# Covering cubes by hyperplanes

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Joint work with Alexander Clifton (Emory).

# My collaborator

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# A naive question

The $n$-dimensional cube $Q^n$ consists of the binary vectors $\{0, 1\}^n$.

An affine hyperplane is:

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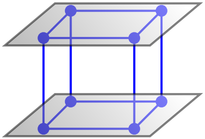
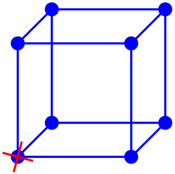


# The Alon-Füredi Theorem

**A new question**

Suppose we would like to avoid exactly one vertex of the cube, how many affine hyperplanes are needed?

For $Q^3$, 3 planes are needed.



**Theorem (Alon, Füredi 1993)**

Any set of affine hyperplanes that covers all the vertices of the $n$-cube $Q^n$ but one contains at least $n$ affine hyperplanes.

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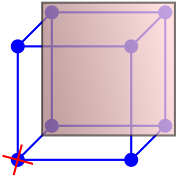

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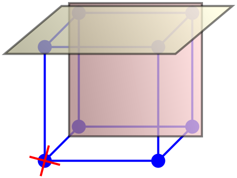

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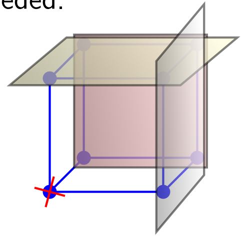

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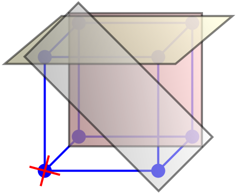

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# An outline of the proof of Alon-Füredi Theorem

**Proof.** Suppose $H_1, \cdots, H_m$ cover all the vertices of $Q^n$ but $(0, \cdots, 0)$, and $H_i$ can be parameterized by

$$\langle \vec{x}, \vec{a}_i \rangle = b_i.$$

Then all $b_i \neq 0$. Let

$$p(\vec{x}) = \prod_{i=1}^{m} (\langle \vec{x}, \vec{a}_i \rangle - b_i).$$

We have $p(\vec{x}) = 0$ at every $\vec{x} \neq \vec{0}$, and non-zero at $\vec{0}$. Polynomials satisfying such property cannot have degree lower than $n$, so

$$m \geq n.$$

# Covering the cube twice

**Question (Bukh's homework assignment at CMU)**

What happens if we would like to cover the vertices of $Q^n$ at least twice, with one vertex uncovered?

Clearly, $2n$ is possible, by taking two copies of $x_i = 1$ for every $i$. But can we do better?

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Such cover is best possible, since removing one affine hyperplane still gives a cover of $Q^n - \{\vec{0}\}$.

# Covering the cube $k$ times

Denote by $f(n,k)$ the minimum number of affine hyperplanes needed to cover every vertex of $Q^n$ at least $k$ times (except for $\vec{0}$ which is not covered at all).

We call such a cover an almost $k$-cover of the $n$-cube.

$$f(n,1) = n.$$

$$f(n,2) = n+1.$$

What is the next?

# Upper and lower bounds

$$f(n,k) \le n + \binom{k}{2}$$

Take

$$x_1 = 1, \cdots, x_n = 1,$$

$$x_1 + \cdots + x_n = 1 \quad \text{for } k-1 \text{ times},$$

$$\vdots$$

$$x_1 + \cdots + x_n = k-1 \quad \text{for 1 time.}$$

$$f(n,k) \ge n + k - 1$$

Note that removing $k-1$ planes from an almost $k$-cover still gives an almost 1-cover.

$$n + 2 \le f(n,3) \le n + 3.$$

# The $k = 3$ case and a natural conjecture

**Theorem (H., Clifton 2019)**

For $n \geq 2$,

$$f(n,3) = n + 3.$$

For $n \geq 3$,

$$f(n,4) \in \{n+5, n+6\}.$$

**Conjecture (H., Clifton 2019)**

For fixed integer $k \geq 1$ and sufficiently large $n$,

$$f(n,k) = n + \binom{k}{2}.$$

### The Nullstellensatz

If $\mathbb{F}$ is an algebraically closed field, and $f, g_1, \cdots, g_m \in \mathbb{F}[x_1, \cdots, x_n]$, where $f$ vanishes over all common zeros of $g_1, \cdots, g_m$, then there exists an integer $k$, and polynomials $h_1, \cdots, h_m \in \mathbb{F}[x_1, \cdots, x_n]$, such that

$$f^k = \sum_{i=1}^{m} h_i g_i.$$

When $m = n$, and $g_i = \prod_{s \in S_i} (x_i - s)$, for some $S_1, \cdots, S_n \subset \mathbb{F}$, a stronger result holds: there are polynomials $h_1, \cdots, h_n$ with $\deg h_i \le \deg f - \deg g_i$, such that

$$f = \sum_{i=1}^{n} h_i g_i.$$

# Punctured Combinatorial Nullstellensatz

We say $\vec{a} = (a_1, \cdots, a_n)$ is a zero of multiplicity $t$ of $f \in \mathbb{F}[x_1, \cdots, x_n]$, if $t$ is the minimum degree of the terms in $f(x_1 + a_1, \cdots, x_n + a_n)$.

For $i = 1, \cdots, n$, let

$$D_i \subset S_i \subset \mathbb{F}. \quad g_i = \prod_{s \in S_i}(x_i - s). \quad \ell_i = \prod_{d \in D_i}(x_i - d).$$

**Theorem (Ball, Serra 2009)**

If $f$ has a zero of multiplicity at least $t$ at all the common zeros of $g_1, \cdots, g_n$, except at least one point of $D_1 \times \cdots \times D_n$ where it has a zero of multiplicity less than $t$, then there are polynomials $h_\tau$ satisfying $\deg(h_\tau) \le \deg(f) - \sum_{i \in \tau} \deg(g_i)$, and a non-zero polynomial $u$ satisfying $\deg(u) \le \deg(f) - \sum_{i=1}^n (\deg(g_i) - \deg(\ell_i))$, such that

$$f = \sum_{\tau \in T(n,t)} g_{\tau(1)} \cdots g_{\tau(t)} h_\tau + u \prod_{i=1}^n \frac{g_i}{\ell_i}.$$

$T(n,t)$ consists of all non-decreasing sequences of length $t$ on $[n]$.

# Outline of our proof using the PCN ($k = 3$)

We prove by contradiction. Suppose $H_1, \cdots, H_{n+2}$ form an almost 3-cover of $Q^n$, and the equation of $H_i$ is $\langle \vec{x}, \vec{a}_i \rangle = 1$. Let $P_i(\vec{x}) = \langle \vec{x}, \vec{a}_i \rangle - 1$, and

$$f = P_1 \cdots P_{n+2}.$$

The Punctured Combinatorial Nullstellensatz gives

$$f = \sum_{1 \le i \le j \le k \le n} x_i(x_i - 1)x_j(x_j - 1)x_k(x_k - 1)h_{ijk} + u \prod_{i=1}^{n} (x_i - 1),$$

with $\deg(u) \le \deg(f) - n = 2$.

The $j$-th order derivatives of $f$ vanish on $Q^n - \{\vec{0}\}$ for $j = 0, 1, 2$
$\Longrightarrow u(e_i) = 0,\ u(e_i + e_j) = 0,\ \partial u / \partial x_i(e_j) = 0.$
$\Longrightarrow u \equiv 0$, contradicting $f(\vec{0}) \neq 0$.

# $f(n,k)$ for fixed $n$ and large $k$

For small $n$, $f(n,k) \neq n + \binom{k}{2}$. Actually,

**Theorem (H., Clifton 2019)**

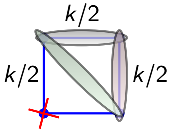
For fixed $n$, and $k$ tends to infinity,

$$f(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + o(1)\right) k.$$

- Upper bound: use every hyperplane

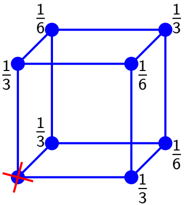
$$x_{i_1} + \cdots + x_{i_j} = 1$$

a total of $\frac{k}{j\binom{n}{j}}$ times. e.g.



# $f(n, k)$ for fixed $n$ and large $k$ (ctd.)

- Lower bound: (e.g. $n = 3$) assign weights to vertices:




Every affine plane covers vertices of total weight at most 1. Therefore one needs at least

$$k \cdot \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3}\right) = \frac{11}{6}k$$

hyperplanes.

For general $n$, assign weight $1/(j\binom{n}{j})$ to vertices whose sum of coordinate is $j$.

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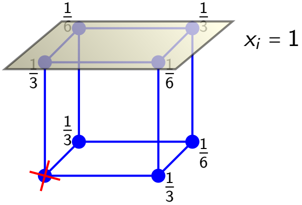


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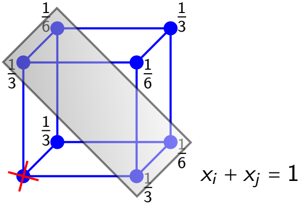


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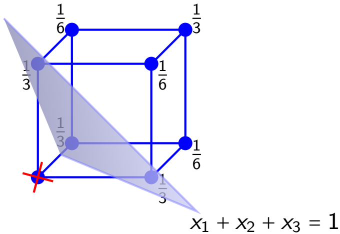


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# An LYM-like inenquality

**The Lubell-Yamato-Meshalkin inequality**

Let $\mathcal{F}$ be a family of subsets in which no set contains another, then

$$\sum_{S\in\mathcal{F}} \frac{1}{\binom{n}{|S|}} \leq 1.$$

**Lemma (H., Clifton 2019)**

Given $n$ real numbers $a_1, \cdots, a_n$, let

$$\mathcal{F} = \{S : \varnothing \neq S \subset [n], \sum_{i\in S} a_i = 1\},$$

then

$$\sum_{S\in\mathcal{F}} \frac{1}{|S|\binom{n}{|S|}} \leq 1.$$

The inequality is tight for all non-zero binary $(a_1, \cdots, a_n)$.

# Proof of the Lemma

We associate every $S \in \mathcal{F}$ (binary vector covered by the plane) with some permutations in $\mathcal{P}_S \subset S_n$.

e.g. When $n = 5$, $S = \{1, 3, 4\}$, it means $a_1 + a_3 + a_4 = 1$, take all permutations in $S_5$ with prefix $(i_1, i_2, i_3)$ satisfying

$$\{i_1, i_2, i_3\} = \{1, 3, 4\}, \quad a_{i_1} < 1, \quad a_{i_1} + a_{i_2} < 1.$$

We can show:

- $\mathcal{P}_S$ are pairwise disjoint.
- $|\mathcal{P}_S| \geq (|S| - 1)!(n - |S|)!$ (the proof uses the *lorry driver puzzle*.)
- Therefore

$$n! \geq \sum_{S \in \mathcal{F}} |\mathcal{P}_S| = \sum_{S \in \mathcal{F}} (|S| - 1)!(n - |S|)!,$$

which simplifies to our desired result.

# Future research problems (I)

**Problem 1**

Prove $f(n,k) = n + \binom{k}{2}$ for large $n$.

**Alon (private communication):** for large $n$, if the almost $k$-cover contains $x_1 = 1, \cdots, x_n = 1$, then it contains at least $n + \binom{k}{2}$ affine hyperplanes in total.

**Problem 2**

Let $g(n,m,k)$ be the minimum number of vertices covered less than $k$ times by $m$ affine hyperplanes not passing through $\vec{0}$. Determine $g(n,m,k)$.

**Alon, Füredi 1993:** $g(n,m,1) = 2^{n-m}$.

# Future research problems (II)

**Question:** Is it true that for all $n, m, k$:

$$g(n, m, k) = 2^{n-d},$$

where $d$ is the maximum integer such that $f(d, k) \leq m$?

**PROBLEM 3**

Does there exist an absolute constant $C > 0$, which does not depend on $n$, such that for a fixed integer $n$, there exists $M_n$, so that whenever $k \geq M_n$,

$$f(n, k) \leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k + C?$$


# Thank you!