Nearly-linear increasing paths in edge-ordered graphs

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ETH Zürich

Joint work with: Matthew Kwan, Alexey Pokrovskiy, Benny Sudakov, Tuan Tran and Adam Zsolt Wagner

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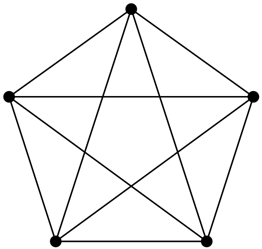

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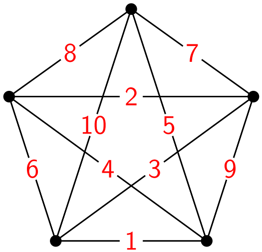

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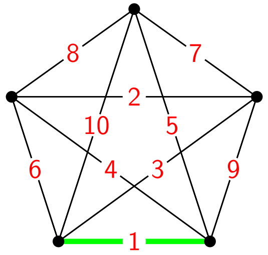

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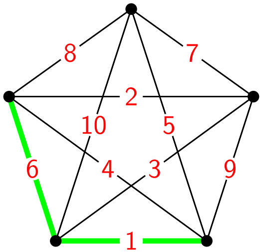

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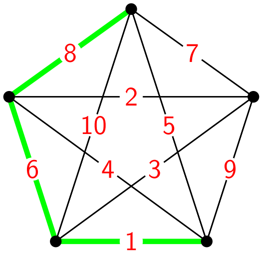

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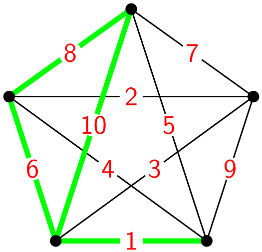

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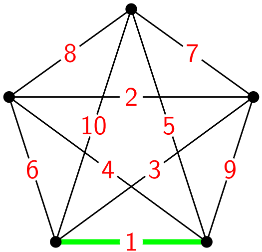

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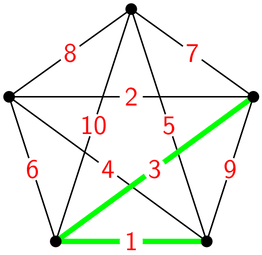

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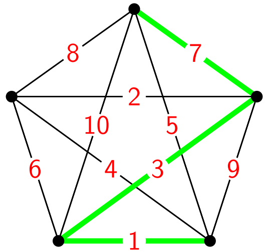

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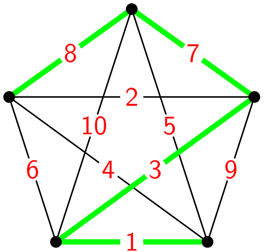

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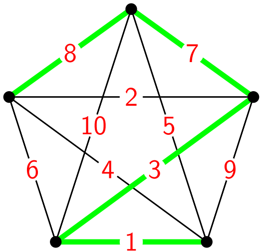

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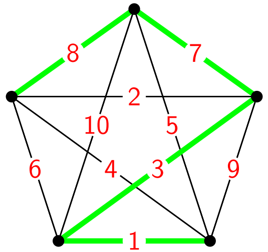

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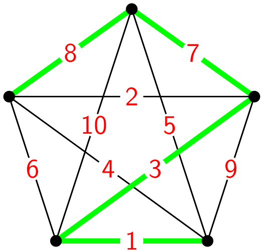

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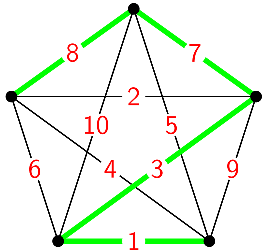

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Theorem 1 (B., Kwan, Pokrovskiy, Sudakov, Tran, Wagner)

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Theorem 2 (B., Kwan, Pokrovskiy, Sudakov, Tran, Wagner)

Let G be a graph with n vertices and average degree $d \geq 2$. Then

$$f(G) \geq \frac{d}{2^{O(\sqrt{\log d \log \log n})}}.$$

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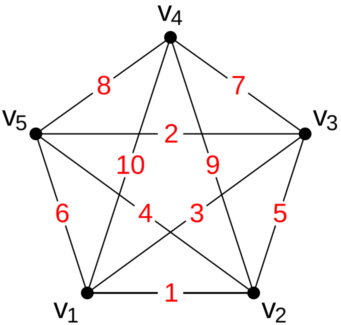

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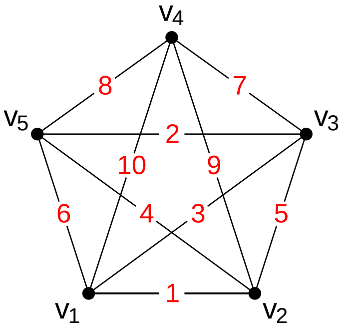

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i \ v	v_1	v_2	v_3	v_4	v_5

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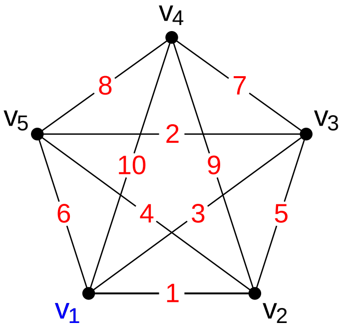

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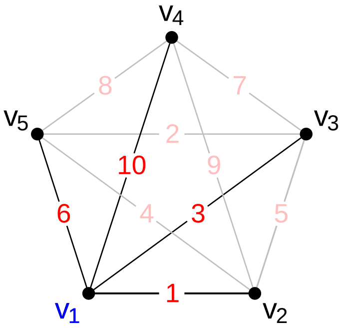

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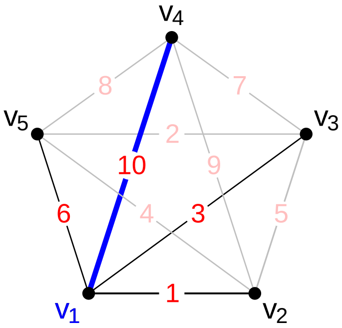

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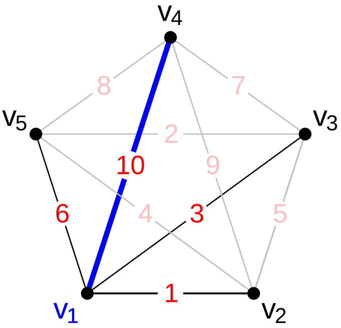

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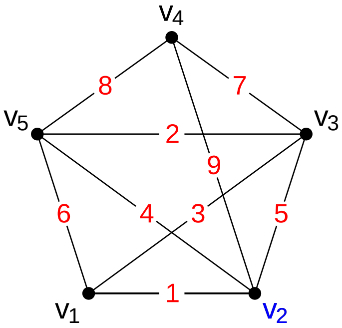

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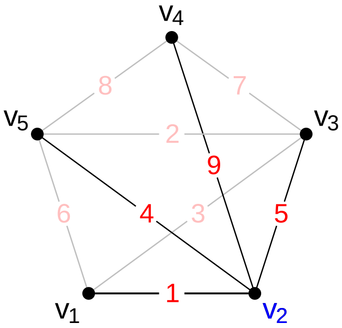

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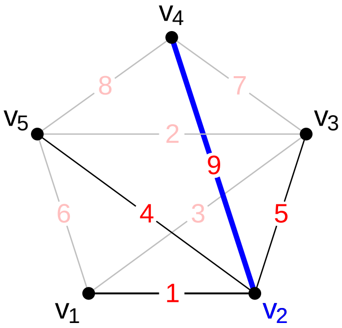

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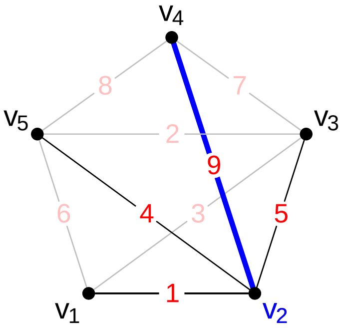

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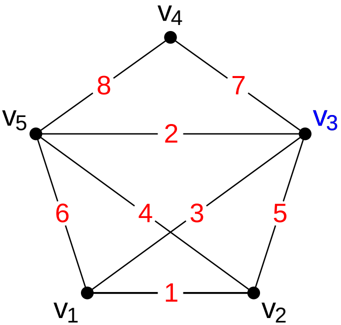

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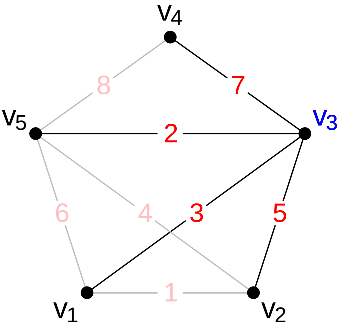

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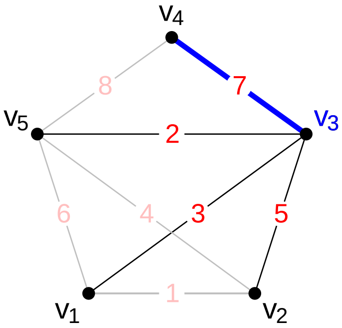

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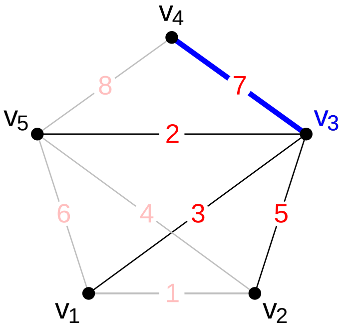

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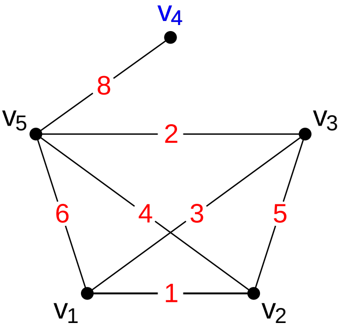

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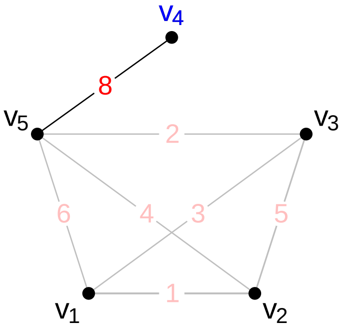

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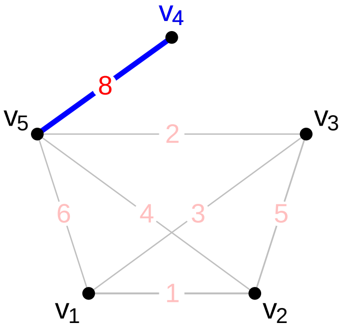

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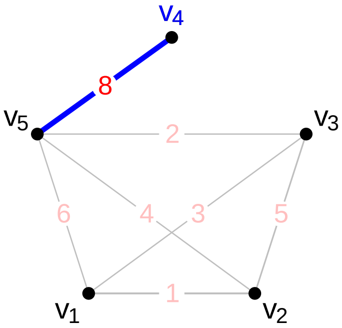

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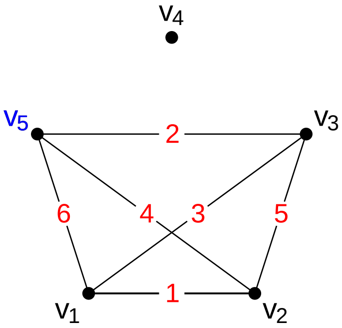

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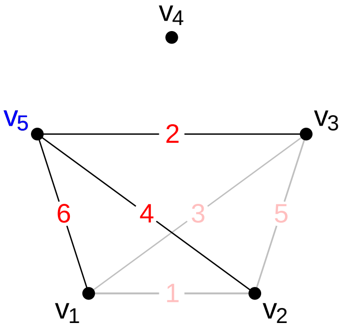

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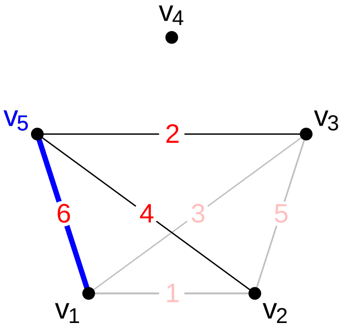

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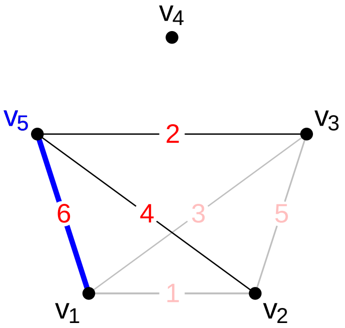

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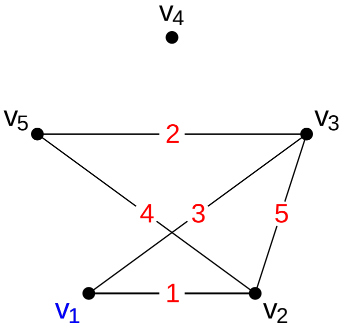
3					
2					
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:

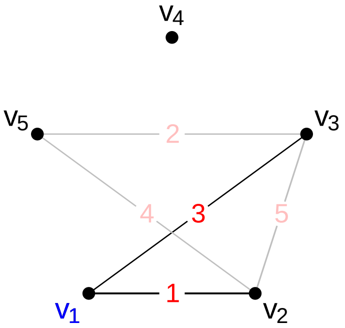


⋮					
3					
2					
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i / v	v_1	v_2	v_3	v_4	v_5

Height tables

Definition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:

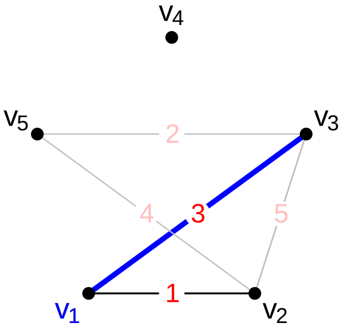


$i \diagdown v$	v_1	v_2	v_3	v_4	v_5
$\vdots$					
3					
2					
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



⋮					
3					
2					
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i / v	v_1	v_2	v_3	v_4	v_5

Height tables

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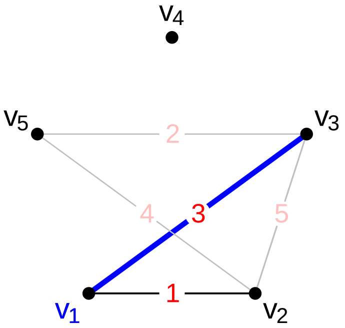
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:



⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

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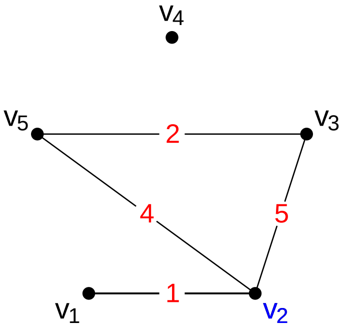
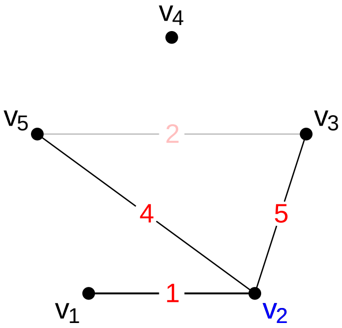


⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $\mathbb{N}$ $V(G)$, constructed as follows:

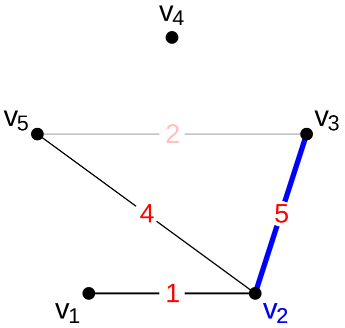


i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



⋮					
3					
2	v_1v_3				
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

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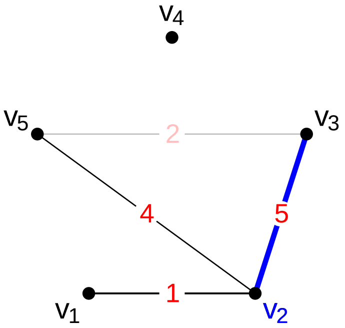

i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3			
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

De nition

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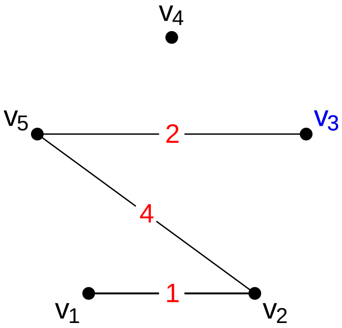
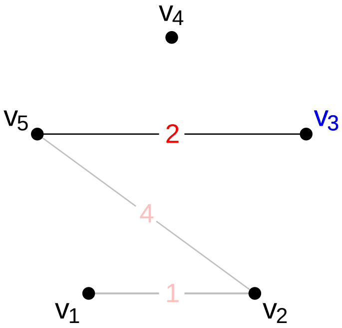


i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3			
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:

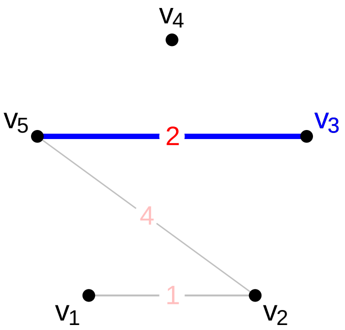


i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3			
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

Definition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:

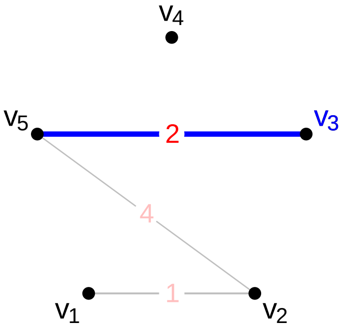


i \ v	v_1	v_2	v_3	v_4	v_5
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
2	v_1v_3	v_2v_3			
3					
⋮					

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:

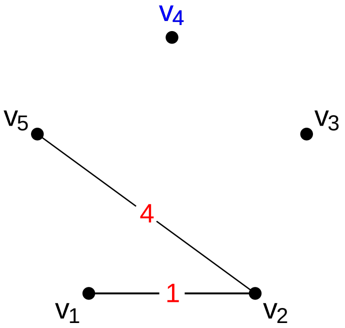


⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:

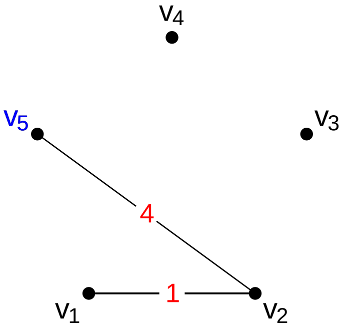


i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $\mathbb{N}$ $V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

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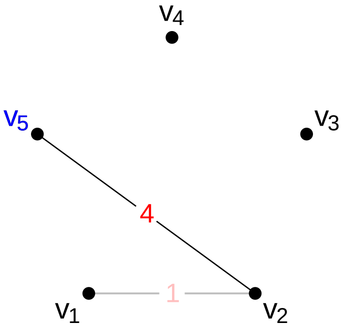
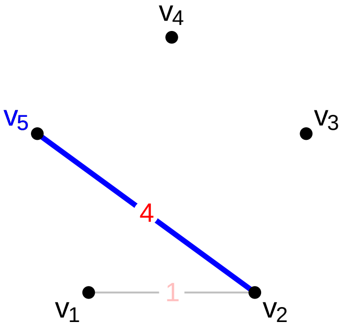
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:

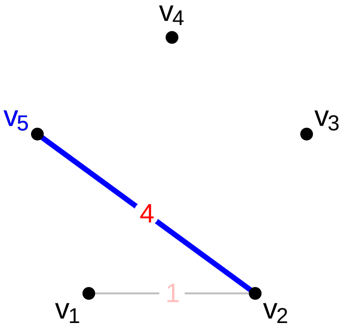


⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i / v	v_1	v_2	v_3	v_4	v_5

Height tables

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by $N \quad V(G)$, constructed as follows:

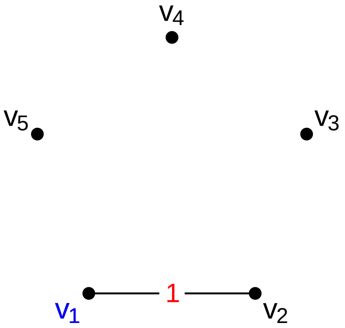


⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Height tables

De nition

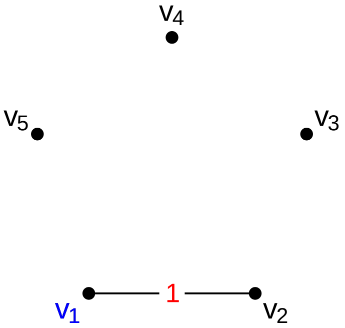
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

De nition

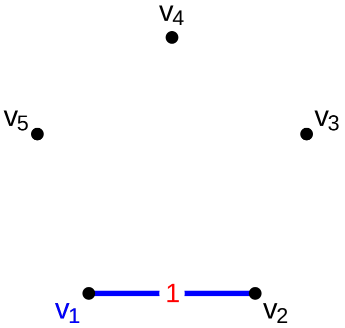
A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:



i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

De nition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially lled array indexed by N $V(G)$, constructed as follows:

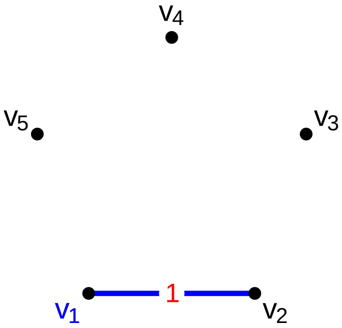


i \ v	v_1	v_2	v_3	v_4	v_5
⋮					
3					
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

Height tables

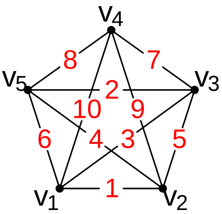
Definition

A height table of an edge ordered graph G with vertex set $[n]$ is a partially filled array indexed by $\mathbb{N} \times V(G)$, constructed as follows:



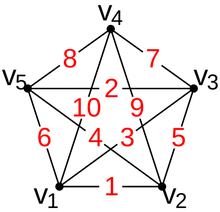
⋮					
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

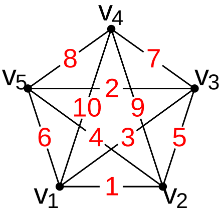
Basic properties of height tables



$i \diagup v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

Basic properties of height tables

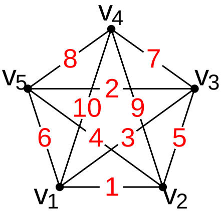


3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Basic properties of height tables



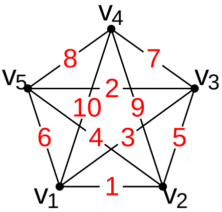
i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j .

Basic properties of height tables



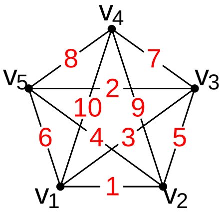
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

Basic properties of height tables



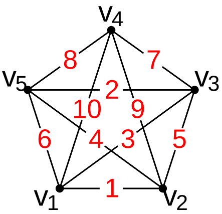
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

Basic properties of height tables

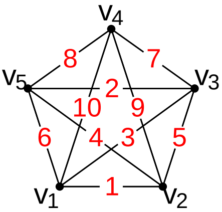


i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge $v_i v_j$ is entered into column v_i or column v_j - column vertex.



$i \backslash v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

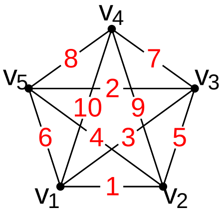
There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
$i \backslash v$	v_1	v_2	v_3	v_4	v_5

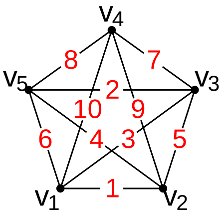
There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty

Basic properties of height tables



$i \backslash v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

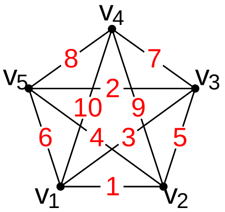
There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty

Basic properties of height tables



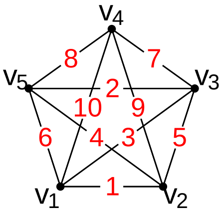
$i \backslash v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1

There are $|E(G)|$ non-empty positions.

The height of e, denoted by $h_G(e)$, is the row index of its position

Any edge v_iv_j is entered into column v_i or column v_j - column vertex.

If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty



i \ v	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
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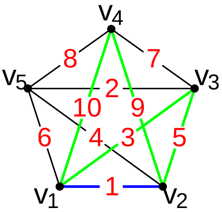
Basic properties of height tables



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3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
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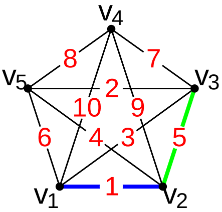

3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
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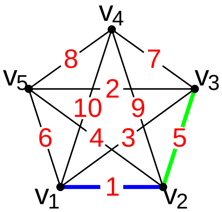
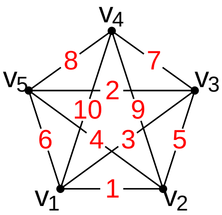


3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
1	v_1v_4	v_2v_4	v_3v_4	v_4v_5	v_5v_1
i \ v	v_1	v_2	v_3	v_4	v_5

- There are $|E(G)|$ non-empty positions.
- The height of e, denoted by $h_G(e)$, is the row index of its position
- Any edge v_iv_j is entered into column v_i or column v_j - column vertex.
- If edge $e = v_iv_j$ is entered at position $(h; v_i)$ all positions $(a; v_i); (a; v_j)$ for $a < h$ are non-empty and contain edges larger than e:
  - Any such position was considered before $(h; v_i)$.
  - At that point edge v_iv_j was unused.
  - Since v_iv_j was not entered, there had to be a larger edge available.

Basic properties of height tables



3	v_1v_2				
2	v_1v_3	v_2v_3	v_3v_5		v_5v_2
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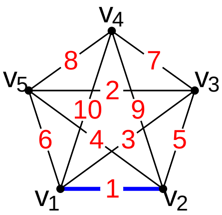
A vertex w is called an extender of an edge vu, entered at position $(h; v)$, if uw is an edge entered at position $(a; u)$ for some $a < h$.

Basic properties of height tables



$i \setminus v$	v_1	v_2	v_3	v_4	v_5
3	v_1v_2				
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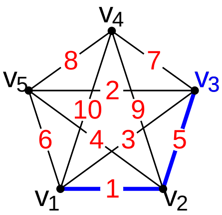

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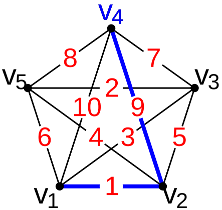

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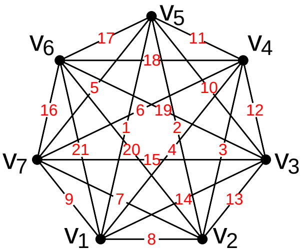
Application of height tables



5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
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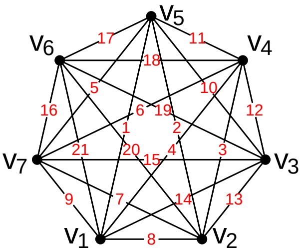

i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
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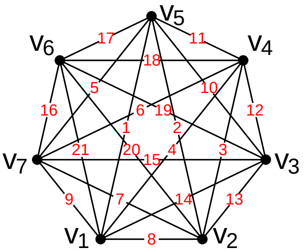
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3

Theorem (Rödl)

In any edge ordered graph there is an increasing path of length $\sqrt{d(G)}$:

Application of height tables



$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
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Theorem (Rödl)

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Proof.

□

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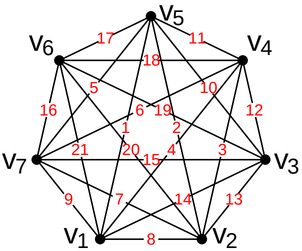

i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
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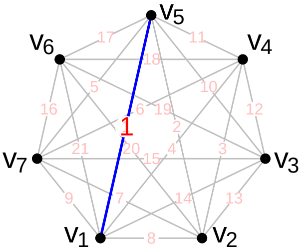
Proof.

There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.

□



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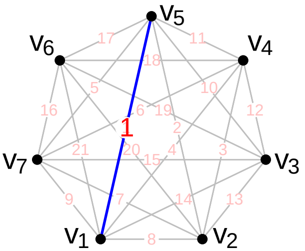

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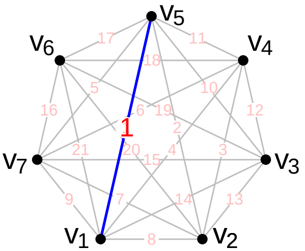
Let u_3 be its highest extender.

□



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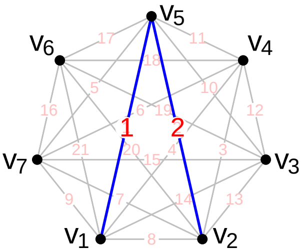

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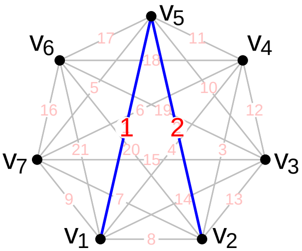

5	v_1v_5						
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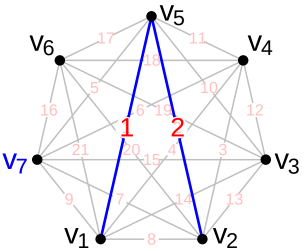
Repeat, let u_{i+1} be the highest extender of $u_{i-1}u_i$

□



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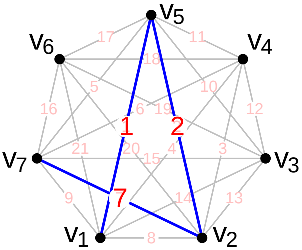

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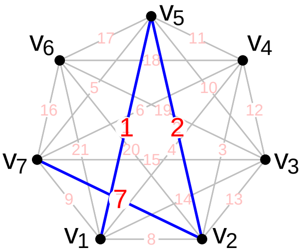

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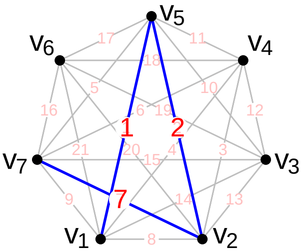
After $d/2$ iterations we obtain an increasing path $u_1 \dots u_{d/2}$.

□

Application of height tables



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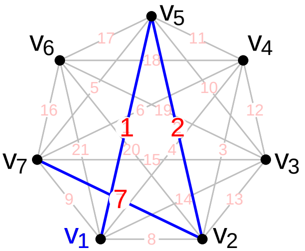
After $d/2$ iterations we obtain an increasing path $u_1 \dots u_{d/2}$. Not quite.

□

Application of height tables



i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3

Theorem (Rödl)

In any edge ordered graph there is an increasing path of length $\sqrt{d(G)}$.

Proof.

There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.

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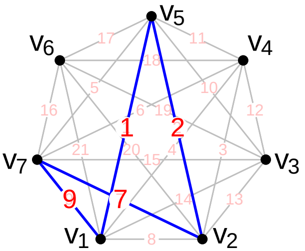

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5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
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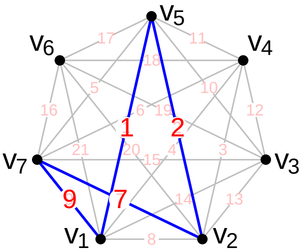

5	v_1v_5						
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3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
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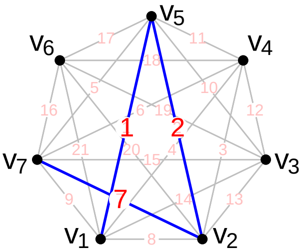

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2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
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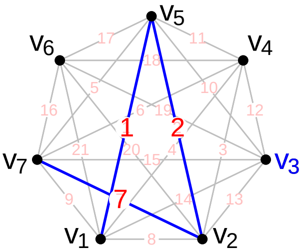

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5	v_1v_5						
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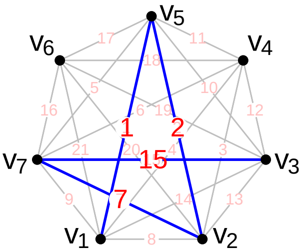

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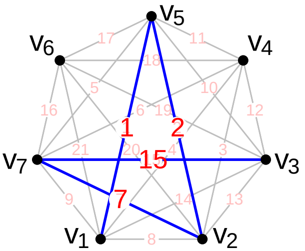

$i \backslash v$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
2	v_1v_3	v_2v_3	v_3v_4	v_4v_5	v_5v_3		v_7v_1
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$h_G(u_iu_{i+1}) \geq h_G(u_{i-1}u_i) - i$

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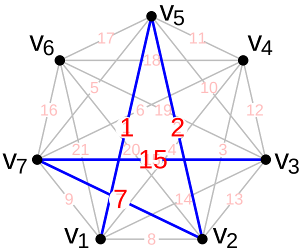

i \ v							
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
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1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
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Theorem (Rödl)

In any edge ordered graph there is an increasing path of length $\sqrt{d(G)}$.

Proof.

- There is an edge u_1u_2 of height at least $|E(G)|/n = d(G)/2$.
- Let u_3 be its highest extender.
- Repeat, let u_{i+1} be the highest extender of $u_{i-1}u_i$ distinct to all u_j
- $h_G(u_iu_{i+1}) \geq h_G(u_{i-1}u_i) - i$
- Repeat as long as $d/2 - 1 - \dots - i = d/2 - \binom{i}{2} > 0$, $\sqrt{d} > i$. □

Our new ingredients

i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
4	v_1v_4	v_2v_4			v_5v_2		
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1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
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5							
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1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
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Lemma (Dropping lemma)

Let G be an ordered graph, $U \subseteq V(G)$; $xy \in E(G)$: $h_G(xy) \geq m = \sqrt{(\;G)|U|}$.

Our new ingredients

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4	v_1v_4	v_2v_4			v_5v_2		
3	v_1v_2	v_2v_7		v_4v_7	v_5v_7		
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1	v_1v_6	v_2v_6	v_3v_6	v_4v_6	v_5v_6	v_6v_7	v_7v_3
i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7

5							
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Let G be an ordered graph, $U \quad V(G); xy\ 2\ E(G):\ h_G(xy) > m = \sqrt{\ \ }\ (\ G)jUj:$
Then $9z; w\ 2\ V(G)\ n\ U:$ xyzw is an increasing path

Our new ingredients

i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
5	v_1v_5						
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i \ v	v_1	v_2	v_3	v_4	v_5	v_6	v_7
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Lemma (Dropping lemma)

Let G be an ordered graph, $U \subseteq V(G)$, $\forall xy \in E(G): h_G(xy) \geq m = \sqrt{\chi(G)|U|}$. Then $\exists z, w \in V(G) \setminus U$: $xyzw$ is an increasing path and

$$h_{G-U}(zw) \geq h_G(xy) - m.$$

Regularisation lemma

Question

Given a graph G with average degree d can we find an almost regular subgraph whose degree is only slightly smaller than d?

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Every graph G has a (possibly non-induced) subgraph whose all degrees in the range $[d', 2d']$, where $d' \geq d(G)/\log n$.

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Given a graph G with average degree d can we find an almost regular subgraph whose degree is only slightly smaller than d?

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Every graph G has a (possibly non-induced) subgraph whose all degrees in the range $[d', 2d']$, where $d' \geq d(G)/\log n$.

Remark: Let $\varepsilon > 0$; then there exists an n vertex graph G with average degree $d(G) = n^{\varepsilon}$ for which this result is tight up to a constant factor.

Proof strategy

Theorem

Let G be an ordered graph, $e \in E(G)$ an edge with $h_G(e) > a$. Then there is an increasing path P starting with e, having length at least

$$a^{1-1/t}/(\log n)^{2t},$$

such that $h_G(f) \geq h_G(e) - a$ for every $f \in E(P)$.

Proof strategy

Theorem

Let G be an ordered graph, $e \in E(G)$ an edge with $h_G(e) \geq a$. Then there is an increasing path P starting with e, having length at least

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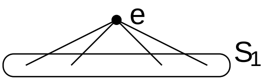
- We assume the theorem is true with paths of length $a^{2/3}$.
- We find a dense *almost regular* subgraph H of G among extenders of e.
- We find a long increasing path within H.

Finding a dense almost regular subgraph of extenders

Finding a dense almost regular subgraph of extenders

Let S_1 be the set of $a = \log n$ highest extenders of e:

Finding a dense almost regular subgraph of extenders



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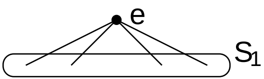

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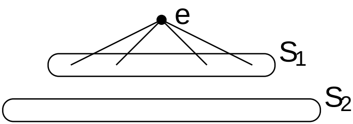

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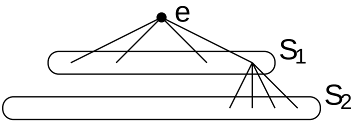

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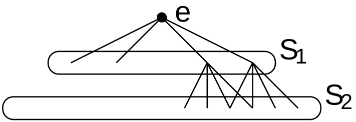

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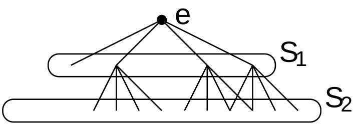

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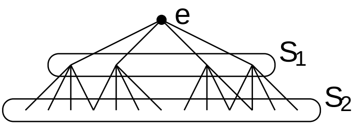

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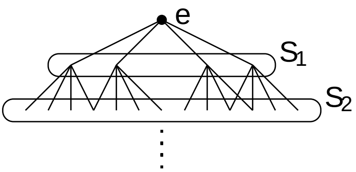
Finding a dense almost regular subgraph of extenders



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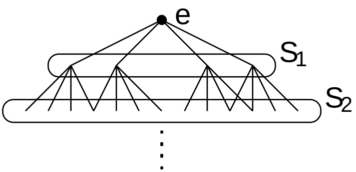

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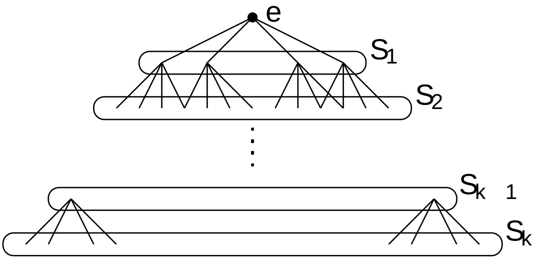

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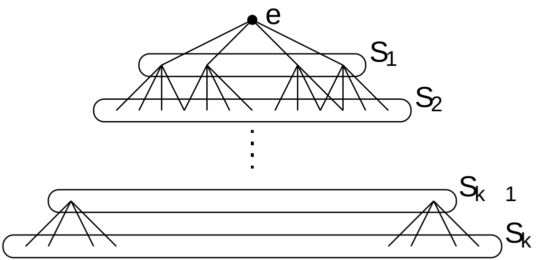

$|S_k| \le 2|S_{k-1}|$

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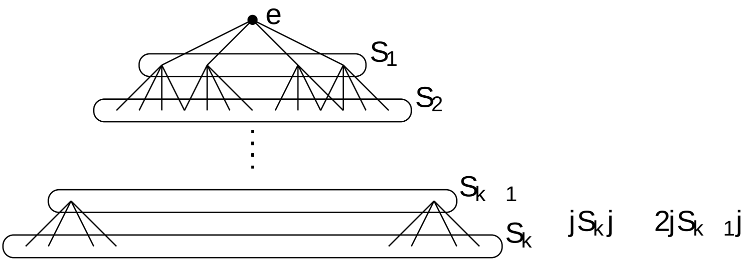

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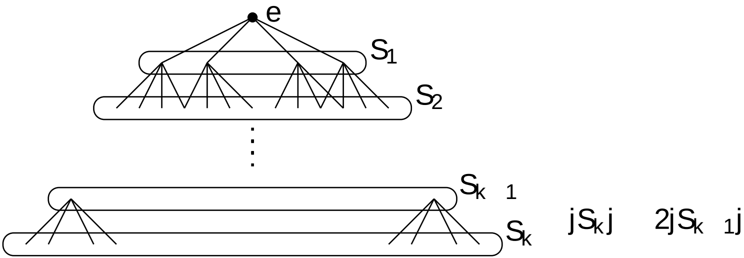

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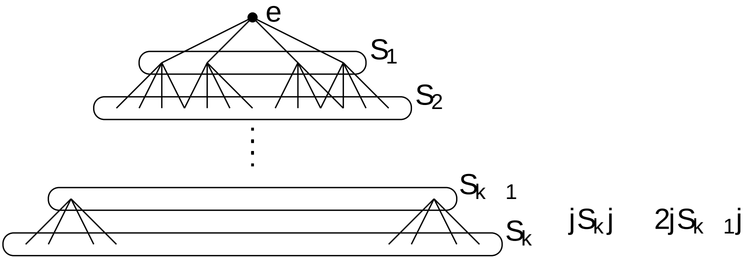
Finding a dense almost regular subgraph of extenders



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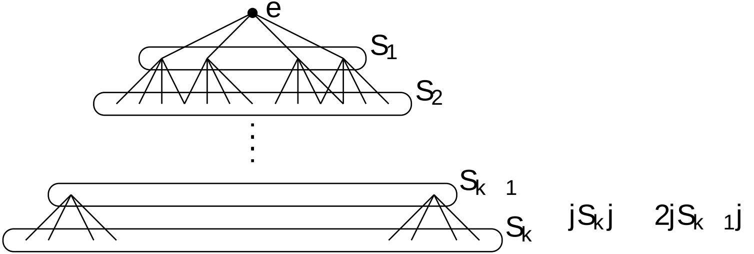
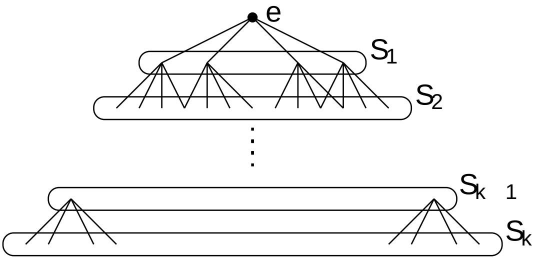
Consider the subgraph G' of G induced by $S_{k-1} \cup S_k$. By construction every vertex in S_{k-1} has degree at least $a = \log n$ in G'.



Let S_1 be the set of $a/\log n$ highest extenders of e. Let S_i be the collection of $a/\log n$ highest extenders of any edge in S_{i-1}.
Let k be the smallest index such that $|S_i| \le 2|S_{i-1}|$. Notice that $k \le \log n$.
Consider the subgraph G' of G induced by $S_{k-1} \cup S_k$. By construction every vertex in S_{k-1} has degree at least $a/\log n$ in G'. Therefore,

$$d(G') \ge a/(6 \log n).$$

Finding a dense almost regular subgraph of extenders



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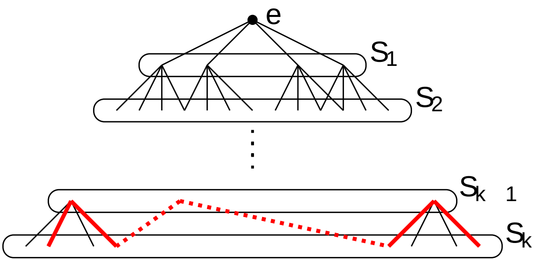
$$d(G^0) \quad a = (6 \log n).$$

Apply regularisation lemma to get an almost regular subgraph H of G^0:

$$a = (6 \log n)^2 \quad d(H) \quad (H) \quad 2d(H)$$

Finding a dense almost regular subgraph of extenders



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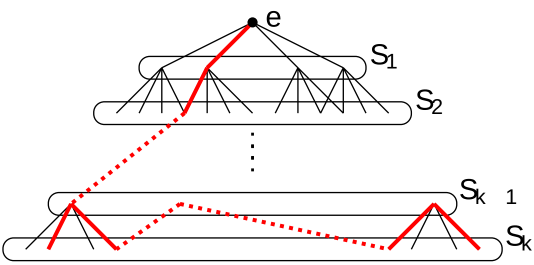
Consider the subgraph G' of G induced by $S_{k-1} \cup S_k$. By construction every vertex in S_{k-1} has degree at least $a/\log n$ in G'. Therefore,

$$d(G') \geq a/(6\log n).$$

Apply regularisation lemma to get an almost regular subgraph H of G'.

$$a/(6\log n)^2 \leq d(H) \leq \Delta(H) \leq 2d(H)$$

Finding a dense almost regular subgraph of extenders



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Finding long increasing paths in almost regular dense g

Apply induction within H using only top $a^{3/4}$ rows.

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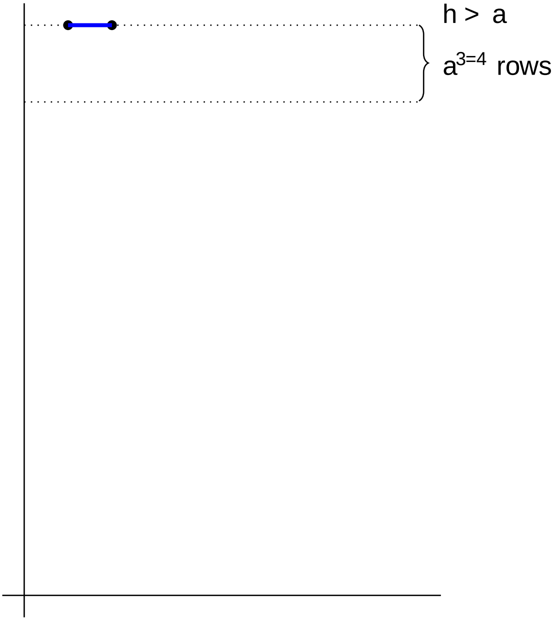
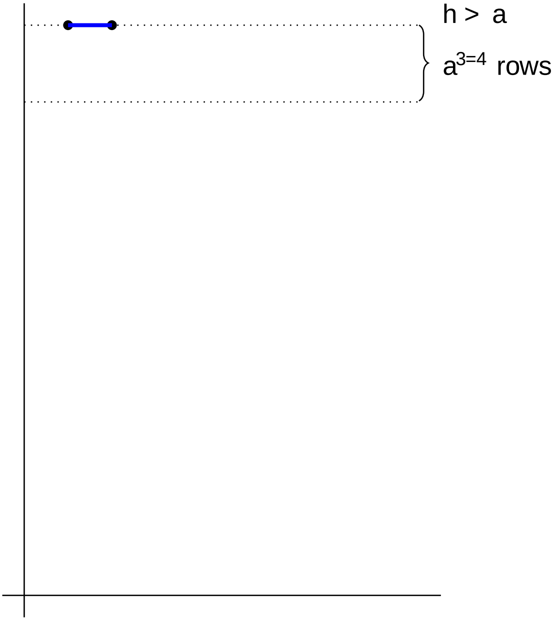


Apply induction within H using only top $a^{3=4}$ rows.
We get an increasing path of length $(a^{3=4})^{2=3} = a^{1=2}$:



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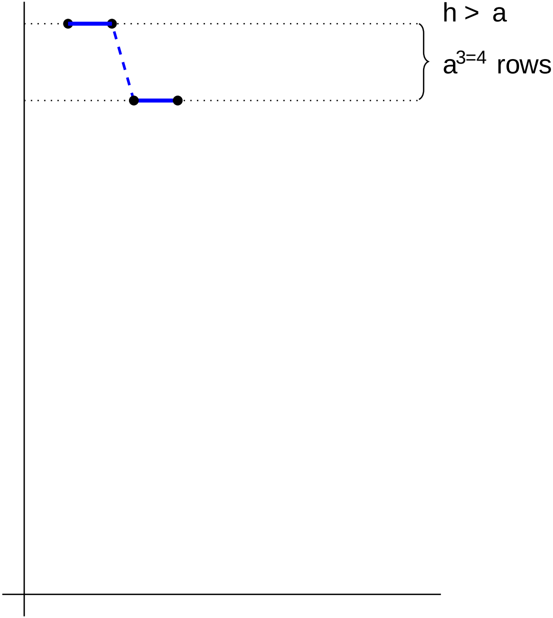

Finding long increasing paths in almost regular dense g

Apply induction within H using only top $a^{3=4}$ rows.
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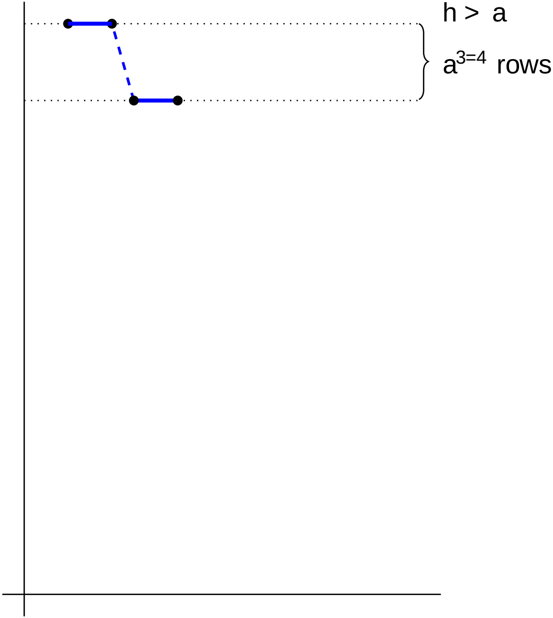
Remove all but its last two vertices.



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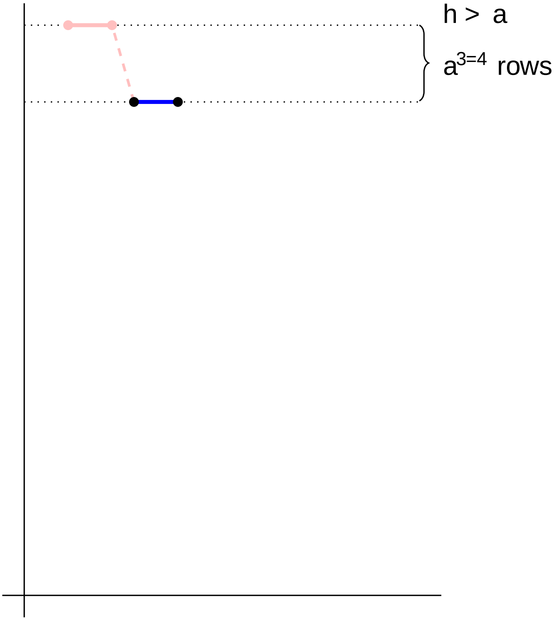

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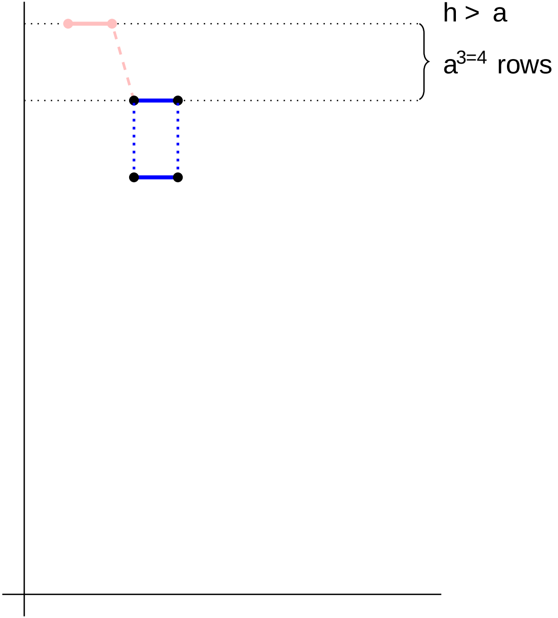

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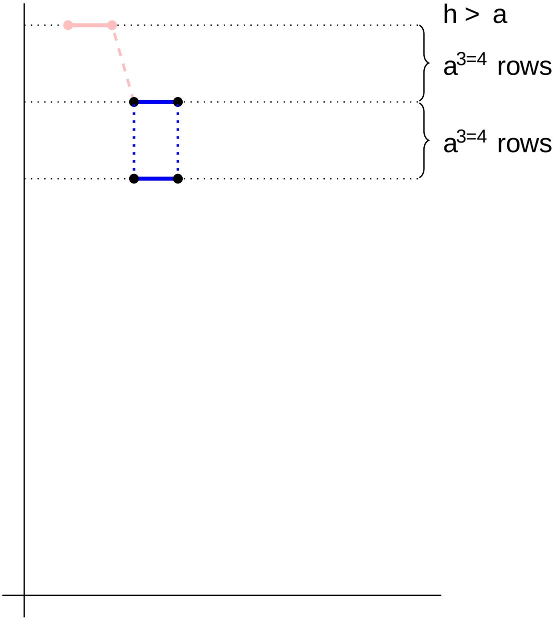

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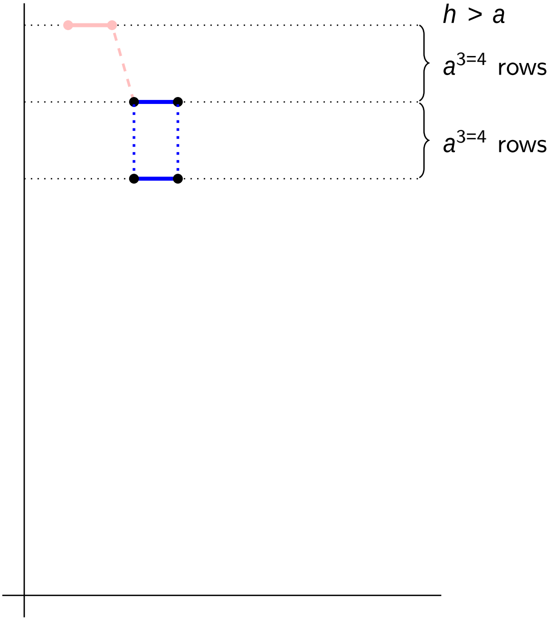

Finding long increasing paths in almost regular dense graphs

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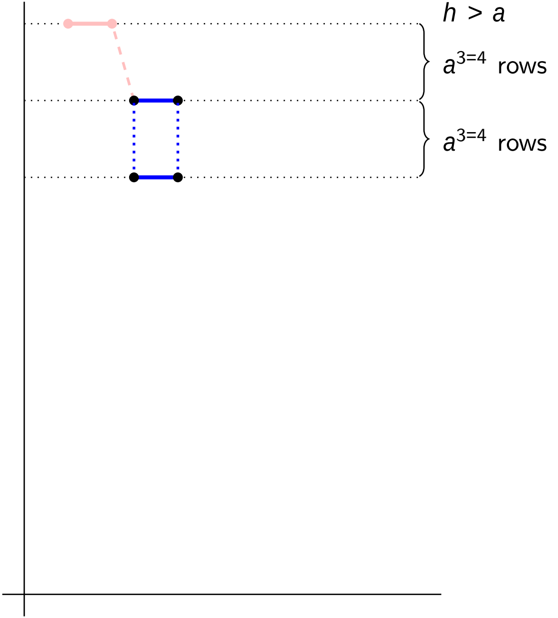

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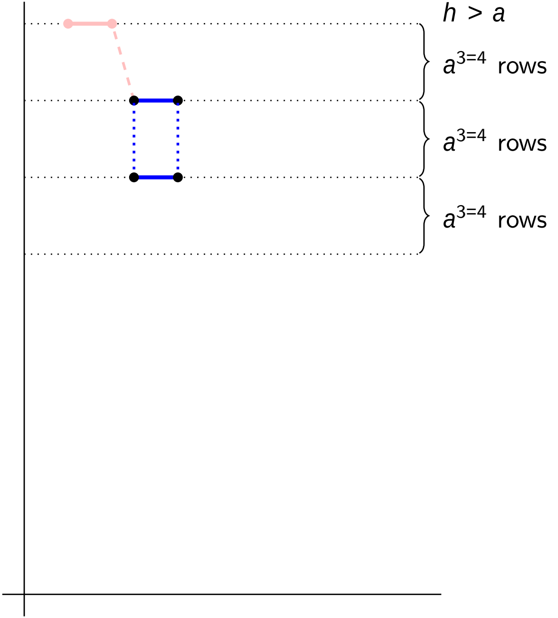

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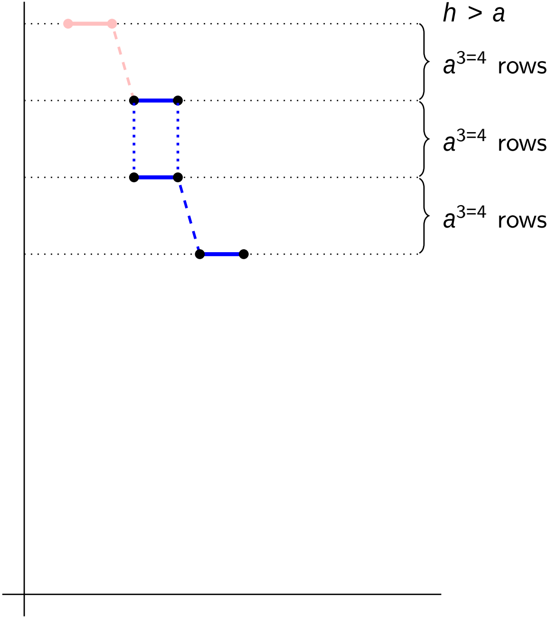

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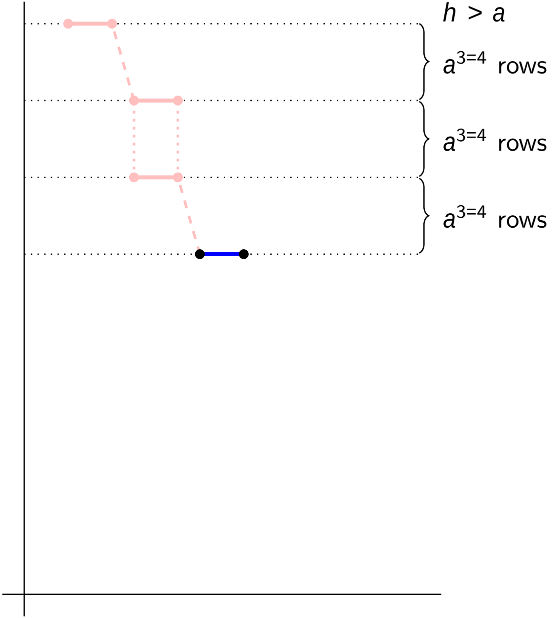

Finding long increasing paths in almost regular dense graphs

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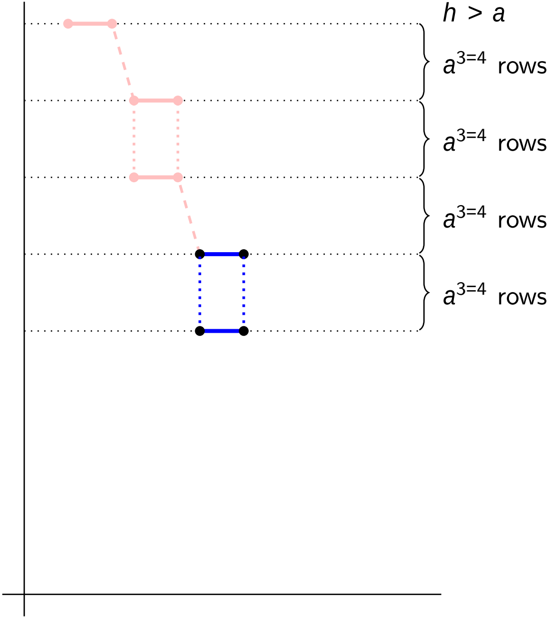

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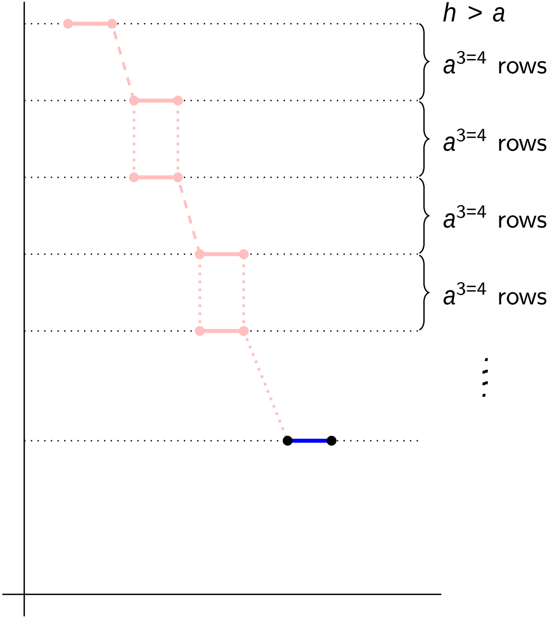

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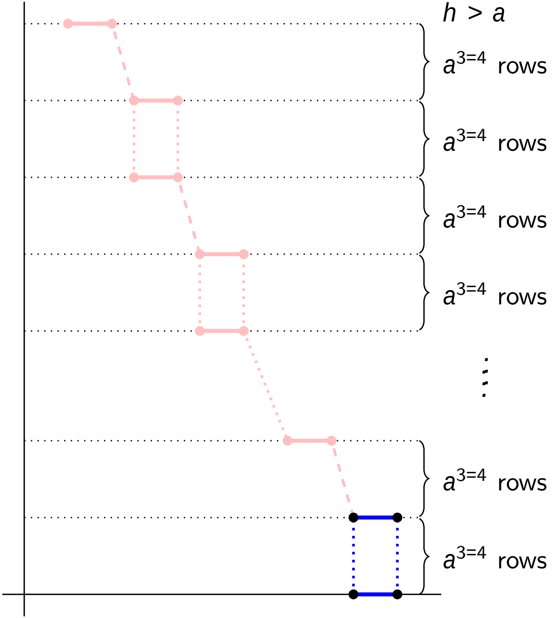

Finding long increasing paths in almost regular dense graphs

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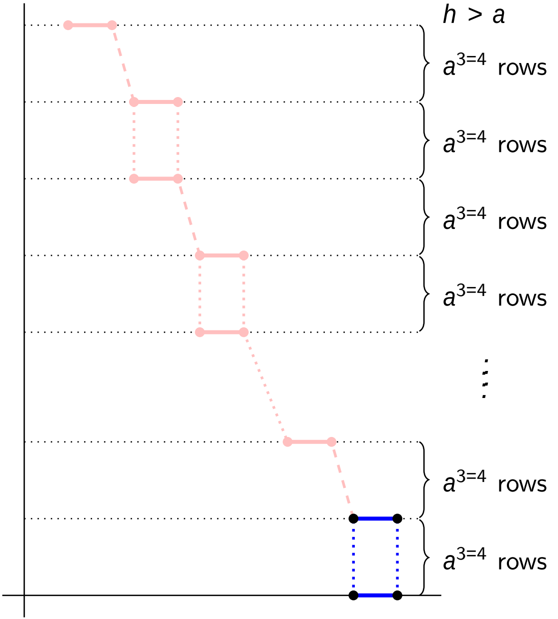
Repeat a=$a^{3=4} = a^{1=4}$ times to obtain a path of length $a^{1=4} \quad a^{1=2} = a^{3=4}:$



Concluding remarks

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Can one improve the bound of $\Omega(\sqrt{d})$ for increasing paths in n vertex graphs with average degree d when d is very small compared to n?

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Proposition

Let G be an edge-ordered graph with average degree d, such that every set of at most εd vertices induces at most $(1/2 - \varepsilon)d$ edges. Then G has an increasing path of length εd.