How redundant is Mantel's Theorem?

Shagnik Das
Freie Universität Berlin

Extremal & probabilistic combinatorics
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Results presented today are joint work with

Ander Lamaison

Tuan Tran

Extremal combinatorics

General extremal problem

Optimise an objective function subject to certain constraints.

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A real-world example

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Forbidden dangerous items

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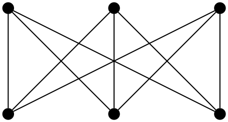

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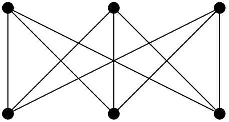

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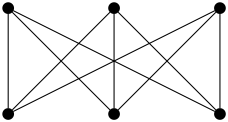

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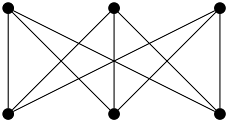

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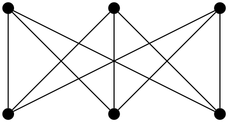

- Several extensions, e.g.:
  - Other forbidden graphs
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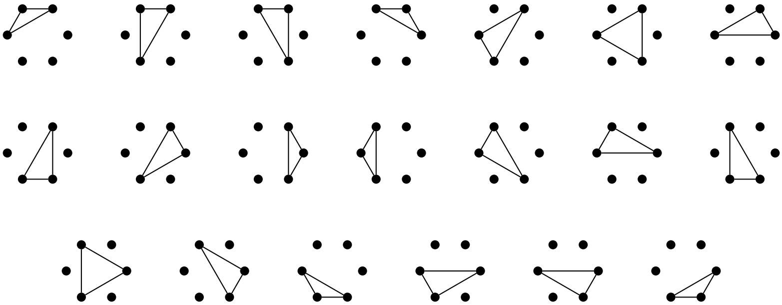
  - Supersaturation
  - Triangle-free subgraphs of $G(n, p)$

Polygona non grata



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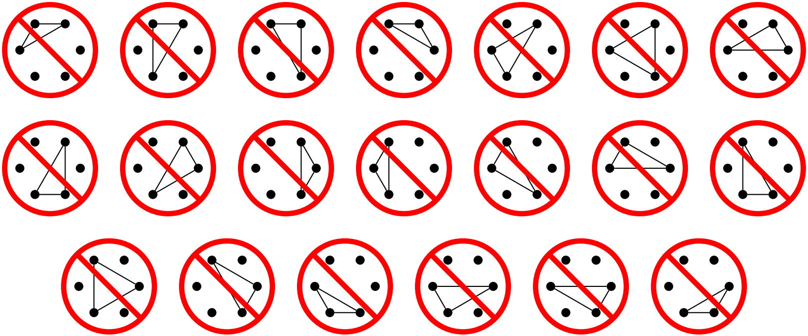

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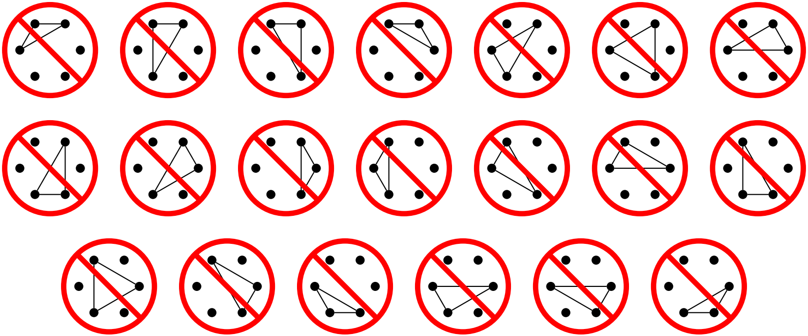

Question (Kalai)

Do we need to forbid <u>all</u> these triangles to achieve Mantel's bound?

Related research

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- Sparse random (edge-)subgraphs of $KG(n,k)$ still satisfy $\alpha(KG(n,k)_p) = \binom{n-1}{k-1}$ [Bollobás–Narayanan–Raigorodskii, Balogh–Bollobás–Narayanan, D.–Tran, Devlin–Kahn, 2015–16]
- Sparsest subgraphs with $\alpha(G) = \binom{n-1}{k-1}$ have $\frac{n-k}{2k}\binom{n}{k}$ edges [D.–Tran, 2016]

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Hales–Jewett

- Find monochromatic combinatorial lines in r-colourings of $[3]^n$ whose active sets are unions of few intervals [Shelah, 1988; Conlon–Kamčev, Leader–Räty, Kamčev–Spiegel, 2018]

A formal restatement

Definition

Let $\mathcal{K}_3(G) \subseteq \binom{[n]}{3}$ be the family of triangles in a graph G.
Given $\mathcal{T} \subseteq \binom{[n]}{3}$, say a graph G is $\mathcal{T}$-avoiding if $\mathcal{K}_3(G) \cap \mathcal{T} = \emptyset$.

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Mantel: $\text{av}\left(\binom{[n]}{3}\right) = \text{av}\left(n, \binom{n}{3}\right) = \lfloor n^2/4 \rfloor$

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Questions

1. What is the smallest m_0 for which $\operatorname{av}(n, m_0) = \lfloor n^2/4 \rfloor$? Which triangles should we forbid?
2. How does $\operatorname{av}(n, m)$ grow as m shrinks below m_0?

A modest first step

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For any $\mathcal{T}$ with $|\mathcal{T}| > \binom{n}{3} - \lfloor n/2 \rfloor$, $\mathrm{av}(\mathcal{T}) = \lfloor n^2/4 \rfloor$.

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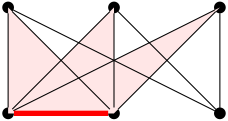
This follows immediately from

Theorem (Rademacher)

Any graph with $\lfloor n^2/4 \rfloor + 1$ edges has at least $\lfloor n/2 \rfloor$ triangles.



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- Take $\mathcal{T} = \mathcal{G}^{(3)}(n, \frac{5}{6} + o(1))$, a random set of triangles of density $\frac{5}{6} + o(1)$
- W.h.p., every pair misses fewer than $n/6$ triangles
- In any graph with $\lfloor n^2/4 \rfloor + 1$ edges, $\mathcal{T}$ contains one of the triangles on Edwards' edge □

Hitting our stride

Theorem (Mubayi, 2012)

If G is a graph with $\lfloor n^2/4 \rfloor + 1$ edges, then either

(i) *G has an edge in at least $(\frac{1}{4} + o(1))n$ edges, or*

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- Can take a union bound over all $2^{\binom{n}{2}} = 2^{O(n^2)}$ graphs $\square$

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  $\Rightarrow \mathrm{av}(\mathcal{T}) \geq \lfloor n^2/4 \rfloor + 1$ □

Closing the gap

Theorem (D.–Lamaison–Tran, 2019+)

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- Idea: use stability for dense graphs with few triangles
- Most dense graphs have many triangles — apply union bound
- Few triangles $\Rightarrow$ close to bipartite — exploit this structure

Robust stability

Theorem (Füredi, 2015)

An n-vertex triangle-free graph G with $\lfloor n^2/4 \rfloor - t$ edges can be made bipartite by removing at most t edges.

Robust stability

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Corollary

The number of n-vertex graphs G with $m \geq n^2/4$ edges and t triangles is at most

$$2^n \binom{n^2/4}{\leq 72t/n} \binom{n^2/4}{\leq 72t/n} = 2^{n + O\left(\frac{t}{n} \log \frac{n^3}{t}\right)}.$$

Upper bound — a few more details

Theorem + Corollary

An n-vertex graph G with $m \geq n^2/4$ edges and t triangles can be made bipartite by removing at most $72t/n$ edges, and there are at most $2^{n+O\left(\frac{t}{n}\log\frac{n^3}{t}\right)}$ such graphs.

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  $2^{n+O\left(\frac{t}{n}\log\frac{n^3}{t}\right)}(1-p)^t = 2^{n+O\left(\frac{t}{n}\log\frac{n^3}{t}\right)-\Omega(t)}$
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- Stability $\Rightarrow O(1)$ edges away from bipartite, classes $X \cup Y$
- Since $m > \lfloor n^2/4 \rfloor$, there is an internal edge $e \subseteq X$
- $d_{\mathcal{T}}(e) \gtrsim \left(\frac{1}{2} + o(1)\right) n \Rightarrow$ miss many edges between e and Y

Fewer forbidden triangles

Question

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Theorem (D.–Lamaison–Tran, 2019+)

For $0 \le k = o\left((n/\log n)^{1/3}\right)$*,*

$$\mathrm{av}\left(n, p\binom{n}{3}\right) = \left\lfloor n^2/4 \right\rfloor + k$$

when $2^{-1/k} \lesssim 1 - p \lesssim 2^{-1/(k+1)}$*.*

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Gives precise result for $m = \tilde{\Omega}(n^{8/3})$

Also obtain meaningful bounds for smaller values of m

The very sparse setting

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Proof.

- Suppose we have $\mathcal{T} \subseteq \binom{[n]}{3}$, $|\mathcal{T}| = m$
- For each triangle in $\mathcal{T}$, delete one of its edges from K_n
- Gives a $\mathcal{T}$-avoiding graph with at least $\binom{n}{2} - m$ edges $\square$

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Bound is tight if triangles in $\mathcal{T}$ are edge-disjoint
$\rightarrow$ partial Steiner Triple System $\rightarrow m \leq \frac{1}{3}\binom{n}{2}$

Beyond the Steiner range

Claim

For $t \geq 0$,

$$\text{av}\left(n, \frac{1}{3}\binom{n}{2} + t\right) \geq \frac{2}{3}\binom{n}{2} - \quad t.$$

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- Destroy at least two triangles of $\mathcal{T}$ in each step

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- Greedily remove such edges one at a time
- Destroy at least two triangles of $\mathcal{T}$ in each step
- Left with a partial Steiner Triple System
- Cannot be too large — gained in previous stage □

Upper bounds

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Theorem (D.–Lamaison–Tran, 2019+)

Let $0 \leq t \leq \frac{7}{15}\binom{n}{2}$. *Then*

$$\frac{2}{3}\binom{n}{2} - \frac{2}{5}t \leq \mathrm{av}\left(n, \frac{1}{3}\binom{n}{2} + t\right) \leq \frac{2}{3}\binom{n}{2} - \frac{1}{7}t + O(n).$$

Blending designs

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- We would like most edges to be covered by a partial STS
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- Decompose K_n into edge-disjoint copies of K_{21}
- Take a STS on most of these 21-cliques
- Decompose the remaining 21-cliques into copies of K_5
  - Take eight triangles from each 5-clique

Redundancy in Turán's Theorem

Theorem (D.–Lamaison–Tran, 2019+)

For fixed $r \geq 3$, there are constants $c, C > 0$ such that, when $m = \alpha n^2$:

(i) *If* $\alpha \lesssim \frac{1}{r(r-1)}$, *then* $\mathrm{av}_r(n, m) = \binom{n}{2} - m$.

(ii) *If* $\alpha \geq \frac{1}{2r(r-1)}$, *then* $\mathrm{av}_r(n, m) \gtrsim t_{r-1}(n) + \max\left\{\frac{cn}{\alpha}, n^2 e^{-C\alpha}\right\}$.

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Corollary

We only need to forbid $O(n^3)$ *copies of* K_r *to achieve Turán's bound.*

Some open problems

Mantel in the sparse range

What is the correct constant c such that
$\mathrm{av}\left(n, \frac{1}{3}\binom{n}{2} + t\right) = \frac{2}{3}\binom{n}{2} - ct$?

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Further generalisations

What happens for other extremal problems? For instance, what if we can only forbid a limited number of four-cycles?

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Thank you!