On the quasirandomness of projective norm graphs

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joint work with
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  Solves the question asymptotically unless H is bipartite.

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- Bukh (random algebraic construction):
  $s \geq f(t) \gg (t-1)! + 1$

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N is a multiplicative $\mathbb{F}_q$-linear $\mathbb{F}_{q^{t-1}} \to \mathbb{F}_q$ function and for $a \in \mathbb{F}_q^*$

$$\left|\mathrm{N}^{-1}(a)\right| = \frac{q^{t-1}-1}{q-1}.$$

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Let $\mathbb{F}$ be a field, $\ell \in \mathbb{N}$, $a_{ij}, b_i \in \mathbb{F}$ for $1 \leq i, j \leq \ell$ such that $a_{i_1 j} \neq a_{i_2 j}$ for all j and $i_1 \neq i_2$. Then the system

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- It is applied for $\ell = t-1$ in a very, very, very special setting:

$$\mathbb{F} = \mathbb{F}_{q^{t-1}}, \quad a_{ij} = -B_i^{q^{j-1}}, \quad b_i \in \mathbb{F}_q \subseteq \mathbb{F}_{q^{t-1}}, \quad x_j = Y^{q^{j-1}}.$$

(Non-)existence of further complete bipartite subgraphs

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For $t \geq 2$ *what is the largest* s_t *such that* $\mathrm{NG}(q, t)$ *contains* K_{t,s_t} *for every large enough prime power* q*? Does* $s_t < (t-1)!$ *hold?*

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- This shows $s_4 = 6$.

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  If T is not generic $\Longrightarrow$ no common neighbours.

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$$\mathrm{N}(A_i + X) = a_i x \qquad \text{for } i = 1, \dots, t. \tag{1}$$

- T is called **generic** if all the first coordinates are distinct. If T is not generic $\implies$ no common neighbours.
- T is called **aligned** if all the second coordinates are the same.

$$\xi(T) = \begin{cases} 1 & \text{if } T \text{ is aligned} \\ 0 & \text{if } T \text{ is not aligned} \end{cases}$$

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$$N(Y + B_i) = b_i, \qquad \text{for } i = 1, \ldots, t-1. \qquad (2)$$

Proposition. $|Solutions((1))| = |Solutions((2))| - \xi(T)$

Pair degrees

Solution set of $N(Y + B_1) = b_1$ is the translate $N^{-1}(b_1) - B_1$. Hence

Proposition (2-neighbourhoods)

Let T be a generic pair of vertices in $\mathrm{NG}(q,t)$. *Then*

$$\deg(T) = \frac{q^{t-1} - 1}{q - 1} - \xi(T) = (1 + o(1))q^{t-2}.$$

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Theorem

For $t \geq 3$ let $S_t(c_1, c_2)$ be the solution set of (3). Then

$$|S_t(c_1, c_2)| = \begin{cases} 1 - \eta_{\mathbb{F}_q}\big((1 + c_1 - c_2)^2 - 4c_1\big) & \text{if } t = 3, \\ 2q + 1 - \eta_{\mathbb{F}_q}(-3) & \text{if } \begin{array}{l} t = 4 \text{ and} \\ (c_1, c_2) = (1, -1), \end{array} \\ q^{t-3} + O(q^{t-3.5}) & \text{otherwise,} \end{cases}$$

where $\eta_{\mathbb{F}_q}$ is the quadratic character of $\mathbb{F}_q$.

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where $\eta_{\mathbb{F}_q}$ is the quadratic character of $\mathbb{F}_q$.
In particular, for $t \geq 4$ we have $\deg(T) = (1 + o(1))q^{t-3}$, unless $t = 4$ and $(c_1, c_2) = (1, -1)$.

Proof ideas

Define the auxiliary polynomial

$$f_{t,c_1,c_2}(Z) = N_{t-2}(Z+1)N_{t-2}(Z) + c_1 N_{t-2}(Z+1) + c_2 N_{t-2}(Z)$$

of degree $2(q^{t-3} + \cdots + q + 1)$ and show that for its set $R_t(c_1, c_2)$ of roots and set $R_t^*(c_1, c_2)$ of multiple roots in $\overline{\mathbb{F}_q}$:

- $S_t(c_1, c_2) \subseteq R_t(c_1, c_2)$
- $S_t(c_1, c_2) \cap \mathbb{F}_q \subseteq R_t^*(c_1, c_2)$
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Consequently: every root of f_{t,c_1,c_2} is contained in $\mathbb{F}_{q^{t-1}} \cup \mathbb{F}_{q^{t-2}}$, multiple roots are contained in $\mathbb{F}_q$ and all have multiplicity two.

Then some double counting, averaging, counting $|R_t(c_1, c_2) \cap \mathbb{F}_{q^{t-2}}|$ classified via the N_{t-2}-norm, more double counting, Weil character sum estimate, finish with induction ...

Bonus from the proof

- The special case $f_{4,1,-1} = h_1 \cdot h_2$ factors over $\mathbb{F}_q$: where $h_1(Z) = Z^{q+1} + Z + 1$ and $h_2(Z) = Z^{q+1} + Z^q + 1$.
- The sets $\mathcal{H}_i = \{Z \in \mathbb{F}_{q^3} : h_i(Z) = 0\}$ are inverses of each other: $\mathcal{H}_2 = \mathcal{H}_1^{-1}$.
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are difference sets[1] in the multiplicative group $N_3^{-1}(1)$. In particular h_1 and h_2 factor over $\mathbb{F}_{q^3}$.

Corollary

$$\{Z \in \mathbb{F}_{q^3} : N_3(Z) = 1,\ N_3(Z+1) = -1\} = \mathcal{H}_1 \cup \mathcal{H}_2$$

[1] Set $D \subseteq G$ is a *difference set* of a multiplicative group of G is every $g \in G \setminus \{1\}$ has a unique representation as a product $d_1 \cdot d_2 = g$, where $d_1 \in D$ and $d_2 \in D^{-1}$.

Quadruple degrees

Theorem (4-neighbourhoods)

Let $q = p^k$ *be a prime power,* $t \geq 2$ *an integer, and* T *a generic set of* 4 *vertices in* $\mathrm{NG}(q,t)$*. Then*

$$\deg(T) \leq 6(q^{t-4} + q^{t-5} + \cdots + q + 1).$$

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- For $t \geq 5$ this was not known to follow from the commutative algebaric proof.
- Full characterization of the 4-neighbourhoods as we did for 2- and 3-neighbourhoods seems difficult.

Finding a $K_{4,6}$ in $\mathrm{NG}(q,4)$: an idea

We hope to find $B_1, B_2, B_3 \in \mathbb{F}_{q^3}$ and $b_1, b_2, b_3 \in \mathbb{F}_q^*$ such that

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Idea: We characterized those (rare) triples which had twice as many common neighbors as the average. Their corresponding system is: $\mathrm{N}_3(Y) = 1,\ N_3(Y + 1) = -1.$

We try to combine two such triples into a quadruple and hope for the best. The corresponding system becomes

$$\mathrm{N}_3(Y) = 1, \qquad \mathrm{N}_3(Y + 1) = -1, \qquad \mathrm{N}_3(Y + A) = -1,$$

where $A \in \mathbb{F}_{q^3}$ and $\mathrm{N}_3(A) = 1$.

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$\mathcal{H}_1, \mathcal{H}_2$ are difference sets, so

- there exists a unique *mixed* product representation $A = Y_1 Y_2$ such that $Y_i \in \mathcal{H}_i$
- there exists *at most one* $\mathcal{H}_i$-product representation for each i

Finding a $K_{4,6}$ in $\mathrm{NG}(q,4)$: the finish

For 6 solutions we need an A with $\mathcal{H}_i$-product representation for both $i = 1$ and 2 that are all distinct.

Let $\mathcal{P}_i := \{hh' : h, h' \in \mathcal{H}_i, h \neq h'\}$, If $\mathcal{P}_1 \cap \mathcal{P}_2 \neq \emptyset$, we found an appropriate A.

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Finish for $q \equiv 2 \pmod 3$. Then $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$.

Was $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$, then $\mathcal{P}_1$ and $\mathcal{P}_2$ partition $N_3^{-1}(1) \setminus \{1\}$. And then

$$-1 = \sum_{N^{-1}(1)\setminus\{1\}} = \sum_{\mathcal{P}_1} + \sum_{\mathcal{P}_2} = 0 + 0$$

A contradiction.

Quasirandomness

A (sequence of) roughly $d = d(n)$-regular graph(s) G on n vertices is **quasirandom** if it satisfies certain properties of the Erdős-Rényi random graph $G(n, \frac{d}{n})$ with probability tending to 1 as $n \to \infty$.

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Theorem

For $t \geq 4$ the projective norm graph $\mathrm{NG}(q,t)$ is H-quasirandom whenever H is a 3-degenerate simple graph.

Generalized Turán numbers

Definition

For $n \in \mathbb{N}$ and simple graphs T and H the **generalized Turán number** $\mathrm{ex}(n, T, H)$ counts the maximum possible number of unlabeled copies of T in an H-free graph on n vertices.

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Let $t \geq 4$ and $s > (t-1)!$. Then $ex(n, T, K_{t,s}) = \Theta\left(n^{v(T) - \frac{e(T)}{t}}\right)$, whenever $T = K_4$ or $T = K_{a,b}$ with $a \leq 3$, $a \leq b < s$.

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Let $t \geq 4$ and $s > (t-1)!$. Then $ex(n, T, K_{t,s}) = \Theta\left(n^{v(T) - \frac{e(T)}{t}}\right)$, whenever $T = K_4$ or $T = K_{a,b}$ with $a \leq 3$, $a \leq b < s$.

New cases that we solve:

- $T = K_4$ and $t = 4, 5$, $(t-1)! < s < f(t)$,
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Generalized Turán numbers

- Kostochka-Mubayi-Verstraëte, Alon-Shikhelman: $s > (t-1)!$, $T = K_3 \Rightarrow$ matching lower bound for $t \geq 2$.
- Ma-Yuan-Zhang: Bukh's random algebraic construction gives a matching lower bound when $T = K_m$, $m \leq t+1$ and $s \geq f(t)$; or $T = K_{a,b}$, $a \leq b$, $a < t, b \leq t$ and $s \geq f(t)$.

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Conjecture

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$ex(n, K_{4,6}) = o(n^{7/4})$