Completion and deficiency problems

Adam Zsolt Wagner

ETH Zürich

Joint work with Rajko Nenadov and Benny Sudakov

Steiner triple sytems

Steiner triple system:

- a family of 3-element subsets of X (called blocks),
- every pair of distinct elements is contained in **precisely** one block.

Steiner triple system:

- a family of 3-element subsets of X (called blocks),
- every pair of distinct elements is contained in **precisely** one block.

Theorem (Kirkman, 1847)

A STS of order n exists if and only if $n \equiv 1$ or $3 \pmod 6$.

Steiner triple sytems

Steiner triple system:

- a family of 3-element subsets of X (called blocks),
- every pair of distinct elements is contained in **precisely** one block.

Theorem (Kirkman, 1847)

A STS of order n exists if and only if $n \equiv 1$ *or* $3 \pmod 6$.

Partial Steiner triple system:

- a family of 3-element subsets of X,
- every pair of distinct elements is contained in **at most** one block.

Embeddings of STSs

A STS $\mathcal{F}$ on set X is **embedded** in a STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

Embeddings of STSs

A STS $\mathcal{F}$ on set X is **embedded** in a STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

Not every partial STS of order n can be embedded in a complete STS of the same order.

Embeddings of STSs

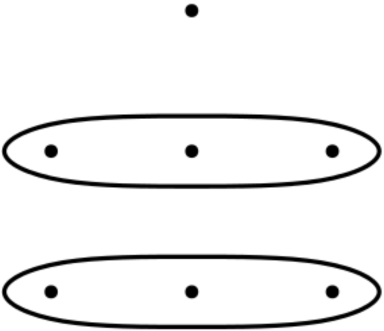
A STS $\mathcal{F}$ on set X is **embedded** in a STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

Not every partial STS of order n can be embedded in a complete STS of the same order. Not even if $n \equiv 1$ or $3 \pmod 6$.

Embeddings of STSs

A STS $\mathcal{F}$ on set X is **embedded** in a STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

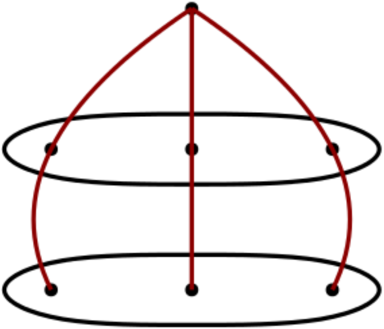
Not every partial STS of order n can be embedded in a complete STS of the same order. Not even if $n \equiv 1$ or $3 \pmod 6$.



Embeddings of STSs

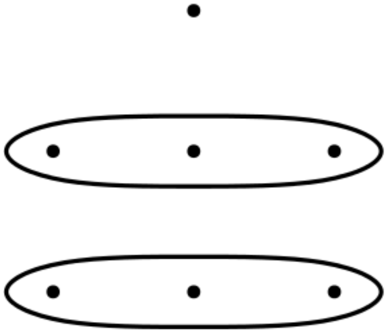
A STS $\mathcal{F}$ on set X is **embedded** in a STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

Not every partial STS of order n can be embedded in a complete STS of the same order. Not even if $n \equiv 1$ or $3 \pmod 6$.



A STS $\mathcal{F}$ on set X is **embedded** in a STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

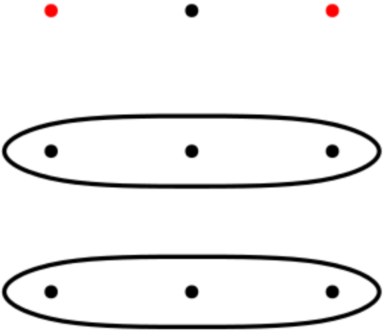
Not every partial STS of order n can be embedded in a complete STS of the same order. Not even if $n \equiv 1$ or $3 \pmod 6$.



Embeddings of STSs

A STS $\mathcal{F}$ on set X is **embedded** in a STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

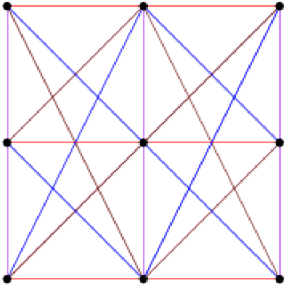
Not every partial STS of order n can be embedded in a complete STS of the same order. Not even if $n \equiv 1$ or $3 \pmod 6$.



Embeddings of STSs

A (partial or complete) STS $\mathcal{F}$ on set X is **embedded** in a (partial or complete) STS $\mathcal{F}'$ on set X' if $\mathcal{F} \subset \mathcal{F}'$ and $X \subset X'$.

Not every partial STS of order n can be embedded in a complete STS of the same order. Not even if $n \equiv 1$ or $3 \pmod 6$.



Small embeddings

Theorem (Treash, 1968)

Every partial STS has a finite completion.

Small embeddings

Theorem (Treash, 1968)

Every partial STS has a finite completion.

Conjecture (Lindner, 1975)

Every partial STS of order n has a completion of order n' for all $n' \geq 2n+1$.

Small embeddings

Theorem (Treash, 1968)

Every partial STS has a finite completion.

Conjecture (Lindner, 1975)

Every partial STS of order n has a completion of order n' for all $n' \geq 2n + 1$.

- Lindner (1975): $6n + 3$ is enough.

Small embeddings

Theorem (Treash, 1968)

Every partial STS has a finite completion.

Conjecture (Lindner, 1975)

Every partial STS of order n has a completion of order n' for all $n' \geq 2n+1$.

- Lindner (1975): $6n+3$ is enough.
- Andersen, Hilton, Mendelsohn (1978): $4n$ is enough.

Small embeddings

Theorem (Treash, 1968)

Every partial STS has a finite completion.

Conjecture (Lindner, 1975)

Every partial STS of order n has a completion of order n' for all $n' \geq 2n + 1$.

- Lindner (1975): $6n + 3$ is enough.
- Andersen, Hilton, Mendelsohn (1978): $4n$ is enough.
- Hilton, Rodger (1987): $2n + 1$ is enough for STS of index $4k$.

Small embeddings

Theorem (Treash, 1968)

Every partial STS has a finite completion.

Conjecture (Lindner, 1975)

Every partial STS of order n has a completion of order n' for all $n' \geq 2n+1$.

- Lindner (1975): $6n+3$ is enough.
- Andersen, Hilton, Mendelsohn (1978): $4n$ is enough.
- Hilton, Rodger (1987): $2n+1$ is enough for STS of index $4k$.
- Bryant (2004): $3n-2$ is enough.

Small embeddings

Theorem (Treash, 1968)

Every partial STS has a finite completion.

Conjecture (Lindner, 1975)

Every partial STS of order n has a completion of order n' for all $n' \geq 2n+1$.

- Lindner (1975): $6n+3$ is enough.
- Andersen, Hilton, Mendelsohn (1978): $4n$ is enough.
- Hilton, Rodger (1987): $2n+1$ is enough for STS of index $4k$.
- Bryant (2004): $3n-2$ is enough.
- Bryant, Horsley (2009): $2n+1$ is enough.

When is $n' < 2n + 1$ enough?

- Colbourn ('83): it is NP-complete to determine whether a partial STS has an embedding of order less than $2n + 1$.

When is $n' < 2n + 1$ enough?

- Colbourn ('83): it is NP-complete to determine whether a partial STS has an embedding of order less than $2n + 1$.
- Colbourn–Colbourn–Rosa ('83), Bryant ('02), Bryant–Maenhaut–Quinn–Webb ('04), Horsley ('04)

When is $n' < 2n + 1$ enough?

- Colbourn ('83): it is NP-complete to determine whether a partial STS has an embedding of order less than $2n + 1$.
- Colbourn–Colbourn–Rosa ('83), Bryant ('02), Bryant–Maenhaut–Quinn–Webb ('04), Horsley ('04)
- Horsley ('04): if a partial STS has at most $n^2/50$ blocks then it has a completion of order $8n/5$.

When is $n' < 2n + 1$ enough?

- Colbourn ('83): it is NP-complete to determine whether a partial STS has an embedding of order less than $2n + 1$.
- Colbourn–Colbourn–Rosa ('83), Bryant ('02), Bryant–Maenhaut–Quinn–Webb ('04), Horsley ('04)
- Horsley ('04): if a partial STS has at most $n^2/50$ blocks then it has a completion of order $8n/5$.

Question

Given that a partial STS has r blocks, how many extra vertices do we need to add to create a completion?

Small embeddings

Question

Given that a partial STS has r blocks, how many extra vertices do we need to add to create a completion?

Small embeddings

Question

Given that a partial STS has r blocks, how many extra vertices do we need to add to create a completion?

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

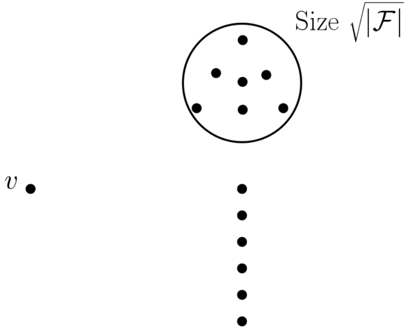
Sharp if $|\mathcal{F}| \geq 2n$:

Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

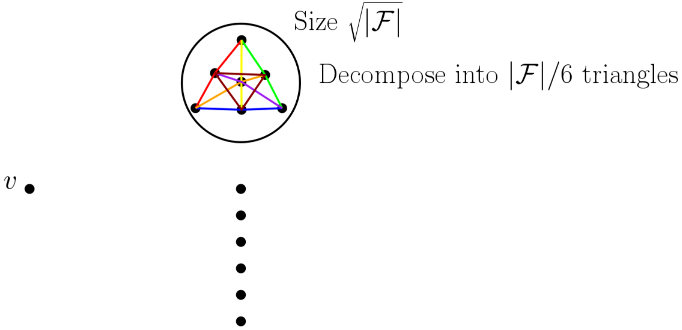
Sharp if $|\mathcal{F}| \geq 2n$:



Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharp if $|\mathcal{F}| \geq 2n$:

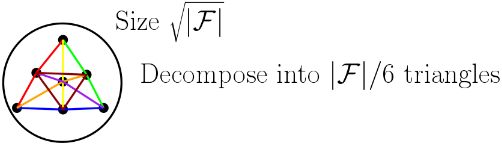


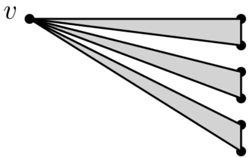
Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharp if $|\mathcal{F}| \geq 2n$:



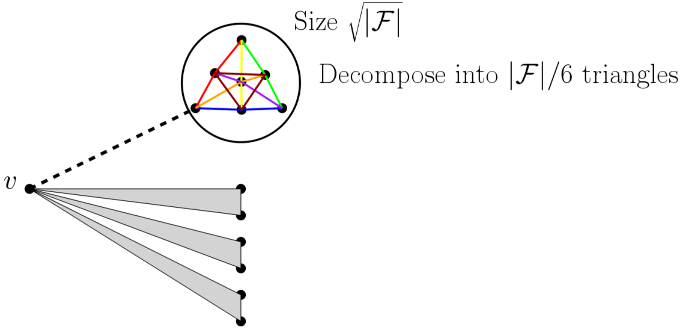


Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharp if $|\mathcal{F}| \geq 2n$:

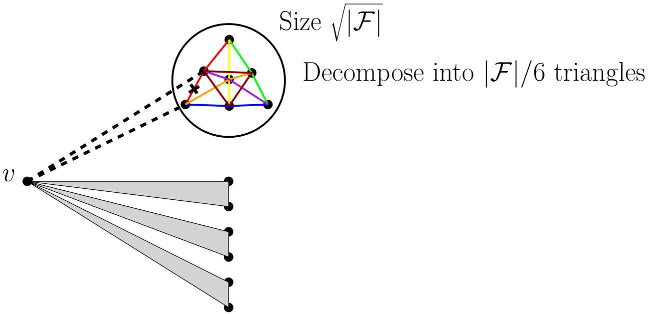


Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharp if $|\mathcal{F}| \geq 2n$:

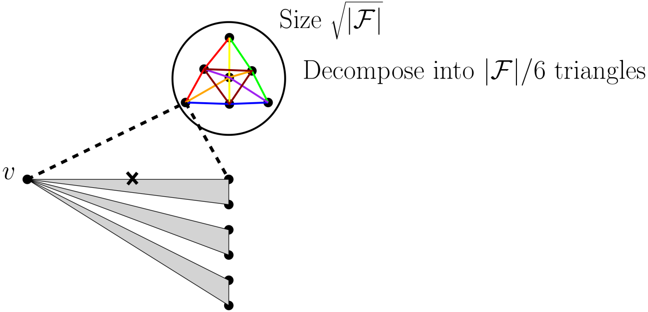


Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharp if $|\mathcal{F}| \geq 2n$:

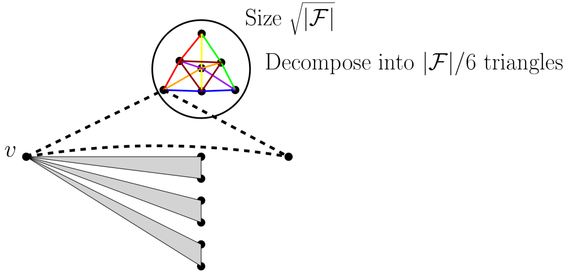


Sharpness of main theorem

Theorem (Nenadov–Sudakov–W)

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Sharp if $|\mathcal{F}| \geq 2n$:



Completing (n, k)-designs

(n, k)-**design:**

- a family of k-element subsets of $[n]$,
- every pair of distinct elements is contained in precisely one block.

(n, k)-**design:**

- a family of k-element subsets of $[n]$,
- every pair of distinct elements is contained in precisely one block.

Question

Given that a partial (n, k)-design has r blocks, how many extra vertices do we need to add to create a completion?

Completing (n,k)-designs

Question

Given that a partial (n,k)-design has r blocks, how many extra vertices do we need to add to create a completion?

Completing (n, k)-designs

Question

Given that a partial (n, k)-design has r blocks, how many extra vertices do we need to add to create a completion?

Theorem (Nenadov–Sudakov–W)

For every $k \geq 3$ there exist absolute constants $\epsilon, n_0 > 0$ such that the following holds. If $\mathcal{F}$ is a partial (n, k)-design of order $n \geq n_0$ with $|\mathcal{F}| \leq \epsilon n^2$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + 7k^2\sqrt{|\mathcal{F}|}$.

Latin squares

Latin square: every element of $[n]$ appears **exactly** once in each row, column.

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

Figure: Leonhard Euler 1707-1783

Latin squares

Partial Latin square: every element of $[n]$ appears **at most** once in each row, column.

Latin squares

Partial Latin square: every element of $[n]$ appears **at most** once in each row, column.
Not every partial Latin square can be completed to a full Latin square:

1	
	2

Latin squares

Partial Latin square: every element of $[n]$ appears **at most** once in each row, column.

Not every partial Latin square can be completed to a full Latin square:

1	
	2

But by adding some rows, columns sometimes they can!

1	3	2
3	**2**	1
2	1	3

Latin squares

Partial Latin square: every element of $[n]$ appears **at most** once in each row, column.
Not every partial Latin square can be completed to a full Latin square:

1	
	2

But by adding some rows, columns sometimes they can!

1	3	2
3	**2**	1
2	1	3

Can one always complete a partial Latin square by adding rows/columns?

Completing partial Latin squares, history

- Marshall, Hall (1945), Ryser(1951): If an r by n rectangle is filled up completely, it can be extended to an n by n Latin square.

Completing partial Latin squares, history

- Marshall, Hall (1945), Ryser(1951): If an r by n rectangle is filled up completely, it can be extended to an n by n Latin square.
- Evans (1960), Lindner (1970), Hilton (1971), Cruse (1973): An n by n partial Latin square can be embedded in a $2n$ by $2n$ Latin square, this is sharp.

Completing partial Latin squares, history

- Marshall, Hall (1945), Ryser(1951): If an r by n rectangle is filled up completely, it can be extended to an n by n Latin square.
- Evans (1960), Lindner (1970), Hilton (1971), Cruse (1973): An n by n partial Latin square can be embedded in a $2n$ by $2n$ Latin square, this is sharp.
- Colbourn (1982): Completing partial Latin squares without adding rows/columns is NP-complete

Completing partial Latin squares, history

- Marshall, Hall (1945), Ryser(1951): If an r by n rectangle is filled up completely, it can be extended to an n by n Latin square.
- Evans (1960), Lindner (1970), Hilton (1971), Cruse (1973): An n by n partial Latin square can be embedded in a $2n$ by $2n$ Latin square, this is sharp.
- Colbourn (1982): Completing partial Latin squares without adding rows/columns is NP-complete

Question

Can we improve the $2n$ in some cases?

When less than $2n$ is enough

Daykin, Häggkvist (1983) conjecture: if each row, column, symbol is used at most $n/4$ times then it can be completed without adding rows/columns.
Chetwynd and Häggkvist (1985), Gustavsson (1991), Bartlett (2014), Barber, Kühn, Lo, Osthus, Taylor (2017): true if $n/4$ replaced by $n/25$.

When less than $2n$ is enough

Daykin, Häggkvist (1983) conjecture: if each row, column, symbol is used at most $n/4$ times then it can be completed without adding rows/columns.
Chetwynd and Häggkvist (1985), Gustavsson (1991), Bartlett (2014), Barber, Kühn, Lo, Osthus, Taylor (2017): true if $n/4$ replaced by $n/25$.

Question

Given the number r of entries, how many rows/columns do we need to add?

When less than $2n$ is enough

Daykin, Häggkvist (1983) conjecture: if each row, column, symbol is used at most $n/4$ times then it can be completed without adding rows/columns. Chetwynd and Häggkvist (1985), Gustavsson (1991), Bartlett (2014), Barber, Kühn, Lo, Osthus, Taylor (2017): true if $n/4$ replaced by $n/25$.

Question

Given the number r of entries, how many rows/columns do we need to add?

Theorem (Nenadov–Sudakov–W)

If L is a partial Latin square of order $n \geq n_0$ with $|L|$ entries, then L has an embedding of order $n + O(\sqrt{|L|})$.

This is sharp up to constant. Similar results for completion of a sequence of orthogonal Latin squares.

Proof

Theorem

If $\mathcal{F}$ is a partial Steiner triple system of order $n \geq n_0$ with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Proof

Theorem

If $\mathcal{F}$ is a partial Steiner triple system of order $n \geq n_0$ with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Equivalently: G is obtained from K_n by removing $|\mathcal{F}|$ triangles. Want to show that by adding $\sqrt{|\mathcal{F}|}$ full degree vertices to G, the graph G' has a K_3-decomposition.

Proof

Theorem

If $\mathcal{F}$ is a partial Steiner triple system of order $n \geq n_0$ with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Equivalently: G is obtained from K_n by removing $|\mathcal{F}|$ triangles. Want to show that by adding $\sqrt{|\mathcal{F}|}$ full degree vertices to G, the graph G' has a K_3-decomposition.

Goal: apply Gustavsson's theorem. If minimum degree at least $(1-\gamma)|V(G')|$ then G' has K_3-decomposition.

Proof

Theorem

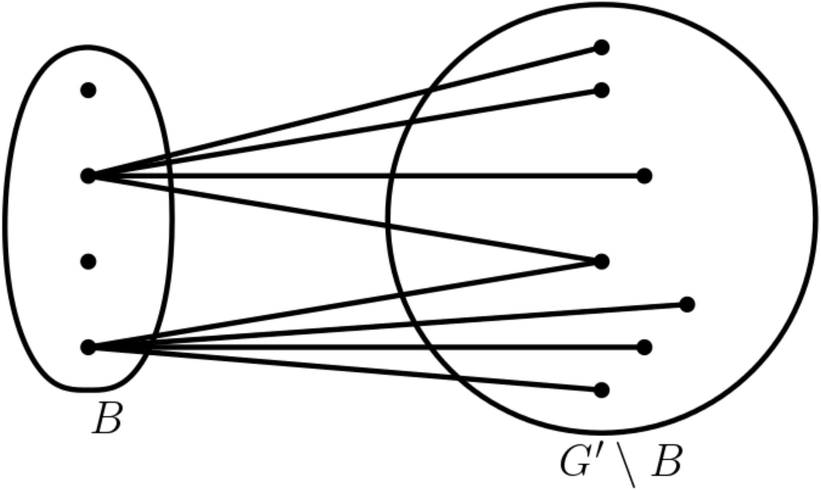
If $\mathcal{F}$ is a partial Steiner triple system of order $n \geq n_0$ with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Equivalently: G is obtained from K_n by removing $|\mathcal{F}|$ triangles. Want to show that by adding $\sqrt{|\mathcal{F}|}$ full degree vertices to G, the graph G' has a K_3-decomposition.

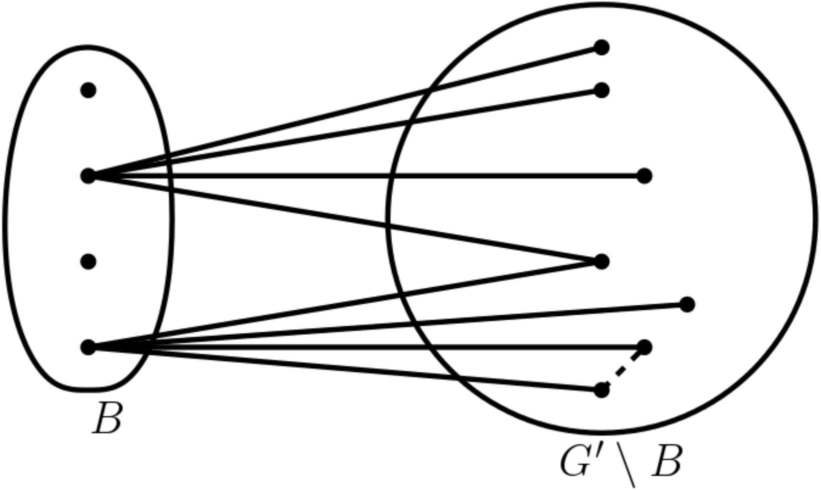
Goal: apply Gustavsson's theorem. If minimum degree at least $(1-\gamma)|V(G')|$ then G' has K_3-decomposition.

Let B be the set of small degree (less than $n - \sqrt{|\mathcal{F}|}$) vertices in G. Cover all edges incident to B with triangles so that rest of the graph has high minimum degree.

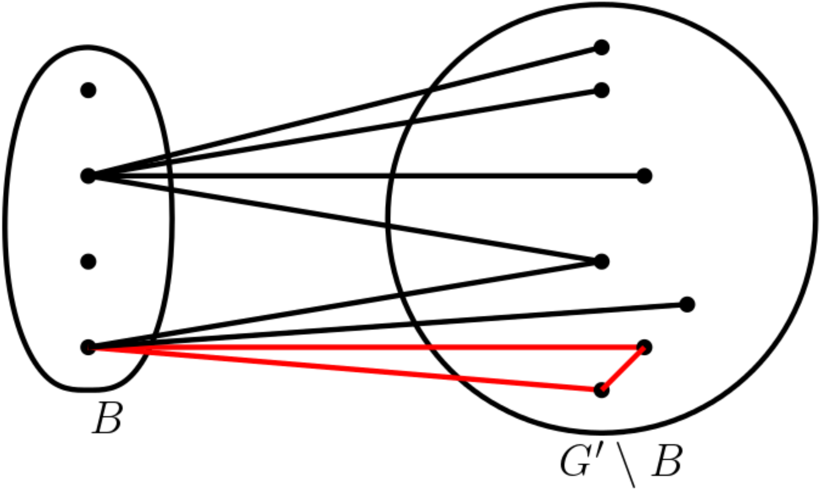
Covering small degree vertices



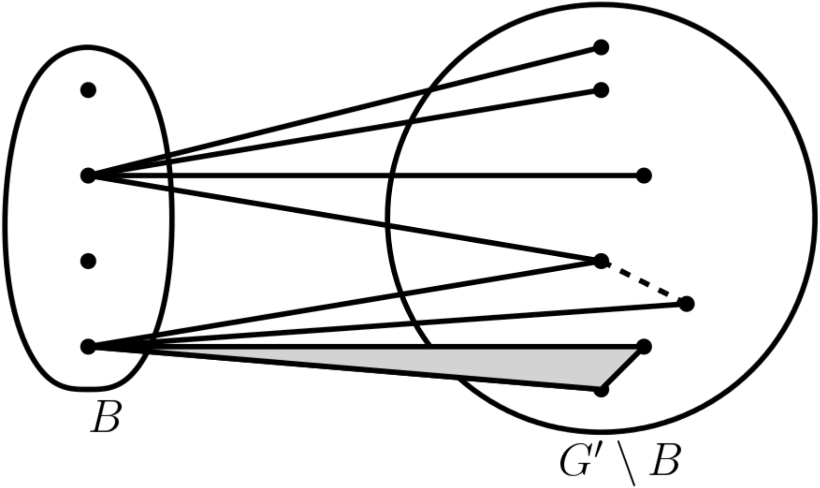
Covering small degree vertices



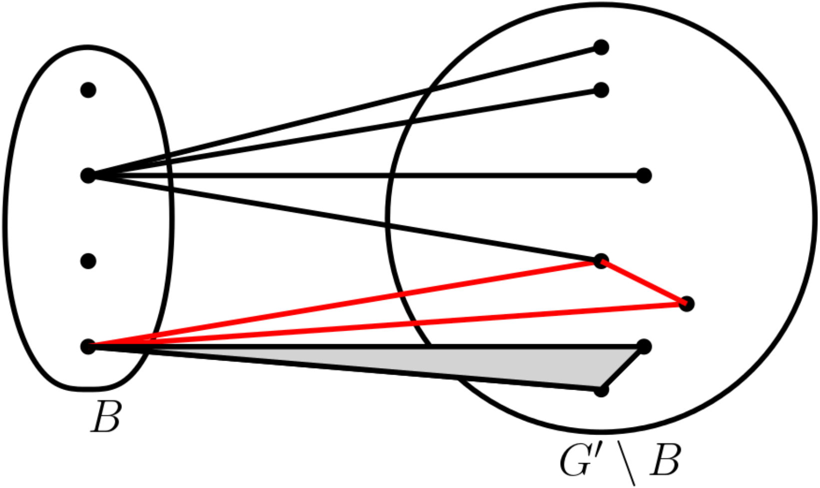
Covering small degree vertices



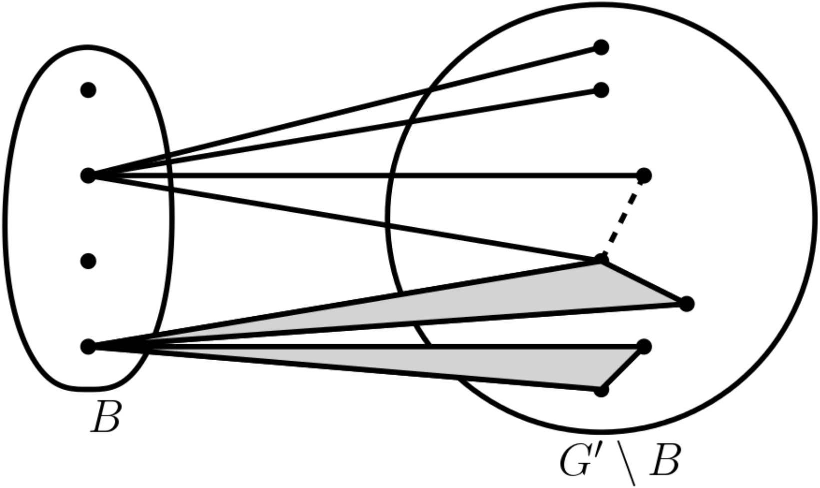
Covering small degree vertices



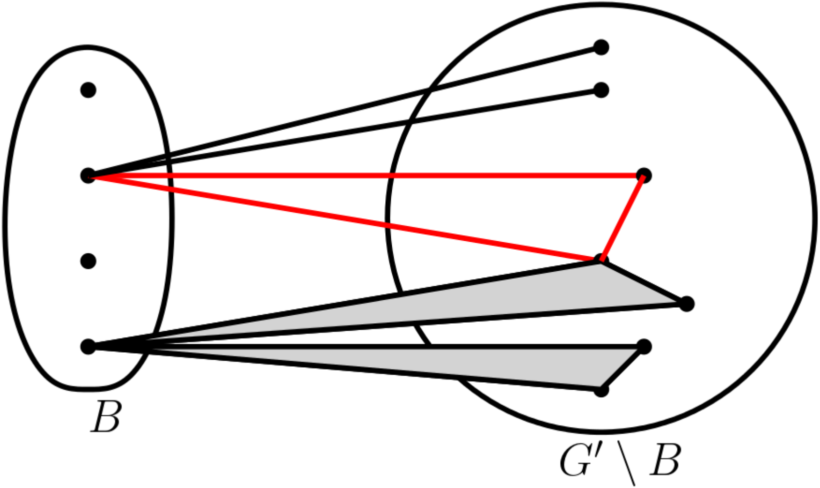
Covering small degree vertices



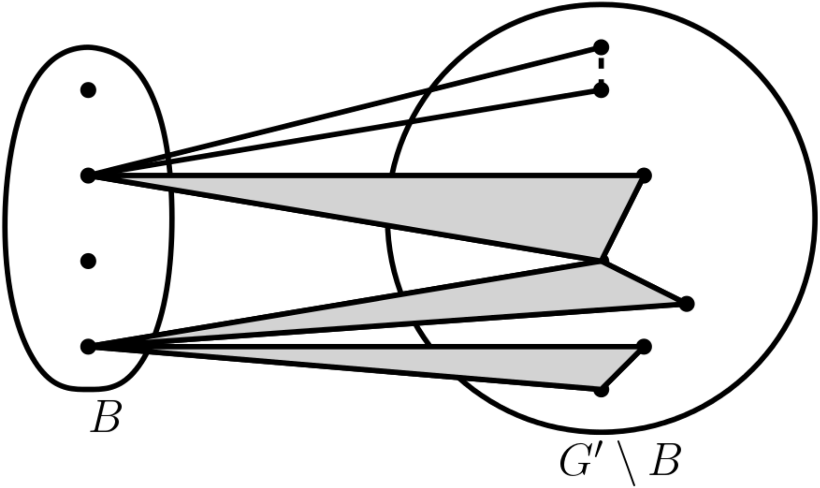
Covering small degree vertices



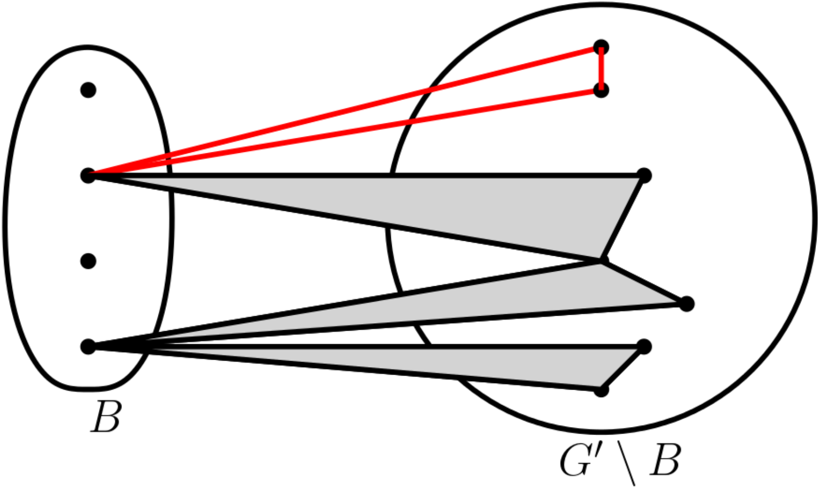
Covering small degree vertices



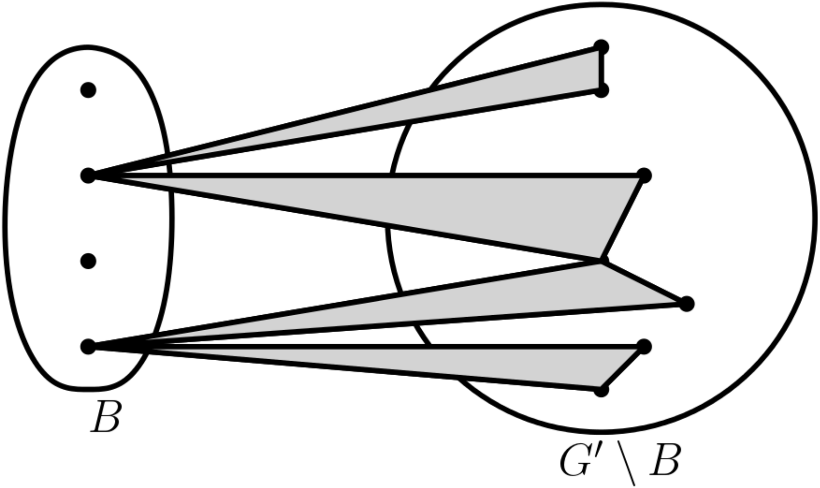
Covering small degree vertices



Covering small degree vertices



Covering small degree vertices



Proof

Lemma

let G be a graph on vertex set $S \dot\cup T$ with $|T| \geq 50\sqrt{r}$, such that the degree of every vertex satisfies $d(v) \geq |V(G)| - \sqrt{r}$. Then no matter how one removes at most r edges from $G[S]$, the resulting graph has a perfect matching.

Proof

Lemma

let G be a graph on vertex set $S \dot\cup T$ with $|T| \geq 50\sqrt{r}$, such that the degree of every vertex satisfies $d(v) \geq |V(G)| - \sqrt{r}$. Then no matter how one removes at most r edges from $G[S]$, the resulting graph has a perfect matching.

Would like to use Dirac's theorem: if $\delta(G) \geq \frac{1}{2}|V(G)|$ then have perfect matching. But vertices in S may have small degree.

Proof

Lemma

let G be a graph on vertex set $S\dot\cup T$ with $|T| \geq 50\sqrt{r}$, such that the degree of every vertex satisfies $d(v) \geq |V(G)| - \sqrt{r}$. Then no matter how one removes at most r edges from $G[S]$, the resulting graph has a perfect matching.

Would like to use Dirac's theorem: if $\delta(G) \geq \frac{1}{2}|V(G)|$ then have perfect matching. But vertices in S may have small degree.

Deal with low degree ($\leq |V(G)| - 2\sqrt{r}$) vertices in S one by one. There are only $2\sqrt{r}$ such vertices, for each we find a vertex in their neighbourhood in T.

Proof

Lemma

let G be a graph on vertex set $S\dot{\cup}T$ with $|T| \geq 50\sqrt{r}$, such that the degree of every vertex satisfies $d(v) \geq |V(G)| - \sqrt{r}$. Then no matter how one removes at most r edges from $G[S]$, the resulting graph has a perfect matching.

Would like to use Dirac's theorem: if $\delta(G) \geq \frac{1}{2}|V(G)|$ then have perfect matching. But vertices in S may have small degree.

Deal with low degree ($\leq |V(G)| - 2\sqrt{r}$) vertices in S one by one. There are only $2\sqrt{r}$ such vertices, for each we find a vertex in their neighbourhood in T.

Remaining graph has minimum degree at least $|V(G)| - 3\sqrt{r} \geq \frac{1}{2}|V(G)|$.

A general class of completion problems

Our results suggest the following new class of extremal problems:

A general class of completion problems

Our results suggest the following new class of extremal problems:

- Given a global, spanning property $\mathcal{P}$ (e.g. Hamiltonicity)

A general class of completion problems

Our results suggest the following new class of extremal problems:

- Given a global, spanning property $\mathcal{P}$ (e.g. Hamiltonicity)
- Typically we can destroy $\mathcal{P}$ by deleting very few edges from K_n, e.g. by isolating a vertex. So Turán problem not so interesting.

A general class of completion problems

Our results suggest the following new class of extremal problems:

- Given a global, spanning property $\mathcal{P}$ (e.g. Hamiltonicity)
- Typically we can destroy $\mathcal{P}$ by deleting very few edges from K_n, e.g. by isolating a vertex. So Turán problem not so interesting.
- Usual solution: add minimum degree condition to avoid these issues.

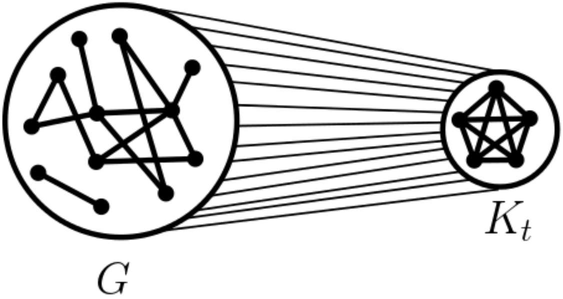
A general class of completion problems

Our results suggest the following new class of extremal problems:

- Given a global, spanning property $\mathcal{P}$ (e.g. Hamiltonicity)
- Typically we can destroy $\mathcal{P}$ by deleting very few edges from K_n, e.g. by isolating a vertex. So Turán problem not so interesting.
- Usual solution: add minimum degree condition to avoid these issues.
- We look at this problem differently.

The join $K * K_t$

$G * K_t =$



Rephrasing our results

If $\mathcal{F}$ is a partial Steiner triple system of order n with $|\mathcal{F}|$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + O(\sqrt{|\mathcal{F}|})$.

Theorem (Rephrased)

If one removes up to r edge-disjoint copies of K_3 from K_n to obtain a graph G, then there exists some t with $t \leq C\sqrt{r}$ so that $G * K_t$ has a K_3-decomposition.

Rephrasing our results

If $\mathcal{F}$ is a partial (n, k)-design with $|\mathcal{F}| \leq \epsilon n^2$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + 7k^2\sqrt{|\mathcal{F}|}$.

Theorem (Rephrased)

If one removes up to $r \leq \varepsilon n^2$ edge-disjoint copies of K_k from K_n to obtain a graph G, then there exists some t with $t \leq 7k^2\sqrt{r}$ so that $G * K_t$ has a K_k-decomposition.

Rephrasing our results

If $\mathcal{F}$ is a partial (n, k)-design with $|\mathcal{F}| \leq \epsilon n^2$ blocks, then there exists an embedding of $\mathcal{F}$ of order at most $n + 7k^2\sqrt{|\mathcal{F}|}$.

Theorem (Rephrased)

If one removes up to $r \leq \varepsilon n^2$ edge-disjoint copies of K_k from K_n to obtain a graph G, then there exists some t with $t \leq 7k^2\sqrt{r}$ so that $G * K_t$ has a K_k-decomposition.

Latin squares results: multipartite analogues of these.

The notion of deficiency

Definition

Given a property $\mathcal{P}$ and graph G, the **deficiency** of G with respect to $\mathcal{P}$ is the smallest t such that $G * K_t$ has property $\mathcal{P}$.

The notion of deficiency

Definition

Given a property $\mathcal{P}$ and graph G, the **deficiency** of G with respect to $\mathcal{P}$ is the smallest t such that $G * K_t$ has property $\mathcal{P}$.

- Previous results: $\mathcal{P}$ is K_k-decomposition

The notion of deficiency

Definition

Given a property $\mathcal{P}$ and graph G, the **deficiency** of G with respect to $\mathcal{P}$ is the smallest t such that $G * K_t$ has property $\mathcal{P}$.

- Previous results: $\mathcal{P}$ is K_k-decomposition
- Concept of deficiency is not completely new: Tutte–Berge formula.

The notion of deficiency

Definition

Given a property $\mathcal{P}$ and graph G, the **deficiency** of G with respect to $\mathcal{P}$ is the smallest t such that $G * K_t$ has property $\mathcal{P}$.

- Previous results: $\mathcal{P}$ is K_k-decomposition
- Concept of deficiency is not completely new: Tutte–Berge formula.
- We propose a systematic study of these problems

Examples: Hamiltonicity

$\mathcal{P}$ = **Hamiltonicity:** $\operatorname{def}(G,\mathcal{P})$ is smallest t such that $G * K_t$ is Hamiltonian

Examples: Hamiltonicity

$\mathcal{P}$ = **Hamiltonicity:** def$(G, \mathcal{P})$ is smallest t such that $G * K_t$ is Hamiltonian

Question

Given that $G * K_t$ does not have a Hamiltonian cycle, at most how many edges can G have?

Examples: Hamiltonicity

$\mathcal{P}$ = **Hamiltonicity:** $\text{def}(G, \mathcal{P})$ is smallest t such that $G * K_t$ is Hamiltonian

Question

Given that $G * K_t$ does not have a Hamiltonian cycle, at most how many edges can G have?

Path cover number $\mu(G)$ is smallest number of vertex-disjoint paths that cover $V(G)$. Note: $\mu(G) = t \iff G * K_t$ is Hamiltonian but $G * K_{t-1}$ is not.

Examples: Hamiltonicity

$\mathcal{P}$ = **Hamiltonicity:** def$(G, \mathcal{P})$ is smallest t such that $G * K_t$ is Hamiltonian

Question

Given that $G * K_t$ does not have a Hamiltonian cycle, at most how many edges can G have?

Path cover number $\mu(G)$ is smallest number of vertex-disjoint paths that cover $V(G)$. Note: $\mu(G) = t \iff G * K_t$ is Hamiltonian but $G * K_{t-1}$ is not.

Question (Rephrased)

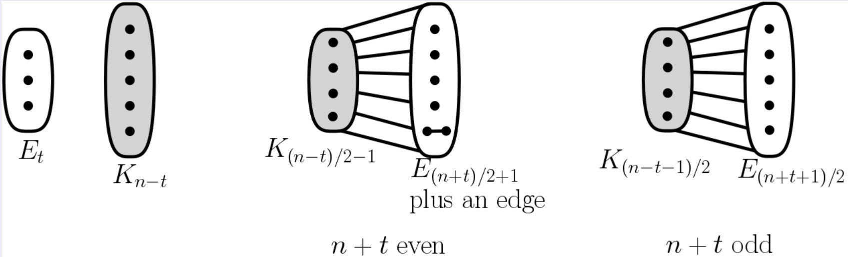
Given $\mu(G)$, how large can $e(G)$ be?

Examples: Hamiltonicity

Question

Given that $G * K_t$ does not have a Hamiltonian cycle, at most how many edges can G have?

Theorem



Prior work by Skupień (1974), we expanded on it.

Examples: K_3-factor

$\mathcal{P}$ = **existence of a K_3-factor:** $\text{def}(G, \mathcal{P})$ is smallest t such that $G * K_t$ has K_3-factor

Examples: K_3-factor

$\mathcal{P}$ = **existence of a K_3-factor:** $\mathrm{def}(G,\mathcal{P})$ is smallest t such that $G * K_t$ has K_3-factor

Question

Given that $G * K_t$ does not have a K_3-factor (and $3|n+t$), at most how many edges can G have?

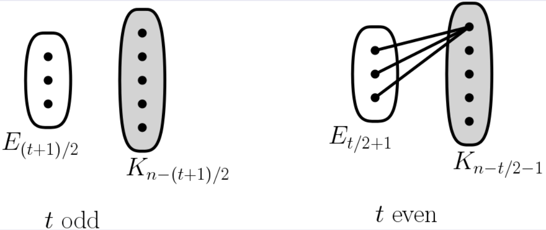
$\mathcal{P}$ = **existence of a K_3-factor:** def$(G, \mathcal{P})$ is smallest t such that $G * K_t$ has K_3-factor

Question

Given that $G * K_t$ does not have a K_3-factor (and $3|n+t$), at most how many edges can G have?

We solve this problem for $t \leq n/1000$.

Theorem (Nenadov–Sudakov–W)



Examples: perfect matching in hypergraphs

$\mathcal{P}$ = **existence of a perfect matching:** let $\mathcal{H}$ be a k-uniform hypergraph, then $\mathrm{def}(\mathcal{H}, \mathcal{P})$ is smallest t such that $\mathcal{H} * K_t$ has perfect matching

Examples: perfect matching in hypergraphs

$\mathcal{P}$ = **existence of a perfect matching:** let $\mathcal{H}$ be a k-uniform hypergraph, then $\mathrm{def}(\mathcal{H}, \mathcal{P})$ is smallest t such that $\mathcal{H} * K_t$ has perfect matching

Question

Given that $\mathcal{H} * K_t$ does not have a perfect matching, at most how many edges can $\mathcal{H}$ have?

Examples: perfect matching in hypergraphs

$\mathcal{P}$ = **existence of a perfect matching:** let $\mathcal{H}$ be a k-uniform hypergraph, then $\text{def}(\mathcal{H}, \mathcal{P})$ is smallest t such that $\mathcal{H} * K_t$ has perfect matching

Question

Given that $\mathcal{H} * K_t$ does not have a perfect matching, at most how many edges can $\mathcal{H}$ have?

This is equivalent to Erdős matching problem!

General problem

Pick a global property $\mathcal{P}$.

Question

Given that $G * K_t$ does not have $\mathcal{P}$, how many edges can G have?

E.g.

- K_k-decomposition,
- containing Hamilton cycle,
- containing power of Hamilton cycle,
- K_k-factor, etc.