Nearly-linear increasing paths in edge-ordered graphs

Matija Bucić

ETH Zürich

Joint work with: Matthew Kwan, Alexey Pokrovskiy, Benny Sudakov, Tuan Tran and Adam Zsolt Wagner

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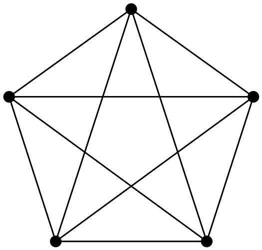

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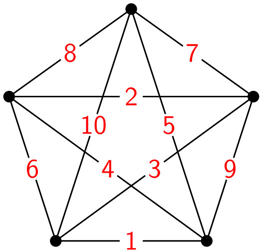

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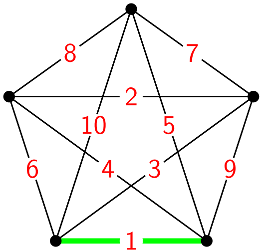

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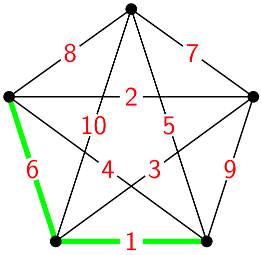

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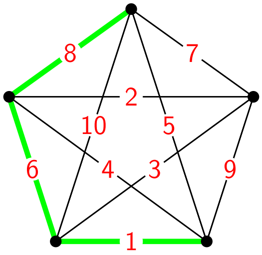

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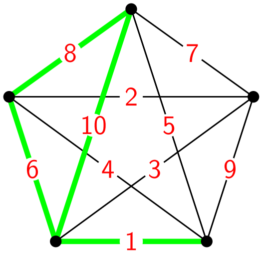

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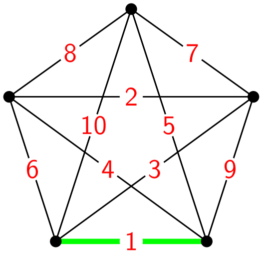

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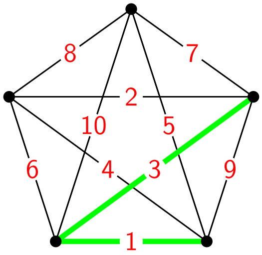

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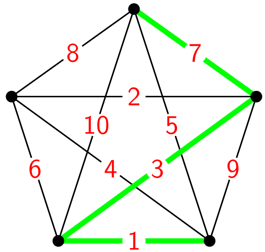

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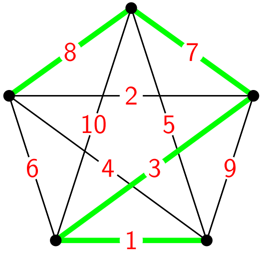

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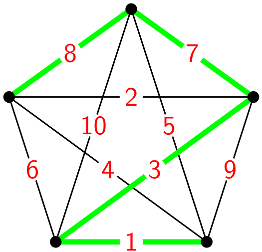

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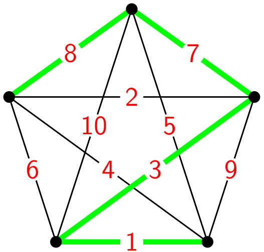

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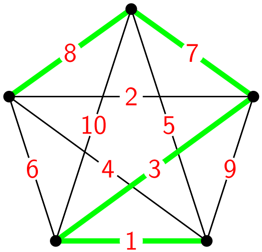

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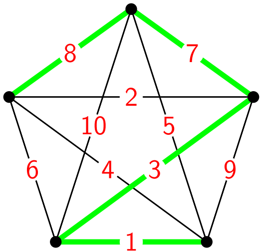

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  - Solved by Martinsson for paths and Angel, Ferber, Sudakov, Tassion for trails

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Theorem 1 (B., Kwan, Pokrovskiy, Sudakov, Tran, Wagner)

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Theorem 2 (B., Kwan, Pokrovskiy, Sudakov, Tran, Wagner)

Let G be a graph with n vertices and average degree $d \geq 2$. Then

$$f(G) \geq \frac{d}{2^{O\left(\sqrt{\log d \log \log n}\right)}}.$$

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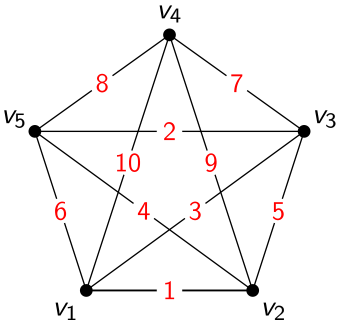

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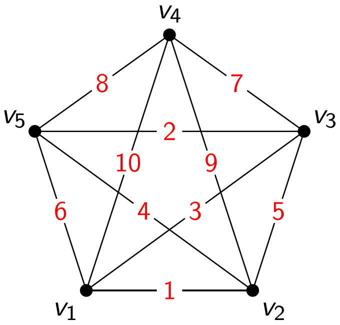

i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
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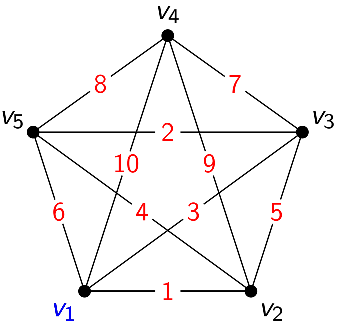

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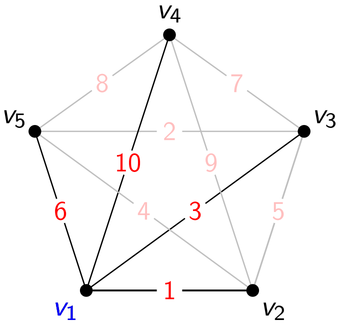

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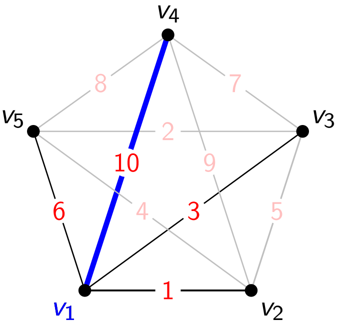

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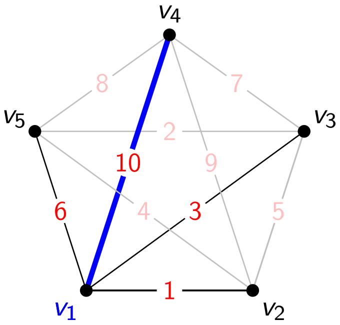

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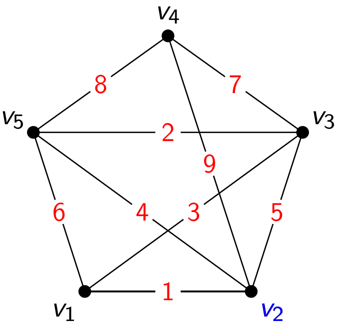

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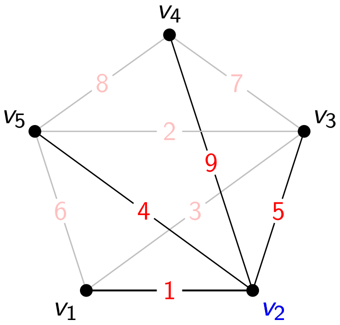

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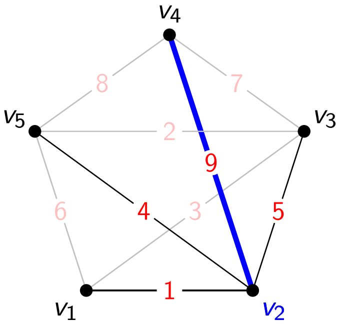

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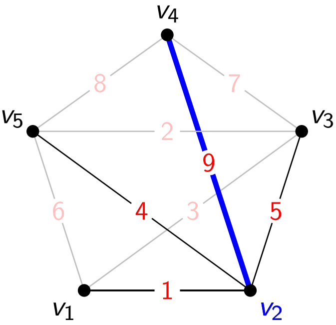

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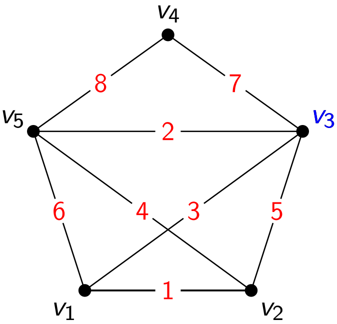

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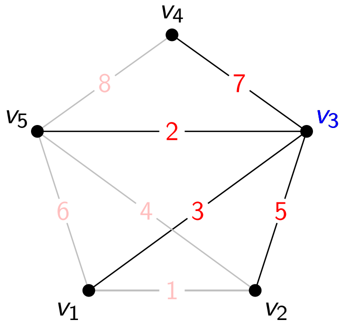

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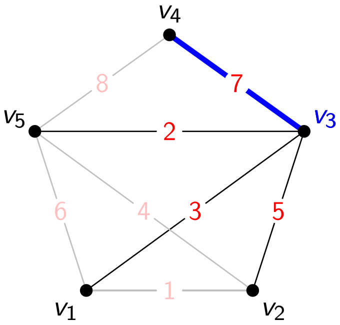

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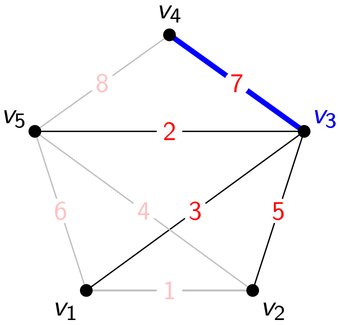

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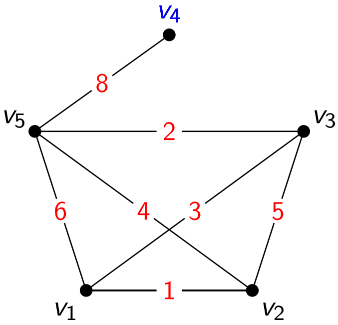

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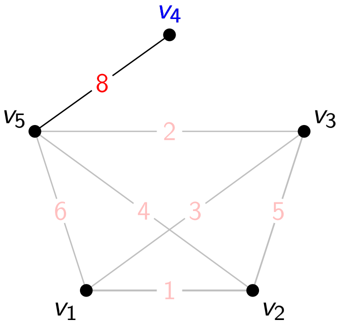

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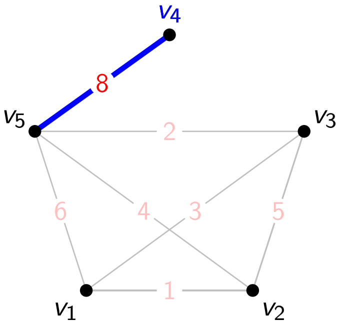

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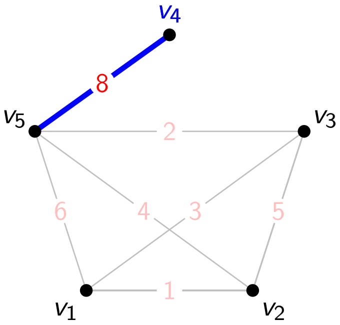

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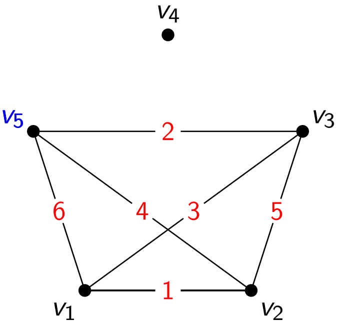

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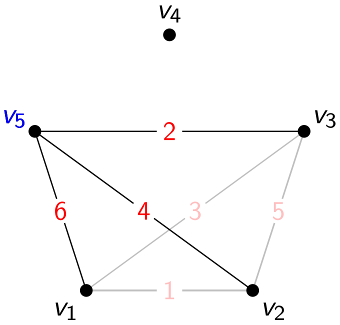

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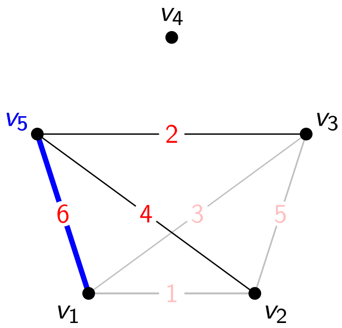

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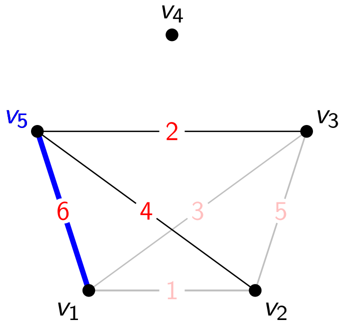

i \ v	v_1	v_2	v_3	v_4	v_5
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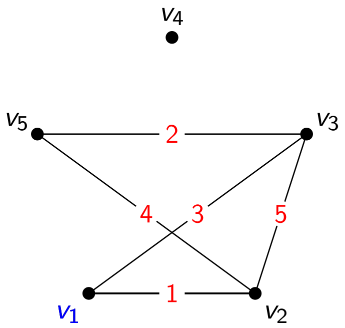

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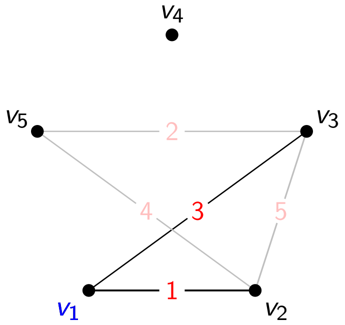

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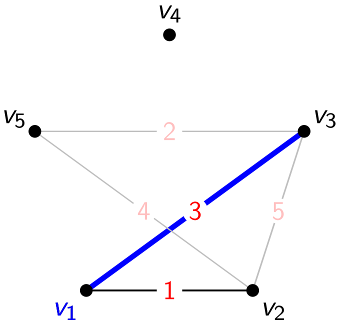

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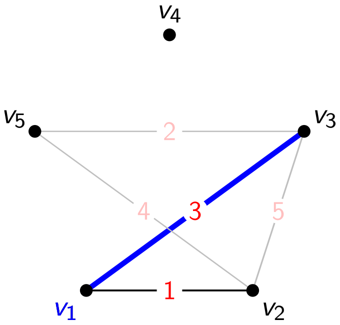

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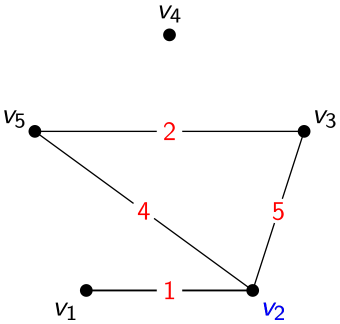

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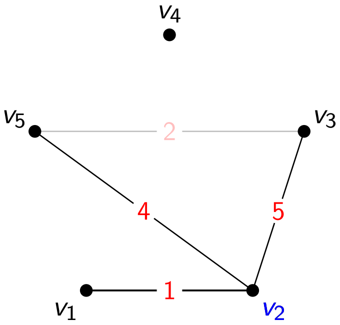

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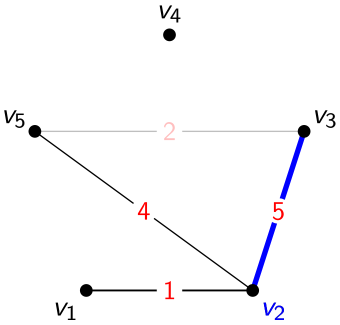

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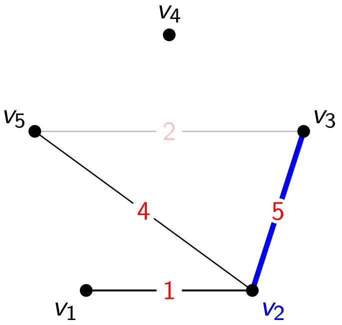

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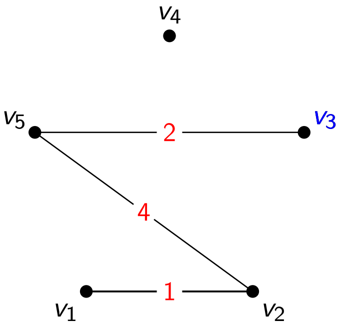

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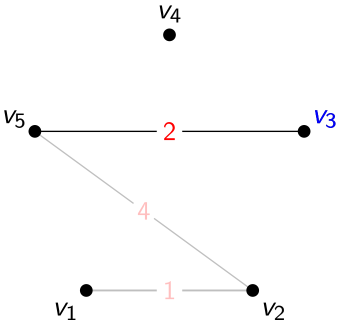

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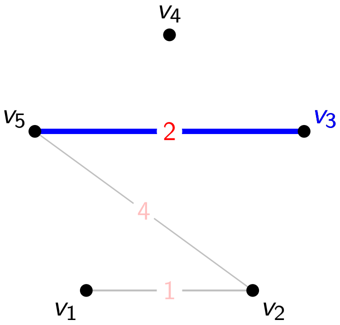

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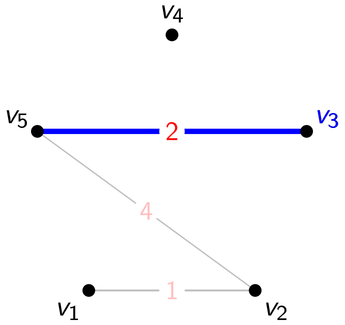

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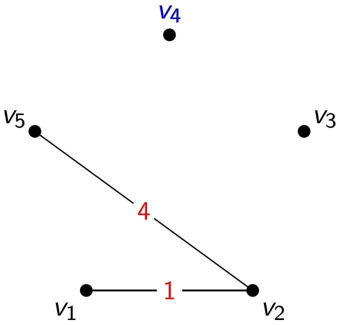

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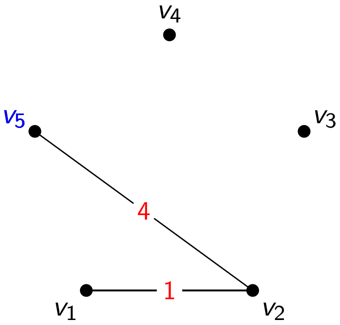

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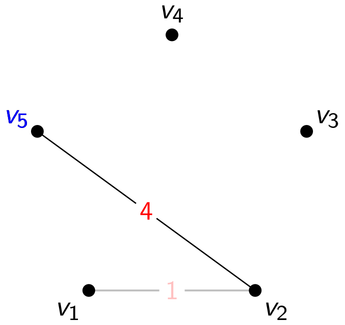

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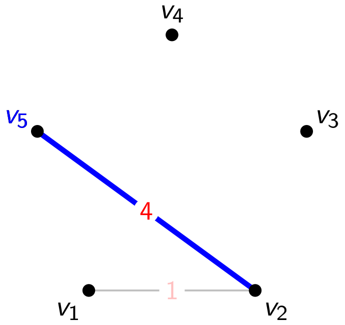

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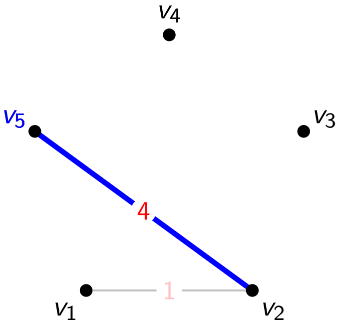

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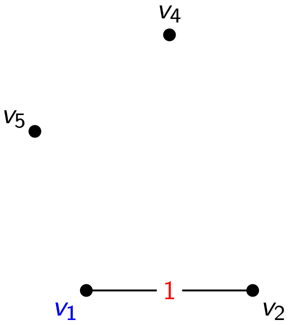

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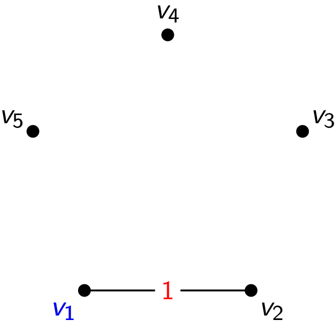

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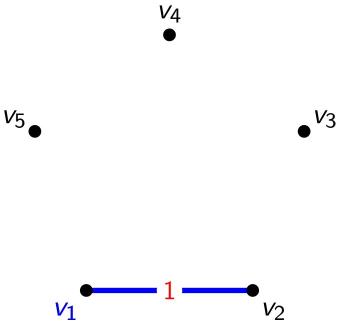

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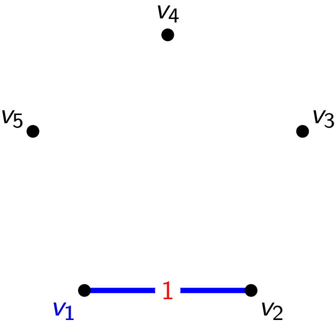

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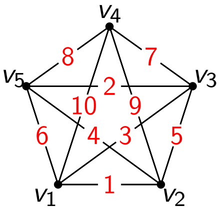
Basic properties of height tables



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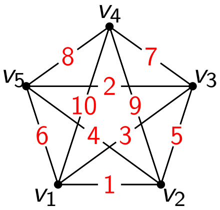

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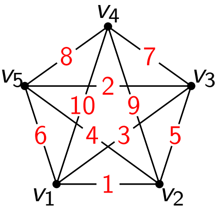

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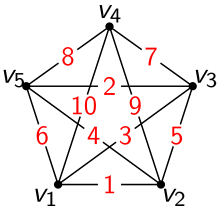

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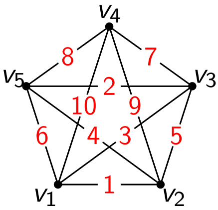

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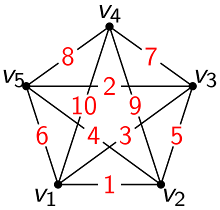

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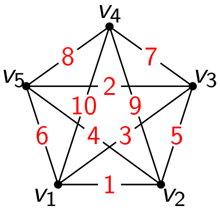

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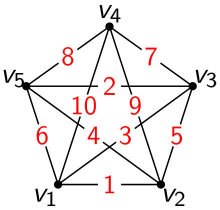

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- If edge $e = v_iv_j$ is entered at position (h, v_i) all positions $(a, v_i), (a, v_j)$ for $a < h$ are non-empty

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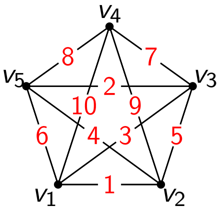
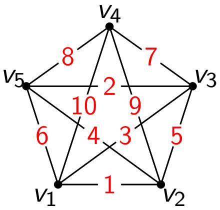


i \ v	v_1	v_2	v_3	v_4	v_5
3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$

- There are $|E(G)|$ non-empty positions.
- The *height* of e, denoted by $\mathrm{h}_G(e)$, is the row index of its position
- Any edge $v_i v_j$ is entered into column v_i or column v_j - *column vertex*.
- If edge $e = v_i v_j$ is entered at position (h, v_i) all positions $(a, v_i), (a, v_j)$ for $a < h$ are non-empty

Basic properties of height tables



i \ v					
3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$
i / v	v_1	v_2	v_3	v_4	v_5

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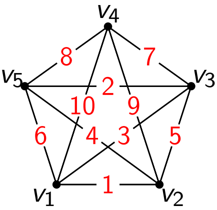

$i \diagup v$	v_1	v_2	v_3	v_4	v_5
3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
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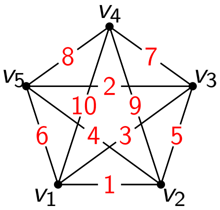

3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$
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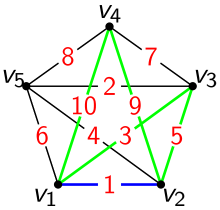

3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$
i \ v	v_1	v_2	v_3	v_4	v_5

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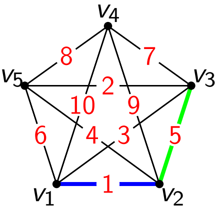
Basic properties of height tables



3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$
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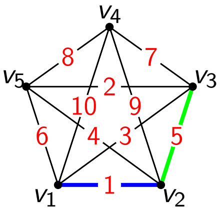

3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$
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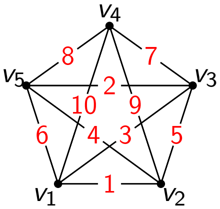
- Any edge $v_i v_j$ is entered into column v_i or column v_j - *column vertex*.
- If edge $e = v_i v_j$ is entered at position (h, v_i) all positions $(a, v_i), (a, v_j)$ for $a < h$ are non-empty and contain edges larger than e.
  Any such position was considered before (h, v_i).
  At that point edge $v_i v_j$ was unused.
  Since $v_i v_j$ was not entered, there had to be a larger edge available.



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
$i \backslash v$	v_1	v_2	v_3	v_4	v_5

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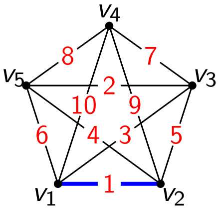
Definition

A vertex w is called an **extender** *of an edge vu, entered at position (h, v), if uw is an edge entered at position (a, u) for some $a < h$.*



3	$v_1 v_2$				
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_5$		$v_5 v_2$
1	$v_1 v_4$	$v_2 v_4$	$v_3 v_4$	$v_4 v_5$	$v_5 v_1$
i \ v	v_1	v_2	v_3	v_4	v_5

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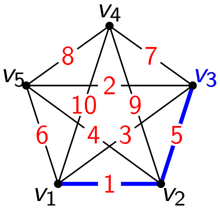
Basic properties of height tables



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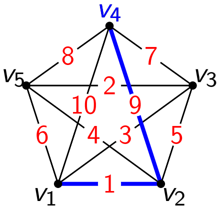

$i \diagup v$	v_1	v_2	v_3	v_4	v_5
3	$v_1 v_2$				
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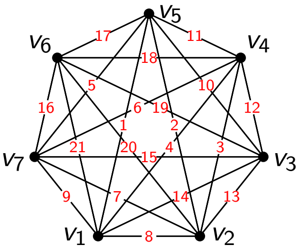
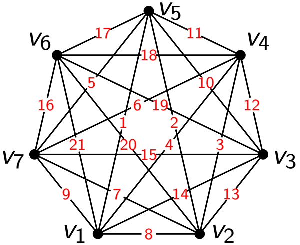
Application of height tables



5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$
i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Application of height tables



5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$
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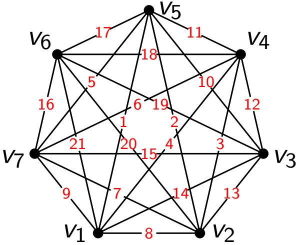
Theorem (Rödl)

In any edge ordered graph there is an increasing path of length $\sqrt{d(G)}$.

Application of height tables



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2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$
i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Theorem (Rödl)

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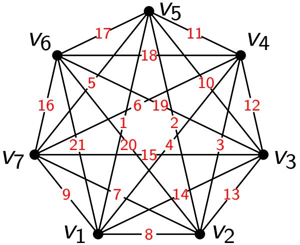
Proof.

□

Application of height tables



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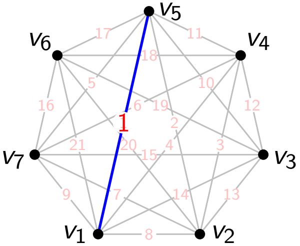
Proof.

- There is an edge $u_1 u_2$ of height at least $|E(G)|/n = d(G)/2$.

□

Application of height tables



$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
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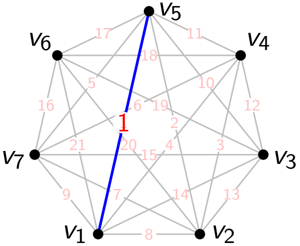

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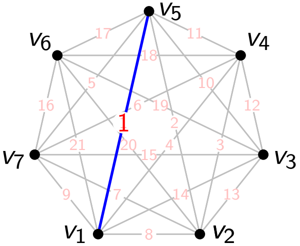
- There is an edge $u_1 u_2$ of height at least $|E(G)|/n = d(G)/2$.
- Let u_3 be its highest extender.

□

Application of height tables



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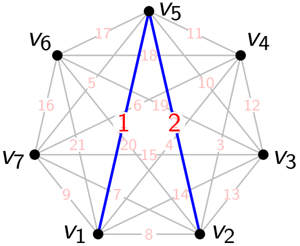

5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
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1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
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Proof.

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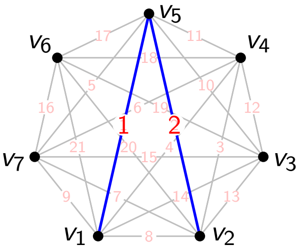
□

Application of height tables



5	$v_1 v_5$						
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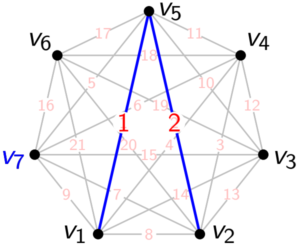
Theorem (Rödl)

In any edge ordered graph there is an increasing path of length $\sqrt{d(G)}$.

Proof.

- There is an edge $u_1 u_2$ of height at least $|E(G)|/n = d(G)/2$.
- Let u_3 be its highest extender.
- Repeat, let u_{i+1} be the highest extender of $u_{i-1} u_i$

□



$i \setminus v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
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3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$

Theorem (Rödl)

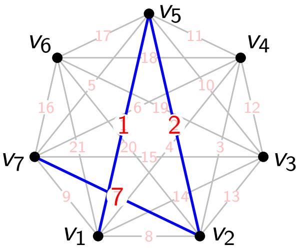
In any edge ordered graph there is an increasing path of length $\sqrt{d(G)}$.

Proof.

- There is an edge $u_1 u_2$ of height at least $|E(G)|/n = d(G)/2$.
- Let u_3 be its highest extender.
- Repeat, let u_{i+1} be the highest extender of $u_{i-1} u_i$

□

Application of height tables



5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
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i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

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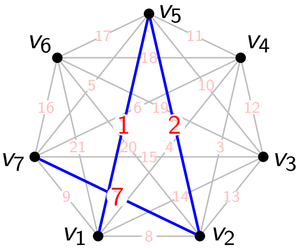

$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$

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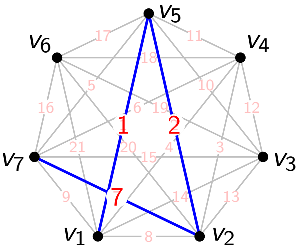
□

Application of height tables



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$

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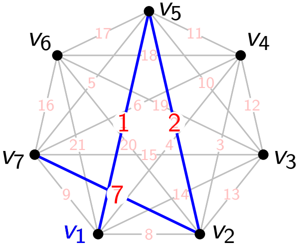
- After $d/2$ iterations we obtain an increasing path $u_1 \dots u_{d/2}$. **Not quite.**

□

Application of height tables



5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$
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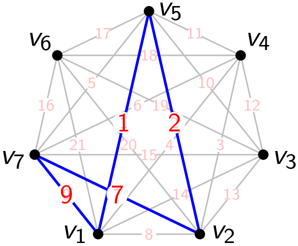
□



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$

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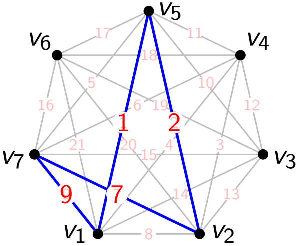
□

Application of height tables



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
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2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
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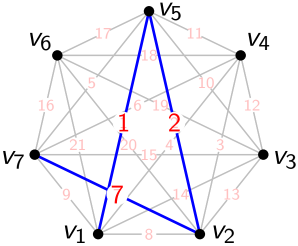
- After $d/2$ iterations we obtain an increasing **trail** $u_1 \dots u_{d/2}$.

□

Application of height tables



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
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2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$

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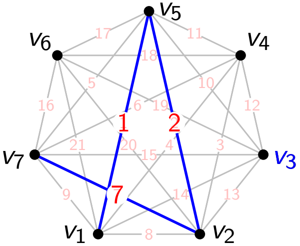
-

□

Application of height tables



5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
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2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
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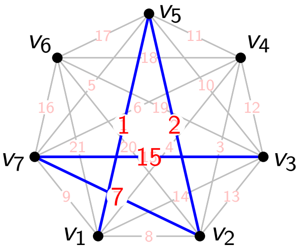
□

Application of height tables



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
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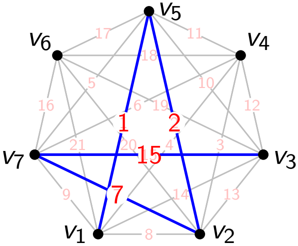
-

□



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
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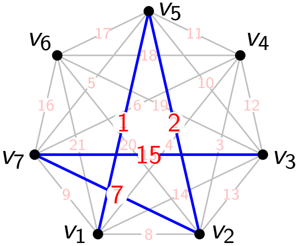
- $\mathrm{h}_G(u_i u_{i+1}) \geq \mathrm{h}_G(u_{i-1} u_i) - i$

□

Application of height tables



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
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- $\mathrm{h}_G(u_i u_{i+1}) \geq \mathrm{h}_G(u_{i-1} u_i) - i$
- Repeat as long as $d/2 - 1 - \ldots - i = d/2 - \binom{i}{2} > 0 \Leftrightarrow \sqrt{d} > i$. □

Our new ingredients

5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$
i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

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i / v							
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2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
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i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

i / v							
5							
4	$v_1 v_4$	$v_2 v_4$					
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$			
2	$v_1 v_3$	$v_2 v_3$		$v_4 v_5$			$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$		$v_4 v_6$		$v_6 v_7$	$v_7 v_3$
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5							
4	$v_1 v_4$	$v_2 v_4$					
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$			
2							$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$		$v_4 v_6$		$v_6 v_7$	
i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

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i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

5							
4							
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i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

5							
4							
3	$v_1 v_4$	$v_2 v_4$					
2	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$			
1	$v_1 v_6$	$v_2 v_6$		$v_4 v_6$		$v_6 v_7$	$v_7 v_1$
i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

Lemma (Dropping lemma)

Let G be an ordered graph, $U \subseteq V(G)$, $xy \in E(G)$: $\mathrm{h}_G(xy) > m = \sqrt{\Delta(G)|U|}$.

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i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	$v_1 v_5$						
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1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$

i / v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5							
4							
3	$v_1 v_4$	$v_2 v_4$					
2	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$			
1	$v_1 v_6$	$v_2 v_6$		$v_4 v_6$		$v_6 v_7$	$v_7 v_1$

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Let G be an ordered graph, $U \subseteq V(G)$, $xy \in E(G)$: $\mathrm{h}_G(xy) > m = \sqrt{\Delta(G)|U|}$. Then $\exists z, w \in V(G) \setminus U$: $xyzw$ is an increasing path

5	$v_1 v_5$						
4	$v_1 v_4$	$v_2 v_4$			$v_5 v_2$		
3	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$	$v_5 v_7$		
2	$v_1 v_3$	$v_2 v_3$	$v_3 v_4$	$v_4 v_5$	$v_5 v_3$		$v_7 v_1$
1	$v_1 v_6$	$v_2 v_6$	$v_3 v_6$	$v_4 v_6$	$v_5 v_6$	$v_6 v_7$	$v_7 v_3$
$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7

5							
4							
3	$v_1 v_4$	$v_2 v_4$					
2	$v_1 v_2$	$v_2 v_7$		$v_4 v_7$			
1	$v_1 v_6$	$v_2 v_6$		$v_4 v_6$		$v_6 v_7$	$v_7 v_1$
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Lemma (Dropping lemma)

Let G be an ordered graph, $U \subseteq V(G)$, $xy \in E(G)$: $\mathrm{h}_G(xy) > m = \sqrt{\Delta(G)|U|}$. Then $\exists z, w \in V(G) \setminus U$: $xyzw$ is an increasing path and

$$\mathrm{h}_{G-U}(zw) \geq \mathrm{h}_G(xy) - m.$$

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Question

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Remark: Let $\varepsilon > 0$, then there exists an n vertex graph G with average degree $d(G) = n^{\varepsilon}$ for which this result is tight up to a constant factor.

Proof strategy

Theorem

Let G be an ordered graph, $e \in E(G)$ an edge with $\mathrm{h}_G(e) > a$. Then there is an increasing path P starting with e, having length at least

$$a^{1-1/t}/(\log n)^{2t},$$

such that $\mathrm{h}_G(f) \geq \mathrm{h}_G(e) - a$ for every $f \in E(P)$.

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- We find a long increasing path within H.

Finding a dense almost regular subgraph of extenders

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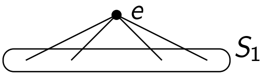

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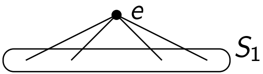

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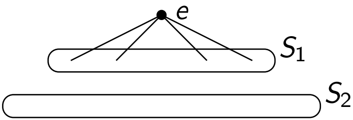

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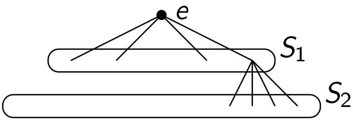

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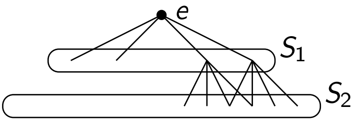

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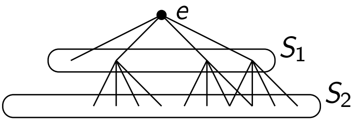

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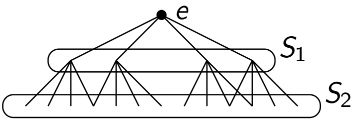

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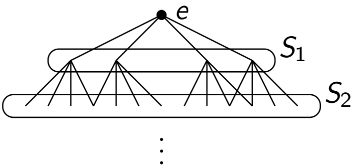

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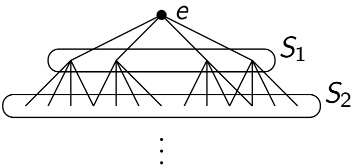

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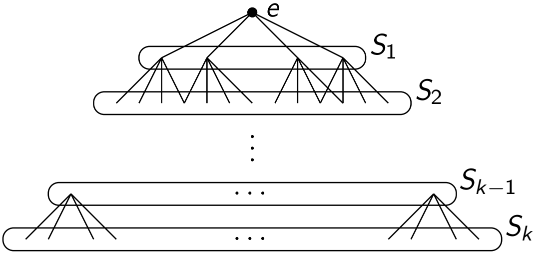

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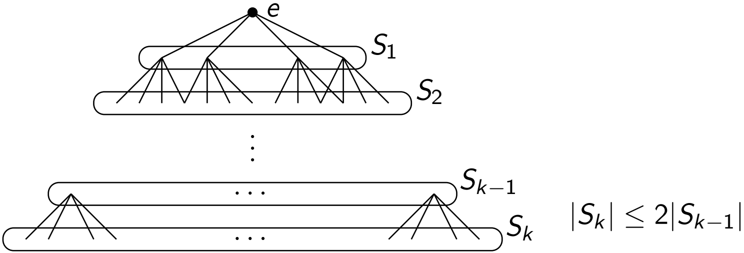

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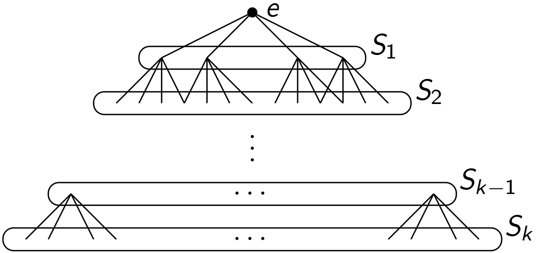

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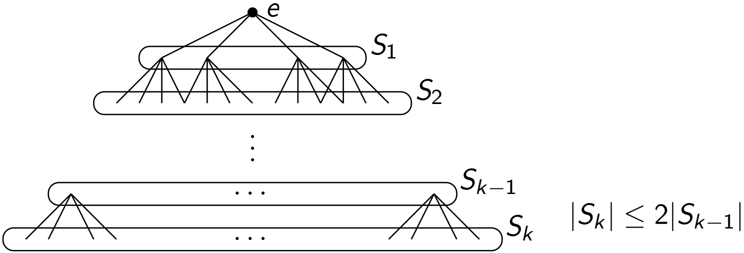

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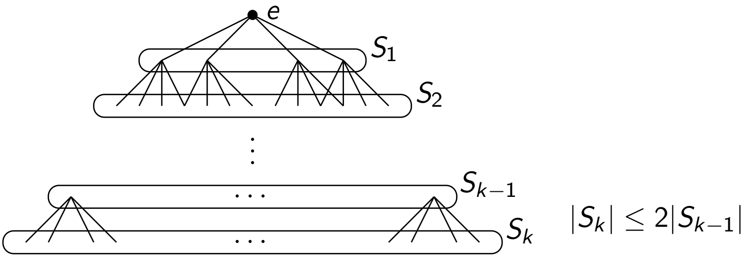

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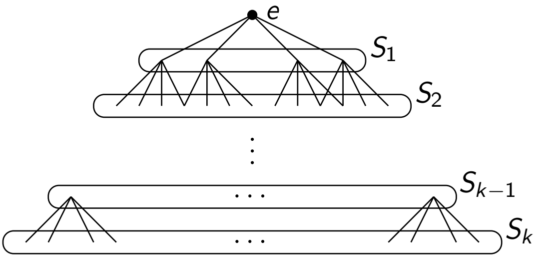

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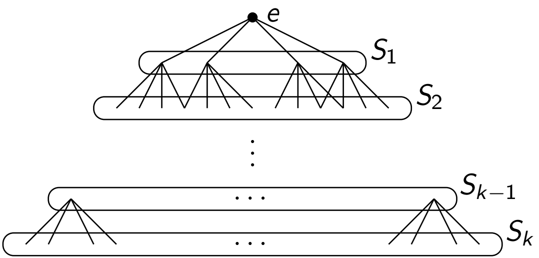

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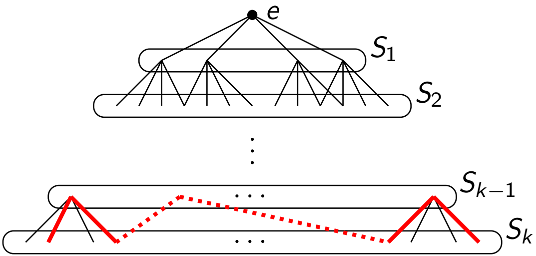

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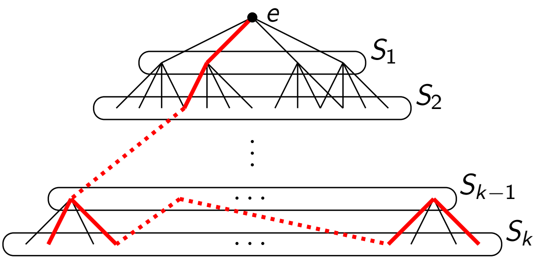

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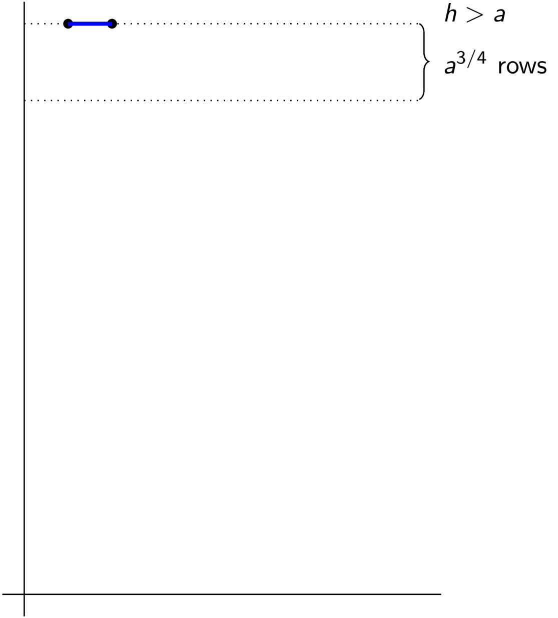

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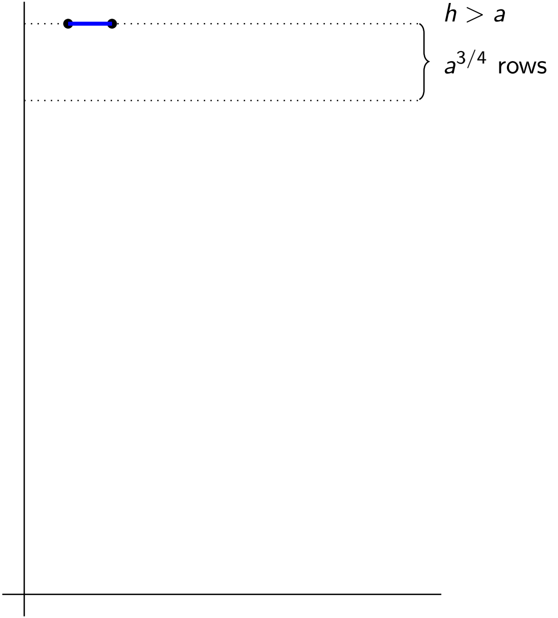

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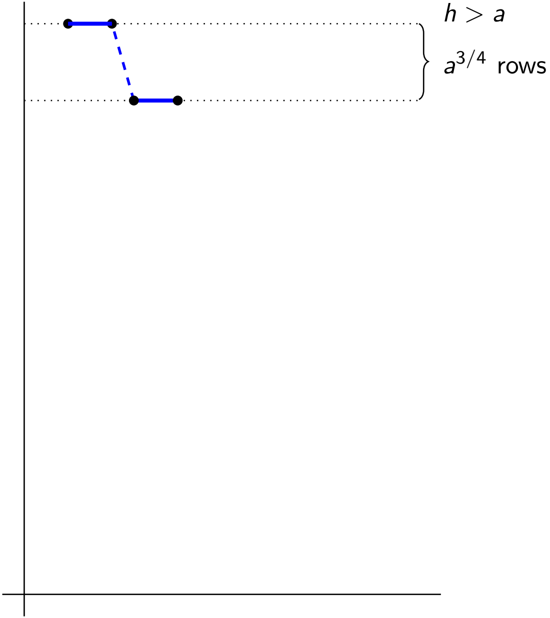

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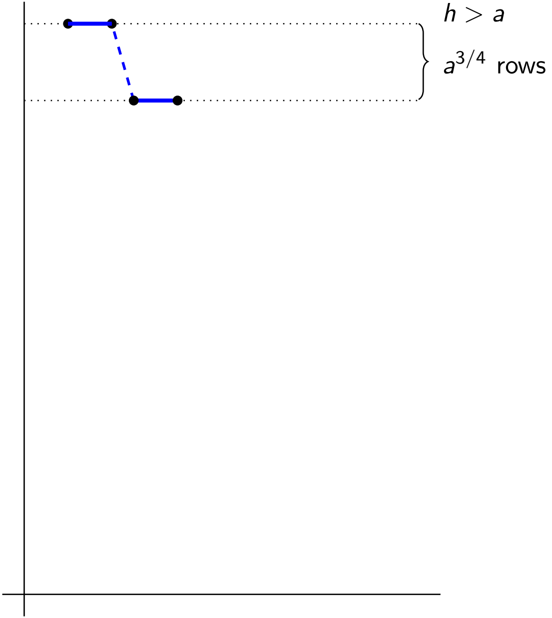

Finding long increasing paths in almost regular dense graphs

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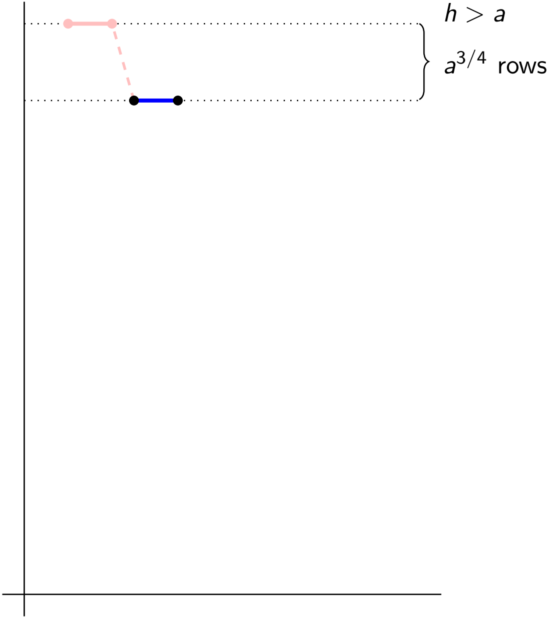

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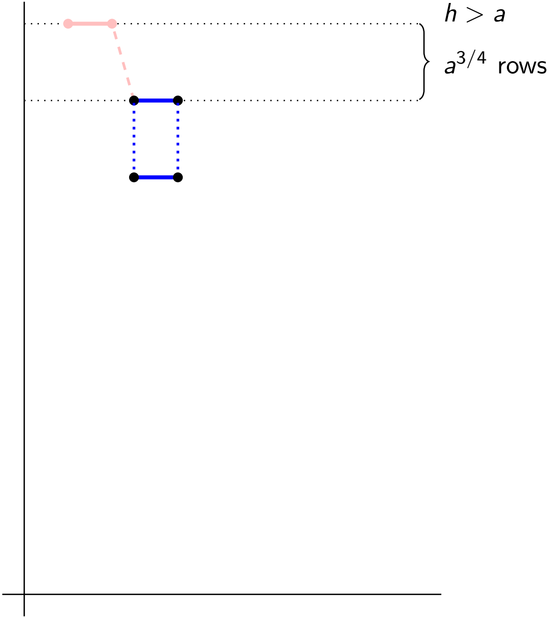

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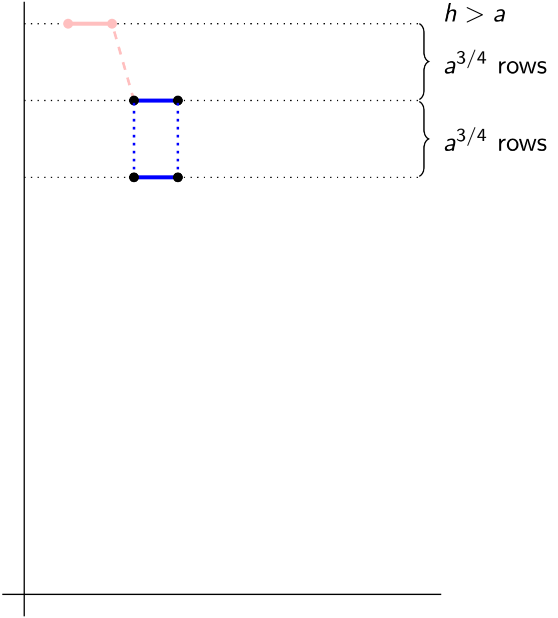

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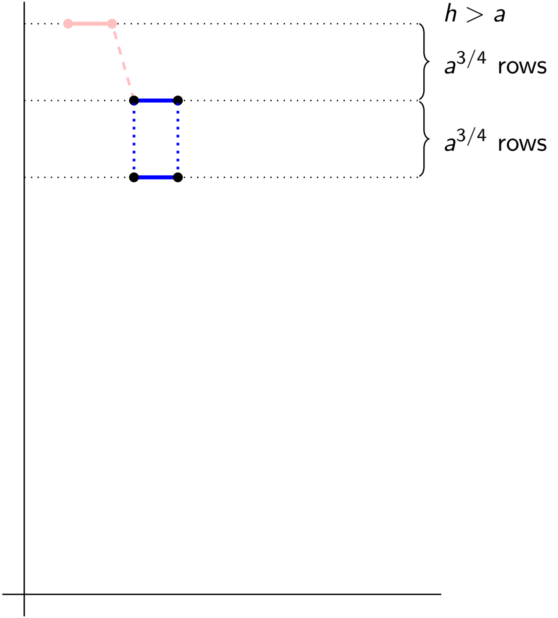

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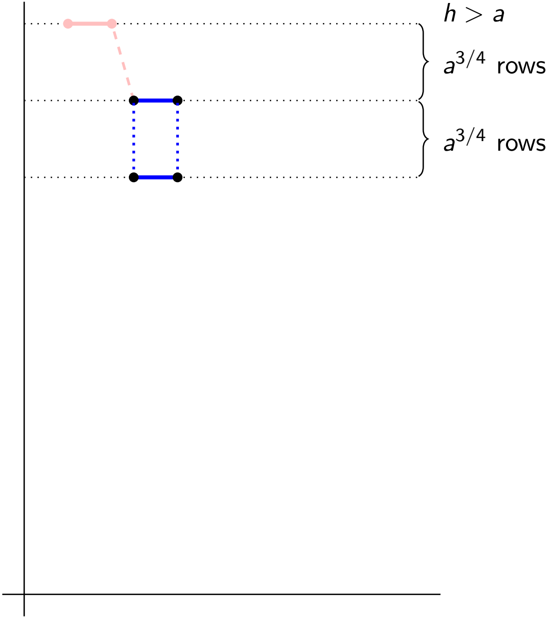

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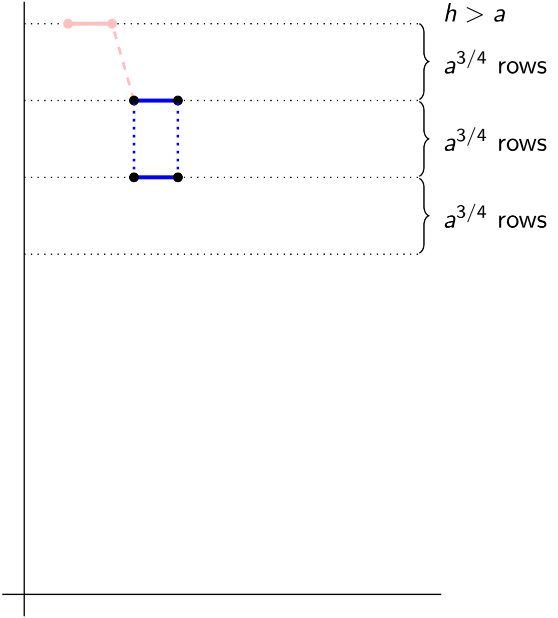

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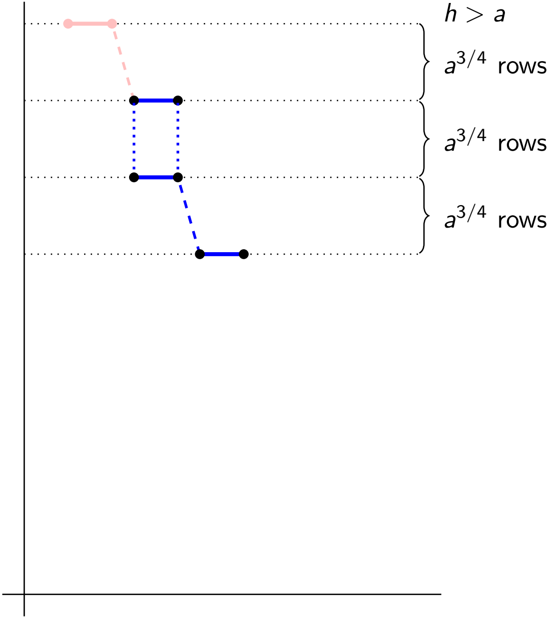

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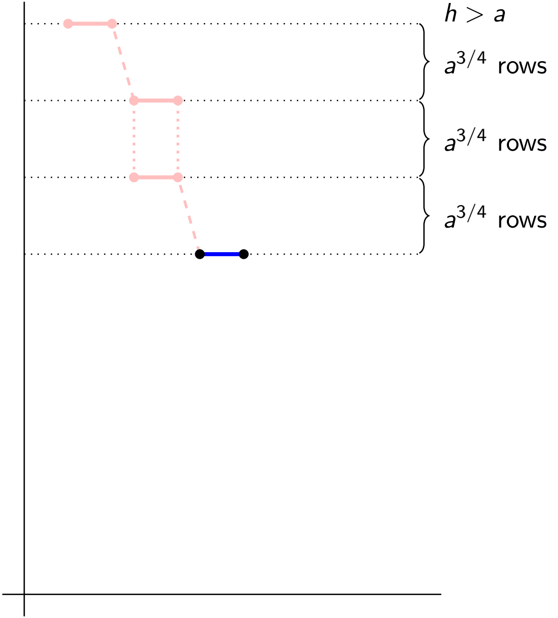

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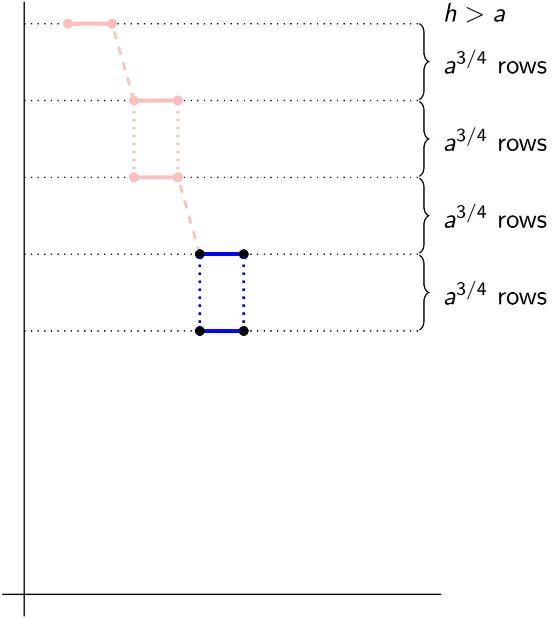

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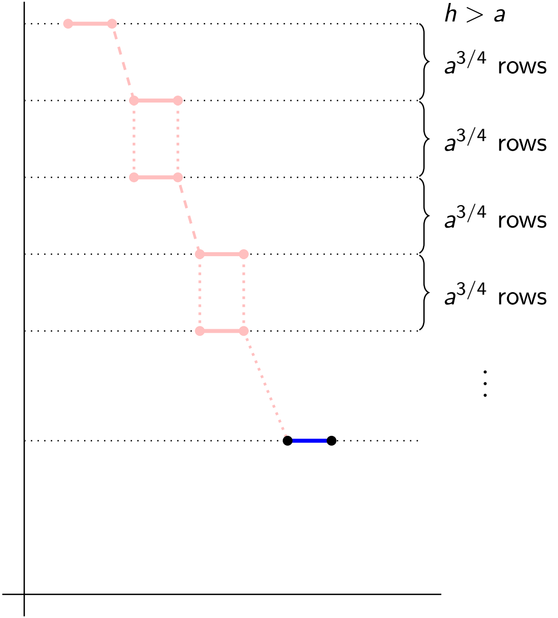

Finding long increasing paths in almost regular dense graphs

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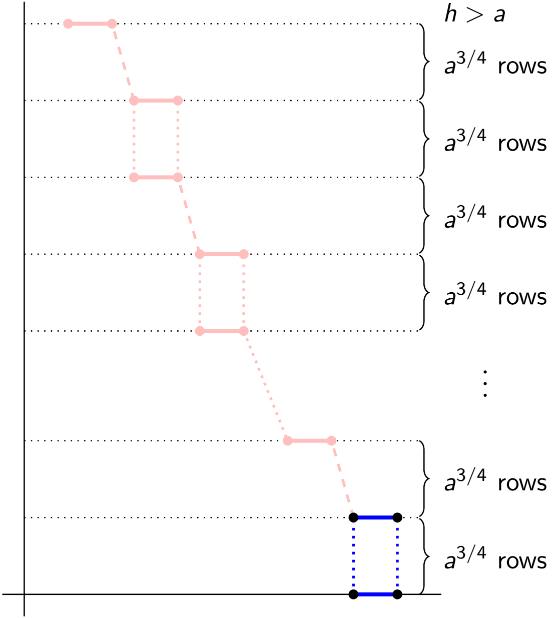
- Remove all but its last two vertices. Dropping lemma shows the last edge falls at most $a^{3/4}$, it applies as

$$(a^{3/4})^2 > |P|\Delta(H) = a^{1/2} \cdot a = a^{3/2}$$

- Repeat $a/a^{3/4} = a^{1/4}$ times



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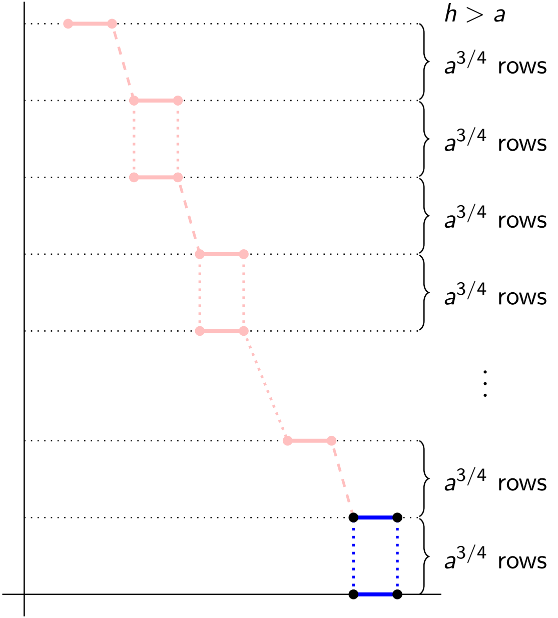

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Proposition

Let G be an edge-ordered graph with average degree d, such that every set of at most εd vertices induces at most $(1/2 - \varepsilon)d$ edges. Then G has an increasing path of length εd.


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
gracias
merci
obrigado
спасибо
dziękuję
bedankt
sukriya
kop khun krap
go raibh maith agat
mochchakkeram
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