Extremal problems concerning cycles in tournaments

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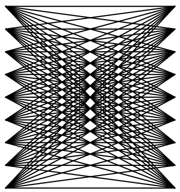
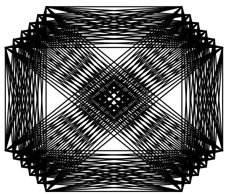
Turán problems

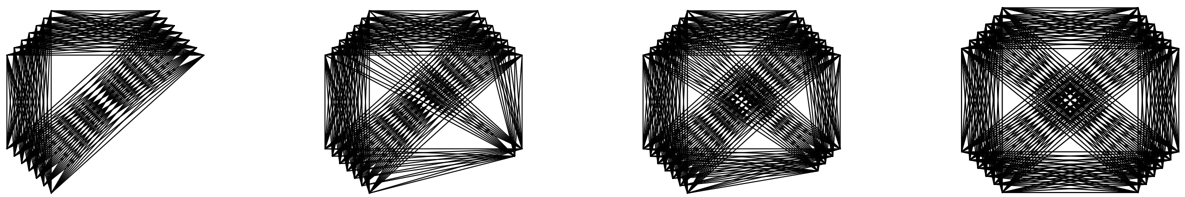
- Maximum edge-density of H-free graph
- Mantel's Theorem (1907): $\frac{1}{2}$ for $H = K_3$ $(K_{\frac{n}{2},\frac{n}{2}})$
- Turán's Theorem (1941): $\frac{\ell-2}{\ell-1}$ for $H = K_\ell$ $(K_{\frac{n}{\ell-1},\dots,\frac{n}{\ell-1}})$
- Erdős-Stone Theorem (1946): $\frac{\chi(H)-2}{\chi(H)-1}$
- extremal examples unique up to $o(n^2)$ edges



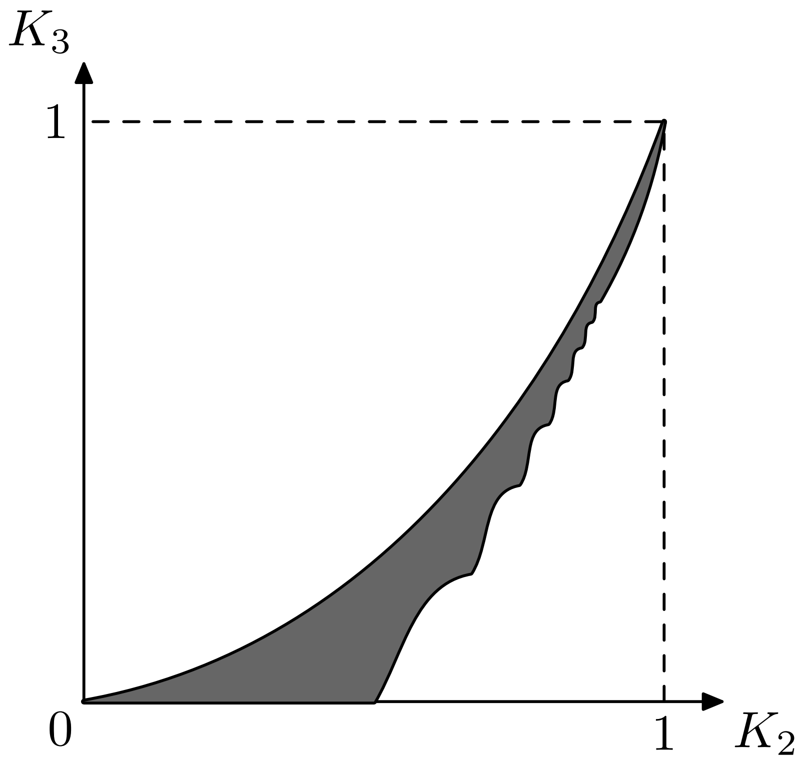


Erdős-Rademacher problem

- Turán's Theorem:
  edge-density $\leq 1/2 \Leftrightarrow$ minimum triangle density $= 0$
- What happens if edge-density $> 1/2$?
- minimum attained by $K_{n,\ldots,n}$ for edge-density $\frac{k-1}{k}$
- smooth transformation from $K_{n,n}$ for $K_{n,n,n}$, from $K_{n,n,n}$ to $K_{n,n,n,n}$, etc.



Erdős-Rademacher problem



solved by Razborov in 2008

Tournaments

- tournament = orientation of a complete graph
- analogue of Erdős-Rademacher Problem
  minimum density of C_4 for a fixed density of C_3
- Conjecture of Linial and Morgenstern (2014)
  blow-up of a transitive tournament (random inside)
  with all but one equal parts and a smaller part
  transitive orientation of $K_{n,\dots,n,\alpha n}$, random inside parts

Tournaments

- minimum density of C_4 for a fixed density of C_3
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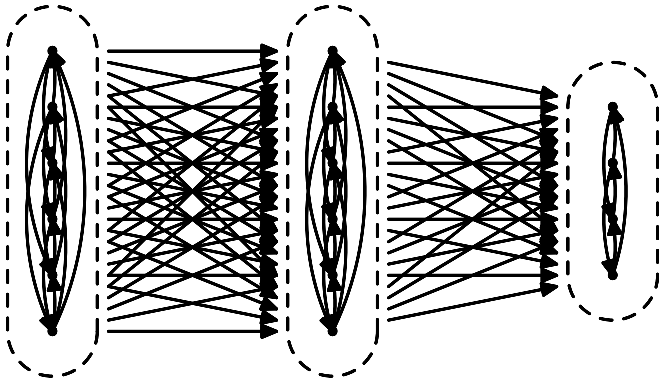
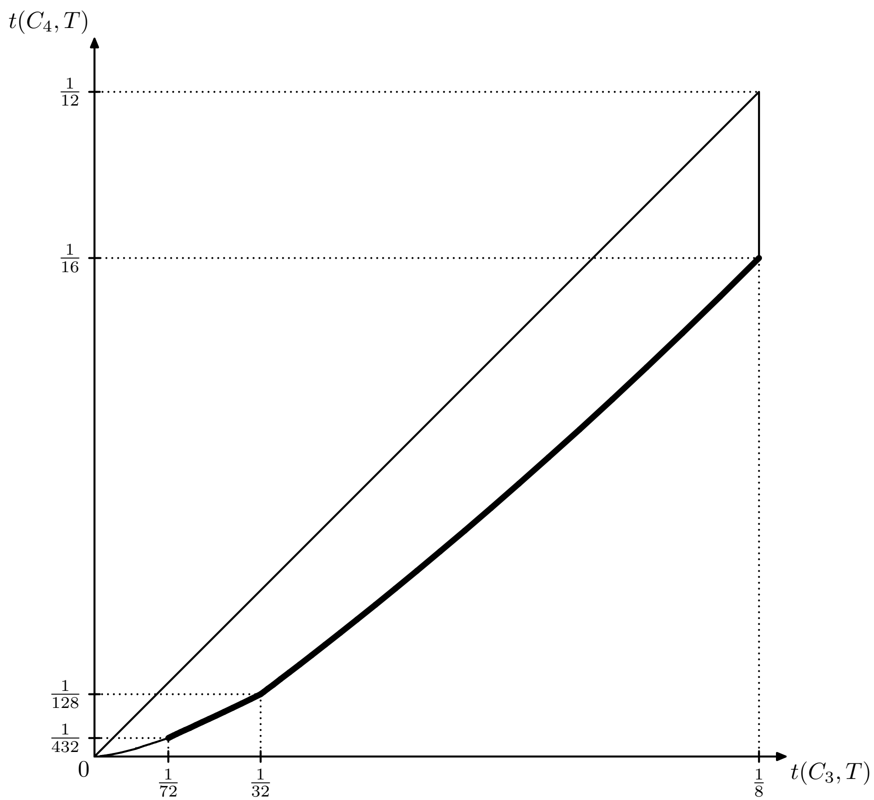


Our results



joint work with Chan, Grzesik and Noel

Approach to the problem

- linear algebra tools

  adjacency matrix $A \in \{0,1\}^{V(G)\times V(G)}$

  $\mathrm{Tr}\, A^k$ = number of closed k-walks

- regularity method

  approximation by an $(n \times n)$-matrix A

  rows and columns $\approx$ parts in regularity decomposition

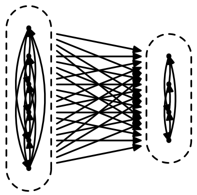
  $A_{ij} \geq 0$ and $A_{ij} + A_{ji} = 1$ for all $i, j \in \{1, \ldots, n\}$

Cases of two and three parts

- non-negative matrix A, s.t. $A + A^T = \mathbb{J}$
- properties of the spectrum of A:
  $\mathrm{Tr}\ A = \lambda_1 + \ldots + \lambda_k = 1/2$
  Perron–Frobenius $\Rightarrow \exists \rho \in \mathbb{R} : \rho = \lambda_1$ and $|\lambda_i| \leq \lambda_1$
  $v^*(A + A^T)v = v^*(\lambda_i + \overline{\lambda_i})v = v^*\mathbb{J}v \geq 0 \Rightarrow \mathrm{Re}\ \lambda_i \geq 0$
- fix $\mathrm{Tr}\ A^3 = \lambda_1^3 + \ldots + \lambda_k^3 \in [1/36, 1/8]$
  minimize $\mathrm{Tr}\ A^4 = \lambda_1^4 + \ldots + \lambda_k^4$
- optimum $\lambda_{\leq k-1} = \rho$ and $\lambda_k = 1/2 - (k-1)\rho$, $k \in \{2, 3\}$

Case of two parts—structure

- $A = (\mathbb{J} + B)/2$, B is antisymmetric, i.e. $B = -B^T$
  analysis of antisymmetric matrix B
- assign $p_v \in [0, 1/2]$ to each vertex v
  orient from v to w with probability $1/2 + (p_v - p_w)$
- conjectured construction: $p_v \in \{0, 1/2\}$



Case of two parts—structure

- $A = (\mathbb{J} + B)/2$, B is antisymmetric, i.e. $B = -B^T$
  A is non-negative and $A + A^T = \mathbb{J}$
- analysis of antisymmetric matrix B
  σ_i and α_i for matrix B with $\sum_i \cos^2 \alpha_i = 1$

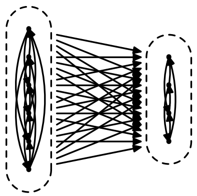
$$B = U^T \begin{pmatrix} 0 & \sigma_1 & 0 & 0 \\ -\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2 \\ 0 & 0 & -\sigma_2 & 0 \end{pmatrix} U$$

Case of two parts—structure

- $A = (\mathbb{J} + B)/2$, B is antisymmetric, i.e. $B = -B^T$
  A is non-negative and $A + A^T = \mathbb{J}$
- analysis of antisymmetric matrix B
  σ_i and α_i for matrix B with $\sum_i \cos^2 \alpha_i = 1$
- $\mathrm{Tr}\, A^3 \approx \mathrm{Tr}\, \mathbb{J}^3 + \mathrm{Tr}\, \mathbb{J} B^2 = \sum_i \sigma_i^2 \cos^2 \alpha_i$
  $\mathrm{Tr}\, A^4 \approx \mathrm{Tr}\, \mathbb{J}^4 + \mathrm{Tr}\, \mathbb{J}^2 B^2 + \mathrm{Tr}\, B^4 \approx \mathrm{Tr}\, \mathbb{J} B^2 + \sum_i \sigma_i^4$
- optimum for $\alpha_1 = 0$, $\alpha_{\geq 2} = \pi/2$ and $\sigma_{\geq 2} = 0$

Case of two parts—structure

- $A = (\mathbb{J} + B)/2$, B is antisymmetric, i.e. $B = -B^T$
  σ_i and α_i for matrix B with $\sum_i \cos^2 \alpha_i = 1$
  optimum for $\alpha_1 = 0$, $\alpha_{\geq 2} = \pi/2$ and $\sigma_{\geq 2} = 0$
- $\Rightarrow$ there exist $\beta_1, \beta_2, \ldots$ such that $B_{ij} = \beta_i - \beta_j$
- $\Rightarrow$ assign $p_v \in [0, 1/2]$ to each vertex v
  orient from v to w with probability $1/2 + (p_v - p_w)$
- conjectured construction: $p_v \in \{0, 1/2\}$



Maximum density of cycles

- work in progress with Grzesik, Lovász Jr. and Volec
- What is maximum density of cycles of length k?

  $k \equiv 1 \mod 4 \Leftrightarrow$ regular tournament

  $k \equiv 2 \mod 4 \Leftrightarrow$ quasirandom tournament

  $k \equiv 3 \mod 4 \Leftrightarrow$ regular tournament

  $k \equiv 4 \mod 4 \Leftrightarrow$????
- "cyclic" tournament for $k = 4$ and $k = 8$

QUASIRANDOM TOURNAMENTS

- When does a tournament look random?

  random tournament = orient each edge randomly

- When does a graph look random?

- Thomason, and Chung, Graham and Wilson (1980's)

  density of K_2 is p, density of C_4 is p^4

  equivalent subgraph density conditions

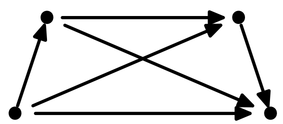
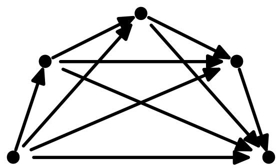
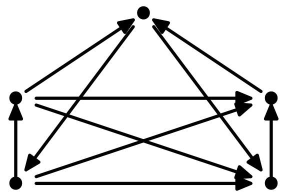
  equivalent uniform density conditions

  equivalent spectral conditions

  ...

QUASIRANDOM TOURNAMENTS

- When does a tournament look random?
- Coregliano, Razborov (2017)
  density of T_4 is $4!/2^6$(unique minimizer)
  density of T_k is $k!/2^{\binom{k}{2}}$ for $k \geq 4$
- Other tournaments forcing quasirandom?
  Coregliano, Parente, Sato (2019)
  unique maximizer of a 5-vertex tournament

Thank you for your attention!