When Ramsey met Brown, Erdős and Sós

Mykhaylo Tyomkyn

Caltech

Joint work with Asaf Shapira

Background and motivation

Question (Brown-Erdős-Sós '73)

For fixed $r \geq 3$, v and e, what is $f_r(n, v, e)$ – the largest size of an n-vertex r-uniform hypergraph without a '(v, e)-configuration', i.e. a set of e edges spanning at most v vertices?

Background and motivation

Question(Brown-Erdős-Sós '73)

For fixed $r \geq 3$, v and e, what is $f_r(n, v, e)$ – the largest size of an n-vertex r-uniform hypergraph without a '(v, e)-configuration', i.e. a set of e edges spanning at most v vertices?

E.g. $f_3(n, 4, 4)$: Turán problem for $K_4^{(3)}$,

Background and motivation

Question(Brown-Erdős-Sós '73)

For fixed $r \geq 3$, v and e, what is $f_r(n, v, e)$ – the largest size of an n-vertex r-uniform hypergraph without a '(v, e)-configuration', i.e. a set of e edges spanning at most v vertices?

E.g. $f_3(n, 4, 4)$: Turán problem for $K_4^{(3)}$, $f_2(n, v, v^2/4)$: KST.

Background and motivation

Question(Brown-Erdős-Sós '73)

For fixed $r \geq 3$, v and e, what is $f_r(n, v, e)$ – the largest size of an n-vertex r-uniform hypergraph without a '(v, e)-configuration', i.e. a set of e edges spanning at most v vertices?

E.g. $f_3(n, 4, 4)$: Turán problem for $K_4^{(3)}$, $f_2(n, v, v^2/4)$: KST.

Theorem(BES)

For $r, e \geq 3$ and $v \geq r + 1$,

$$\Omega(n^{\frac{er-v}{e-1}}) = f_r(n, v, e) = O(n^{\lceil \frac{er-v}{e-1} \rceil}).$$

Background and motivation

Question(Brown-Erdős-Sós '73)

For fixed $r \geq 3$, v and e, what is $f_r(n, v, e)$ – the largest size of an n-vertex r-uniform hypergraph without a '(v, e)-configuration', i.e. a set of e edges spanning at most v vertices?

E.g. $f_3(n, 4, 4)$: Turán problem for $K_4^{(3)}$, $f_2(n, v, v^2/4)$: KST.

Theorem(BES)

For $r, e \geq 3$ and $v \geq r+1$,

$$\Omega(n^{\frac{er-v}{e-1}}) = f_r(n, v, e) = O(n^{\lceil \frac{er-v}{e-1} \rceil}).$$

So, $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, for any integer $2 \leq k < r$.

Background and motivation

Question(Brown-Erdős-Sós '73)

For fixed $r \geq 3$, v and e, what is $f_r(n, v, e)$ – the largest size of an n-vertex r-uniform hypergraph without a '(v, e)-configuration', i.e. a set of e edges spanning at most v vertices?

E.g. $f_3(n, 4, 4)$: Turán problem for $K_4^{(3)}$, $f_2(n, v, v^2/4)$: KST.

Theorem(BES)

For $r, e \geq 3$ and $v \geq r + 1$,

$$\Omega(n^{\frac{er-v}{e-1}}) = f_r(n, v, e) = O(n^{\lceil \frac{er-v}{e-1} \rceil}).$$

So, $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, for any integer $2 \leq k < r$.

Upper bound: double-counting.

Background and motivation

Question(Brown-Erdős-Sós '73)

For fixed $r \geq 3$, v and e, what is $f_r(n, v, e)$ – the largest size of an n-vertex r-uniform hypergraph without a '(v, e)-configuration', i.e. a set of e edges spanning at most v vertices?

E.g. $f_3(n, 4, 4)$: Turán problem for $K_4^{(3)}$, $f_2(n, v, v^2/4)$: KST.

Theorem(BES)

For $r, e \geq 3$ and $v \geq r + 1$,

$$\Omega(n^{\frac{er-v}{e-1}}) = f_r(n, v, e) = O(n^{\lceil \frac{er-v}{e-1} \rceil}).$$

So, $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, for any integer $2 \leq k < r$.

Upper bound: double-counting. Lower bound: alteration method.

Background and motivation

Given $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, it is natural to ask the following

Background and motivation

Given $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, it is natural to ask the following

Conjecture (Brown-Erdős-Sós '73)

For any integers $r, e \geq 3$ and $2 \leq k < r$,

$$f_r(n, e(r-k)+k+1, e) = o(n^k).$$

Background and motivation

Given $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, it is natural to ask the following

Conjecture (Brown-Erdős-Sós '73)

For any integers $r, e \geq 3$ and $2 \leq k < r$,

$$f_r(n, e(r-k)+k+1, e) = o(n^k).$$

The case $k=2$, i.e. $v=(r-2)e+3$ is of special interest, as the problem reduces to linear r-graphs.

Background and motivation

Given $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, it is natural to ask the following

Conjecture (Brown-Erdős-Sós '73)

For any integers $r, e \geq 3$ and $2 \leq k < r$,

$$f_r(n, e(r-k)+k+1, e) = o(n^k).$$

The case $k = 2$, i.e. $v = (r-2)e + 3$ is of special interest, as the problem reduces to linear r-graphs.

Conjecture (BES: quadratic regime)

For any $\varepsilon > 0$ and integers $r, e \geq 3$ there exists $n_0 = n_0(r, e, \varepsilon)$ such that every linear r-graph with $n \geq n_0$ vertices and at least εn^2 edges contains an $((r-2)e+3, e)$-configuration.

Background and motivation

Given $f_r(n, e(r-k)+k, e) = \Theta(n^k)$, it is natural to ask the following

Conjecture (Brown-Erdős-Sós '73)

For any integers $r, e \geq 3$ and $2 \leq k < r$,

$$f_r(n, e(r-k)+k+1, e) = o(n^k).$$

The case $k = 2$, i.e. $v = (r-2)e + 3$ is of special interest, as the problem reduces to linear r-graphs.

Conjecture (BES: quadratic regime)

For any $\varepsilon > 0$ and integers $r, e \geq 3$ there exists $n_0 = n_0(r, e, \varepsilon)$ such that every linear r-graph with $n \geq n_0$ vertices and at least εn^2 edges contains an $((r-2)e+3, e)$-configuration.

Holds easily in Steiner systems

State of affairs

What is known

State of affairs

What is known

- $f_3(n, 6, 3) = o(n^2)$: Ruzsa–Szemerédi '78
- $f_r(n, 3(r-2)+3, 3) = o(n^2)$: Erdős–Frankl–Rödl '86
- $f_r(n, 3(r-k)+k+1, 3) = o(n^k)$: Alon–Shapira '06
- $f_r(n, 4(r-k)+k+1, 4) = o(n^k)$ for $r > k \geq 3$, and
- $f_r(n, 3(r-k)+k+\lfloor \log_2 e \rfloor, e) = o(n^k)$: Sárközy–Selkow '05

State of affairs

What is known

- $f_3(n, 6, 3) = o(n^2)$: Ruzsa–Szemerédi '78
- $f_r(n, 3(r-2)+3, 3) = o(n^2)$: Erdős–Frankl–Rödl '86
- $f_r(n, 3(r-k)+k+1, 3) = o(n^k)$: Alon–Shapira '06
- $f_r(n, 4(r-k)+k+1, 4) = o(n^k)$ for $r > k \geq 3$, and
- $f_r(n, 3(r-k)+k+\lfloor \log_2 e \rfloor, e) = o(n^k)$: Sárközy–Selkow '05

The remaining values are unknown, even $f_3(n, 7, 4)$.

State of affairs

What is known

- $f_3(n, 6, 3) = o(n^2)$: Ruzsa–Szemerédi '78
- $f_r(n, 3(r-2)+3, 3) = o(n^2)$: Erdős–Frankl–Rödl '86
- $f_r(n, 3(r-k)+k+1, 3) = o(n^k)$: Alon–Shapira '06
- $f_r(n, 4(r-k)+k+1, 4) = o(n^k)$ for $r > k \geq 3$, and
- $f_r(n, 3(r-k)+k+\lfloor \log_2 e \rfloor, e) = o(n^k)$: Sárközy–Selkow '05

The remaining values are unknown, even $f_3(n, 7, 4)$. Possible approaches:

State of affairs

What is known

- $f_3(n,6,3) = o(n^2)$: Ruzsa–Szemerédi '78
- $f_r(n, 3(r-2)+3, 3) = o(n^2)$: Erdős–Frankl–Rödl '86
- $f_r(n, 3(r-k)+k+1, 3) = o(n^k)$: Alon–Shapira '06
- $f_r(n, 4(r-k)+k+1, 4) = o(n^k)$ for $r > k \geq 3$, and
- $f_r(n, 3(r-k)+k+\lfloor \log_2 e \rfloor, e) = o(n^k)$: Sárközy–Selkow '05

The remaining values are unknown, even $f_3(n,7,4)$. Possible approaches:

- BES in groups (Solymosi, Solymosi–Wong, Nenadov–Sudakov–T., Long, Wong)

State of affairs

What is known

- $f_3(n,6,3) = o(n^2)$: Ruzsa–Szemerédi '78
- $f_r(n, 3(r-2)+3, 3) = o(n^2)$: Erdős–Frankl–Rödl '86
- $f_r(n, 3(r-k)+k+1, 3) = o(n^k)$: Alon–Shapira '06
- $f_r(n, 4(r-k)+k+1, 4) = o(n^k)$ for $r > k \geq 3$, and
- $f_r(n, 3(r-k)+k+\lfloor \log_2 e \rfloor, e) = o(n^k)$: Sárközy–Selkow '05

The remaining values are unknown, even $f_3(n,7,4)$. Possible approaches:

- BES in groups (Solymosi, Solymosi–Wong, Nenadov–Sudakov–T., Long, Wong)
- Improving on the Sárközy–Selkow bound (Conlon, Gishboliner–Levanzov–Shapira)

Enter Ramsey

Conjecture (BES, quadratic)

For any $\varepsilon > 0$ and integers $r, e \geq 3$ there exists $n_0 = n_0(r, e, \varepsilon)$ such that every linear r-graph with $n \geq n_0$ vertices and at least εn^2 edges contains an $((r-2)e+3, e)$-configuration.

Enter Ramsey

Conjecture (BES, quadratic)

For any $\varepsilon > 0$ and integers $r, e \geq 3$ there exists $n_0 = n_0(r, e, \varepsilon)$ such that every linear r-graph with $n \geq n_0$ vertices and at least εn^2 edges contains an $((r-2)e+3, e)$-configuration.

A new take (Gyárfás, Nenadov, Conlon): study the corresponding Ramsey problem in complete linear r-graphs, a.k.a. Steiner systems

Enter Ramsey

Conjecture (BES, quadratic)

For any $\varepsilon > 0$ and integers $r, e \geq 3$ there exists $n_0 = n_0(r, e, \varepsilon)$ such that every linear r-graph with $n \geq n_0$ vertices and at least εn^2 edges contains an $((r-2)e+3, e)$-configuration.

A new take (Gyárfás, Nenadov, Conlon): study the corresponding Ramsey problem in complete linear r-graphs, a.k.a. Steiner systems

Conjecture (BES, Ramsey version)

For any integers $c \geq 2$ and $r, e \geq 3$ there exists $n_0 = n_0(c, r, e)$ such that for all $n \geq n_0$ every c-colouring of a complete linear r-graph of order n contains a monochromatic $((r-2)e+3, e)$-configuration.

Enter Ramsey

Conjecture (BES, quadratic)

For any $\varepsilon > 0$ and integers $r, e \geq 3$ there exists $n_0 = n_0(r, e, \varepsilon)$ such that every linear r-graph with $n \geq n_0$ vertices and at least εn^2 edges contains an $((r-2)e+3, e)$-configuration.

A new take (Gyárfás, Nenadov, Conlon): study the corresponding Ramsey problem in complete linear r-graphs, a.k.a. Steiner systems

Conjecture (BES, Ramsey version)

For any integers $c \geq 2$ and $r, e \geq 3$ there exists $n_0 = n_0(c, r, e)$ such that for all $n \geq n_0$ every c-colouring of a complete linear r-graph of order n contains a monochromatic $((r-2)e+3, e)$-configuration.

Ramsey's theorem gives this immediately for $e = 3$ and any c, r

Our results

Our results

Theorem (Shapira–T. '19+)

For every $c \geq 2$ there exists $r_0 = r_0(c)$ such that for every $r \geq r_0$, $e \geq 3$ and $n \geq n_0(c, r, e)$ in every edge-colouring of a complete linear r-graph on n vertices with c colours there is a monochromatic $((r-2)e+3, e)$-configuration.

Our results

Theorem (Shapira–T. '19+)

For every $c \geq 2$ there exists $r_0 = r_0(c)$ such that for every $r \geq r_0$, $e \geq 3$ and $n \geq n_0(c, r, e)$ in every edge-colouring of a complete linear r-graph on n vertices with c colours there is a monochromatic $((r-2)e+3, e)$-configuration.

For $c = 2$ we show that one can take $r_0(2) = 4$

Our results

Theorem (Shapira–T. '19+)

For every $c \geq 2$ there exists $r_0 = r_0(c)$ such that for every $r \geq r_0$, $e \geq 3$ and $n \geq n_0(c, r, e)$ in every edge-colouring of a complete linear r-graph on n vertices with c colours there is a monochromatic $((r-2)e+3, e)$-configuration.

For $c = 2$ we show that one can take $r_0(2) = 4$

Theorem (Shapira–T. '19+)

For any $r \geq 4$, $e \geq 3$ and $n \geq n_0(r, e)$ in every edge-colouring of a complete linear r-graph on n vertices with 2 colours there is a monochromatic $((r-2)e+3, e)$-configuration.

Proof idea

We focus here on the case $(c, r) = (2, 4)$:

Proof idea

We focus here on the case $(c, r) = (2, 4)$:

Theorem (ST)

For any $e \geq 3$ and $n \geq n_0(e)$ in every edge-colouring of a complete linear 4-graph on n vertices with 2 colours there is a colour class containing a set of e edges spanning at most $2e + 3$ vertices.

Proof idea

We focus here on the case $(c, r) = (2, 4)$:

Theorem (ST)

For any $e \geq 3$ and $n \geq n_0(e)$ in every edge-colouring of a complete linear 4-graph on n vertices with 2 colours there is a colour class containing a set of e edges spanning at most $2e + 3$ vertices.

Idea: for a 4-graph $\mathcal{G}$ define an auxiliary graph $B(\mathcal{G})$, suited for finding $(2e + 3, e)$-configurations.

Proof idea

We focus here on the case $(c, r) = (2, 4)$:

Theorem (ST)

For any $e \geq 3$ and $n \geq n_0(e)$ in every edge-colouring of a complete linear 4-graph on n vertices with 2 colours there is a colour class containing a set of e edges spanning at most $2e + 3$ vertices.

Idea: for a 4-graph $\mathcal{G}$ define an auxiliary graph $B(\mathcal{G})$, suited for finding $(2e + 3, e)$-configurations.

In a 2-colouring of an SQS one colour $\mathcal{G}$, will have a rich $B(\mathcal{G})$

Proof idea

We focus here on the case $(c, r) = (2, 4)$:

Theorem (ST)

For any $e \geq 3$ and $n \geq n_0(e)$ in every edge-colouring of a complete linear 4-graph on n vertices with 2 colours there is a colour class containing a set of e edges spanning at most $2e + 3$ vertices.

Idea: for a 4-graph $\mathcal{G}$ define an auxiliary graph $B(\mathcal{G})$, suited for finding $(2e + 3, e)$-configurations.

In a 2-colouring of an SQS one colour $\mathcal{G}$, will have a rich $B(\mathcal{G})$

'Explore' $\mathcal{G}$ along $B(\mathcal{G})$ to exhibit a $(2e + 3, e)$-configuration in it.

Tools: Auxiliary graph

Tools: Auxiliary graph

Definition (Bowtie graph)

For a linear 4-graph $\mathcal{G}$, define $B(\mathcal{G}) := (V, E)$, where

- $V = \{\{S, T\} : S, T \in E(\mathcal{G}), |S \cap T| = 1\}$,
- $E = \{\{b_1, b_2\} : b_1 = S_1 T, b_2 = S_2 T, |S_1 \cup S_2 \cup T| = 9\}$.

Tools: Auxiliary graph

Definition (Bowtie graph)

For a linear 4-graph $\mathcal{G}$, define $B(\mathcal{G}) := (V, E)$, where

- $V = \{\{S, T\} : S, T \in E(\mathcal{G}), |S \cap T| = 1\}$,
- $E = \{\{b_1, b_2\} : b_1 = S_1 T, b_2 = S_2 T, |S_1 \cup S_2 \cup T| = 9\}$.

Observation

Edges of B correspond to 'non-trivial triangles' in the underlying graph of $\mathcal{G}$. In particular, $\Delta(B) \leq 18$.

Tools: Auxiliary graph

Definition (Bowtie graph)

For a linear 4-graph $\mathcal{G}$, define $B(\mathcal{G}) := (V, E)$, where

- $V = \{\{S, T\} : S, T \in E(\mathcal{G}), |S \cap T| = 1\}$,
- $E = \{\{b_1, b_2\} : b_1 = S_1 T, b_2 = S_2 T, |S_1 \cup S_2 \cup T| = 9\}$.

Observation

Edges of B correspond to 'non-trivial triangles' in the underlying graph of $\mathcal{G}$. In particular, $\Delta(B) \leq 18$.

Large components in B are good for us:

Tools: Auxiliary graph

Definition (Bowtie graph)

For a linear 4-graph $\mathcal{G}$, define $B(\mathcal{G}) := (V, E)$, where

- $V = \{\{S, T\} : S, T \in E(\mathcal{G}), |S \cap T| = 1\}$,
- $E = \{\{b_1, b_2\} : b_1 = S_1T, b_2 = S_2T, |S_1 \cup S_2 \cup T| = 9\}$.

Observation

Edges of B correspond to ‘non-trivial triangles’ in the underlying graph of $\mathcal{G}$. In particular, $\Delta(B) \leq 18$.

Large components in B are good for us:

Lemma

If B has a connected component of order at least 2^{100e^3}, then $\mathcal{G}$ contains a $(2e+3, e)$-configuration.

Tools: Ramsey multiplicity

Proposition(Goodman inspired)

For large n, in every 2-edge-colouring of K_n there is a colour class G satisfying

$$T(G) \geq \left(\frac{1}{6} - o(1)\right) \sum_{u \in K_n} \binom{d_G(u)}{2} = \Theta(n^3)$$

Tools: Ramsey multiplicity

Proposition(Goodman inspired)

For large n, in every 2-edge-colouring of K_n there is a colour class G satisfying

$$T(G) \geq (\frac{1}{6} - o(1)) \sum_{u \in K_n} \binom{d_G(u)}{2} = \Theta(n^3)$$

That is, about half of the 'cherries' in G are contained in a triangle.

Tools: Ramsey multiplicity

Proposition(Goodman inspired)

For large n, in every 2-edge-colouring of K_n there is a colour class G satisfying

$$T(G) \geq (\frac{1}{6} - o(1)) \sum_{u \in K_n} \binom{d_G(u)}{2} = \Theta(n^3)$$

That is, about half of the ‘cherries’ in G are contained in a triangle.

Corollary

For large n, in every 2-colouring of a complete linear 4-graph of order n there is a colour class $\mathcal{G}$ satisfying $d_{avg}(B(\mathcal{G})) > 9 - o(1)$.

Tools: Ramsey multiplicity

Proposition(Goodman inspired)

For large n, in every 2-edge-colouring of K_n there is a colour class G satisfying

$$T(G) \geq (\frac{1}{6} - o(1)) \sum_{u \in K_n} \binom{d_G(u)}{2} = \Theta(n^3)$$

That is, about half of the 'cherries' in G are contained in a triangle.

Corollary

For large n, in every 2-colouring of a complete linear 4-graph of order n there is a colour class $\mathcal{G}$ satisfying $d_{avg}(B(\mathcal{G})) > 9 - o(1)$.

For $c \geq 3$, use Ramsey multiplicity instead.

Dense components

Definition

Call a component $C \subseteq B$ *dense* if $d_{avg}(C) \geq 9$.

Dense components

Definition

Call a component $C \subseteq B$ *dense* if $d_{avg}(C) \geq 9$.

Since $d_{avg}(B) > 9 - o(1)$, $\Delta(B) \leq 18$, and assuming all components are smaller than 2^{100e^3}, averaging gives

Lemma

B has $\Theta(n^3)$ dense components.

Dense components

Definition

Call a component $C \subseteq B$ *dense* if $d_{avg}(C) \geq 9$.

Since $d_{avg}(B) > 9 - o(1)$, $\Delta(B) \leq 18$, and assuming all components are smaller than 2^{100e^3}, averaging gives

Lemma

B has $\Theta(n^3)$ dense components.

More averaging:

Dense components

Definition

Call a component $C \subseteq B$ *dense* if $d_{avg}(C) \geq 9$.

Since $d_{avg}(B) > 9 - o(1)$, $\Delta(B) \leq 18$, and assuming all components are smaller than 2^{100e^3}, averaging gives

Lemma

B has $\Theta(n^3)$ dense components.

More averaging:

Lemma

There exist a vertex, a hyperedge $u_0 \in T_0 \in \mathcal{G}$, and $\Theta(n)$ further hyperedges $T_1^0, T_2^0 \ldots, \in E(\mathcal{G})$ such that, for each i we have that $u_0 \in T_i^0$ and all bowties $\{T_0, T_i^0\}$ belong to distinct dense components.

Inductive configurations

Definition

Call a $(2i+3, i)$-configuration $\mathcal{F}$ *inductive* if either $i = 2$, or $i > 2$ and there exists a hyperedge $T \in \mathcal{F}$ such that:

- T is contained in a $(9, 3)$-configuration,
- T has 2 vertices of degree 1, and
- $\mathcal{F} \setminus \{T\}$ is inductive.

Inductive configurations

Definition

Call a $(2i+3, i)$-configuration $\mathcal{F}$ *inductive* if either $i = 2$, or $i > 2$ and there exists a hyperedge $T \in \mathcal{F}$ such that:

- T is contained in a $(9, 3)$-configuration,
- T has 2 vertices of degree 1, and
- $\mathcal{F} \setminus \{T\}$ is inductive.

Main idea

Dense components in B give rise to inductive configurations.

Inductive configurations

Definition

Call a $(2i+3, i)$-configuration $\mathcal{F}$ *inductive* if either $i = 2$, or $i > 2$ and there exists a hyperedge $T \in \mathcal{F}$ such that:

- T is contained in a $(9, 3)$-configuration,
- T has 2 vertices of degree 1, and
- $\mathcal{F} \setminus \{T\}$ is inductive.

Main idea

Dense components in B give rise to inductive configurations. 'Explore' a dense component in a bootstrap percolation manner, until one of the following happens.

Inductive configurations

Definition

Call a $(2i+3, i)$-configuration $\mathcal{F}$ *inductive* if either $i = 2$, or $i > 2$ and there exists a hyperedge $T \in \mathcal{F}$ such that:

- T is contained in a $(9, 3)$-configuration,
- T has 2 vertices of degree 1, and
- $\mathcal{F} \setminus \{T\}$ is inductive.

Main idea

Dense components in B give rise to inductive configurations. 'Explore' a dense component in a bootstrap percolation manner, until one of the following happens.

- We create a $(2i+2, i)$-configuration $\rightarrow$ continue in a new component.

Inductive configurations

Definition

Call a $(2i+3, i)$-configuration $\mathcal{F}$ *inductive* if either $i = 2$, or $i > 2$ and there exists a hyperedge $T \in \mathcal{F}$ such that:

- T is contained in a $(9, 3)$-configuration,
- T has 2 vertices of degree 1, and
- $\mathcal{F} \setminus \{T\}$ is inductive.

Main idea

Dense components in B give rise to inductive configurations. 'Explore' a dense component in a bootstrap percolation manner, until one of the following happens.

- We create a $(2i+2, i)$-configuration $\rightarrow$ continue in a new component.
- We reach $i = e$, i.e. a $(2e+3, e)$-configuration.

Main lemma

Recall: we have $u_0 \in T_0 \in \mathcal{G}$, and a set $\mathcal{C}$ of $\Theta(n)$ dense B-components, such that each $C \in \mathcal{C}$ contains a bowtie $\{T_0, T\}$, for some $T \ni u_0$.

Main lemma

Recall: we have $u_0 \in T_0 \in \mathcal{G}$, and a set $\mathcal{C}$ of $\Theta(n)$ dense B-components, such that each $C \in \mathcal{C}$ contains a bowtie $\{T_0, T\}$, for some $T \ni u_0$.

Lemma

For each $2 \leq i \leq e$ there exists a $(2i+3, i)$-configuration $\mathcal{F}_i \subset \mathcal{G}$ of one of the following two types:

(a) $\mathcal{F}_i$ is an $(2i+2, i)$-configuration with $T_0 \in E(\mathcal{F}_i)$.

(b) There exist a subhypergraph $\mathcal{E}_i \subseteq \mathcal{F}_i$ and a component $C_i \in \mathcal{C}$ such that:

1. $\mathcal{E}_i$ is an inductive $(2j+3, j)$-configuration for some $j \geq 2$ with $T_0 \in E(\mathcal{E}_i)$,
2. $V(\mathcal{E}_i) \cap V(\mathcal{F}_i \setminus \mathcal{E}_i) \subseteq T_0$,
3. The set $A_i = \{b \in V(C_i) : b = \{T, S\}; T, S \in \mathcal{E}_i\}$ satisfies $d_{avg}(B[A_i]) < 9$.

Main lemma

Recall: we have $u_0 \in T_0 \in \mathcal{G}$, and a set $\mathcal{C}$ of $\Theta(n)$ dense B-components, such that each $C \in \mathcal{C}$ contains a bowtie $\{T_0, T\}$, for some $T \ni u_0$.

Lemma

For each $2 \leq i \leq e$ there exists a $(2i+3, i)$-configuration $\mathcal{F}_i \subset \mathcal{G}$ of one of the following two types:

(a) $\mathcal{F}_i$ is an $(2i+2, i)$-configuration with $T_0 \in E(\mathcal{F}_i)$.

(b) There exist a subhypergraph $\mathcal{E}_i \subseteq \mathcal{F}_i$ and a component $C_i \in \mathcal{C}$ such that:

1. $\mathcal{E}_i$ is an inductive $(2j+3, j)$-configuration for some $j \geq 2$ with $T_0 \in E(\mathcal{E}_i)$,
2. $V(\mathcal{E}_i) \cap V(\mathcal{F}_i \setminus \mathcal{E}_i) \subseteq T_0$,
3. The set $A_i = \{b \in V(C_i) : b = \{T, S\}; T, S \in \mathcal{E}_i\}$ satisfies $d_{avg}(B[A_i]) < 9$.

In particular, $A_i \subsetneq C_i$, and we can continue the process

Component exploration

How to make sure that $d_{avg}(B[A_i]) < 9$ at each step?

Component exploration

How to make sure that $d_{avg}(B[A_i]) < 9$ at each step?

Reversing the roles, it suffices to show the following

Component exploration

How to make sure that $d_{avg}(B[A_i]) < 9$ at each step?

Reversing the roles, it suffices to show the following

Claim

Suppose that $i \geq 2$, $\mathcal{F}$ is an inductive $(2i+3, i)$-configuration, and $B = B(\mathcal{F})$. Then for any $A \subset V(B)$ we have $d_{avg}(B[A]) < 9$. In particular, B has no dense components.

Component exploration

How to make sure that $d_{avg}(B[A_i]) < 9$ at each step?

Reversing the roles, it suffices to show the following

Claim

Suppose that $i \geq 2$, $\mathcal{F}$ is an inductive $(2i+3, i)$-configuration, and $B = B(\mathcal{F})$. Then for any $A \subset V(B)$ we have $d_{avg}(B[A]) < 9$. In particular, B has no dense components.

Proof: True for $i = 2$, as $|B| = 1$.

Component exploration

How to make sure that $d_{avg}(B[A_i]) < 9$ at each step?

Reversing the roles, it suffices to show the following

Claim

Suppose that $i \geq 2$, $\mathcal{F}$ is an inductive $(2i+3, i)$-configuration, and $B = B(\mathcal{F})$. Then for any $A \subset V(B)$ we have $d_{avg}(B[A]) < 9$. In particular, B has no dense components.

Proof: True for $i = 2$, as $|B| = 1$.
$i \to i+1$: the added vertices of $B[A]$ are indexed by two 3-uniform matchings: R and G,

Component exploration

How to make sure that $d_{avg}(B[A_i]) < 9$ at each step?

Reversing the roles, it suffices to show the following

Claim

Suppose that $i \geq 2$, $\mathcal{F}$ is an inductive $(2i+3, i)$-configuration, and $B = B(\mathcal{F})$. Then for any $A \subset V(B)$ we have $d_{avg}(B[A]) < 9$. In particular, B has no dense components.

Proof: True for $i = 2$, as $|B| = 1$.
$i \to i+1$: the added vertices of $B[A]$ are indexed by two 3-uniform matchings: R and G, resulting in $|R| + |G|$ new vertices and at most

$$3|V(R) \cap V(G)| \leq \frac{9}{2}(|R| + |G|)$$

new edges.

Open questions

Conjecture (BES-R)

For any integers $c \geq 2$ and $r, e \geq 3$ there exists $n_0 = n_0(c, r, e)$ such that for all $n \geq n_0$ every c-colouring of a complete linear r-graph of order n contains a monochromatic $((r-2)e+3, e)$-configuration.

Open questions

Conjecture (BES-R)

For any integers $c \geq 2$ and $r, e \geq 3$ there exists $n_0 = n_0(c, r, e)$ such that for all $n \geq n_0$ every c-colouring of a complete linear r-graph of order n contains a monochromatic $((r-2)e+3, e)$-configuration.

More generally:

Meta-question

Study Ramsey and Turán type problems in Steiner systems

Open questions

Conjecture (BES-R)

For any integers $c \geq 2$ and $r, e \geq 3$ there exists $n_0 = n_0(c, r, e)$ such that for all $n \geq n_0$ every c-colouring of a complete linear r-graph of order n contains a monochromatic $((r-2)e+3, e)$-configuration.

More generally:

Meta-question

Study Ramsey and Turán type problems in Steiner systems

Gyárfás et. al., Füredi–Gyárfás, DeBiasio–Tait