diagonal graph Ramsey numbers

Theorem: (Ajtai-Komlós-Szemerédi 1980, Kim 1995)

$$r(3, n) = \Theta\left(\frac{n^2}{\log n}\right).$$

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Central problem in the development of the probabilistic method:

- Alterations (Erdős 1961)
- Lovász local lemma (Spencer 1975)
- Large deviation inequalities (Krivelevich 1995)
- Rödl nibble (Kim 1995)
- H-free random graph process (Erdős-Suen-Winkler 1995, Bohman-Keevash 2010).

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However, for almost all H, $r(H, K_n)$ is not well understood.

Off-diagonal hypergraph Ramsey numbers

Improving earlier results of Erdős-Hajnal and Erdős-Rado:

Theorem: (Conlon-F.-Sudakov 2010)

For $4 \leq s \leq n$,

$$2^{\Omega(sn\log(2n/s))} \leq r_3(s,n) \leq 2^{O(n^{s-2}\log n)}.$$

Off-diagonal hypergraph Ramsey numbers

$K_4^{(3)} - e$ is the 3-graph with 4 vertices and 3 edges.

Theorem: (Erdős-Hajnal 1972)

$$2^{\Omega(n)} \leq r(K_4^{(3)} - e, K_n^{(3)}) \leq 2^{O(n \log n)}.$$

Lower bound construction: Let T be a tournament on N vertices with no transitive subtournament of order $2 \log N + 1$.

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Question: (Erdős-Hajnal 1972)

Does $r(K_4^{(3)} - e, K_n^{(3)})$ grow only exponentially in n?

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Theorem: (F.-He 2019)

$$r(K_4^{(3)} - e, K_n^{(3)}) = 2^{\Theta(n \log n)}.$$

That is, every 3-graph on N vertices in which any four vertices contains at most two edges has independence number $\Omega\left(\frac{\log N}{\log \log N}\right)$, and this is tight.

Link hypergraphs versus cliques

Definition:

For graph G, the *link hypergraph* L_G is the 3-graph on $V(G) \cup \{w\}$ whose edges are the triples $\{u, v, w\}$ with $\{u, v\} \in E(G)$.

Note $K_4^{(3)} - e = L_{K_3}$.

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Proposition: (Conlon-F.-Sudakov 2010)

If G is bipartite, then $r(L_G, K_n^{(3)}) = n^{\Theta(1)}$.
If G is nonbipartite, then $r(L_G, K_n^{(3)}) = 2^{\Omega(n)}$.

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Definition:

The *link* F_v of a vertex v in a k-graph F is the $(k-1)$-graph on $V(F) \setminus \{v\}$ where $e \in E(F_v)$ if $e \cup \{v\} \in E(F)$.

Theorem: (F.-He 2019)

$\forall g$, there is a 3-graph on $N = n^{c_g n}$ vertices with independence number $< n$ and the link of each vertex has odd girth at least g.

Theorem: (F.-He 2019)

For $s, n \geq 3$,

$$r(L_{K_s}, K_n^{(3)}) \leq (2n)^{sn}.$$

A hypergraph Ramsey problem of Erdős and Hajnal

Definition

$f_k(N, s, t) :=$ max. n such that every k-graph on N vertices has s vertices with $\geq t$ edges or has independence number at least n.

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Let $t(0) = t(1) = 0$ and $t(s) = s_1 s_2 s_3 + t(s_1) + t(s_2) + t(s_3)$, where $s = s_1 + s_2 + s_3$ with s_1, s_2, s_3 as equal as possible.

Erdős-Hajnal 1972: If $t \leq t(s)$, then $f_3(N, s, t) = N^{\Theta(1)}$.

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Theorem: (F.-He 2019)

If $s > s_0$ and $.26\binom{s}{3} < t < .46\binom{s}{3}$, then $f_3(N, s, t) = \Theta(\frac{\log N}{\log \log N})$.

Theorem: (F.-He)

For $s, n \geq 3$,

$$r(L_{K_s}, K^{(3)}_{n,n,n}) = \binom{n+s}{s}^{\Theta(n)}.$$

Lower bound proof for $s \geq 14$: Let $N = \binom{n+s}{s}^{n/1000}$.

$\exists$ 3-graph Γ on N vertices which is L_{K_s}-free and $\bar{\Gamma}$ is $K^{(3)}_{n,n,n}$-free.

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Link hypergraphs

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Let $A = G(m, p)$ with $m = \binom{n+s}{s}^{2/13}$ and $p = m^{-2/(s-1)}$.

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A triple $\{i, j, k\} \in \binom{[N]}{3}$ of distinct vertices is an edge of Γ if $\chi(i, j) \sim \chi(i, k)$, $\chi(j, i) \sim \chi(j, k)$, and $\chi(k, i) \sim \chi(k, j)$ in A.

Thank you!