Super-pancyclic hypergraphs and bipartite graphs

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The incidence graph $I = I(\mathcal{H})$ is the bipartite graph with parts $V(\mathcal{H})$ and $E(\mathcal{H})$ where $ve \in E(I)$ iff $v \in e$ in $\mathcal{H}$.

Then $\mathcal{H}$ has a cycle of length k iff I has a cycle of length $2k$.

Jackson's results and conjecture

For integers n, m, and δ with $\delta \leq m$, let $\mathcal{G}(n, m, \delta)$ be the set of all bipartite graphs with partition (X, Y) s. t. $|X| = n \geq 2, |Y| = m$ and for every $x \in X$, $d(x) \geq \delta$.

Theorem 1 [Jackson, 1981]: If a graph $G \in \mathcal{G}(n, m, \delta)$ satisfies $n \leq \delta$ and $m \leq 2\delta - 2$, then it contains a cycle of length $2n$, i.e., a cycle that covers X.

The bound $m \leq 2\delta - 2$ is exact.

Examples

Example 1 : For $\delta = n$, let $G_1(n) \in \mathcal{G}(\delta, 2\delta - 1, \delta)$ be obtained from a copy of $K_{\delta,\delta-1}$ where every vertex in X has an additional neighbor of degree 1.

Example 2 : Fix $a \geq b > 0$ such that $a + b = n$. Let $G_2(a, b) \in \mathcal{G}(n, 2\delta - 1, \delta)$ be the bipartite graph obtained from a copy H_1 of $K_{a,\delta}$ and a copy H_2 of $K_{b,\delta}$ by gluing together a vertex of H_1 in a part of size δ with a vertex of H_2 in a part of size δ.

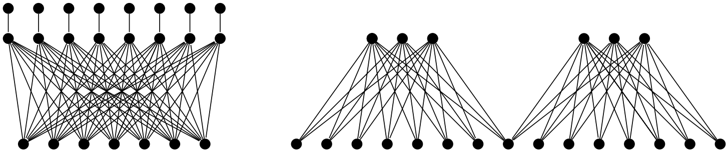

Figure: Examples 1 and 2.

A conjecture

Conjecture 1 [Jackson, 1981]: If some $G \in \mathcal{G}(n, m, \delta)$ is 2-connected and (i) $m \le 3\delta - 5$ if $n \le \delta$, or (ii) $m \le \lfloor \frac{2(n-\alpha)}{\delta-1-\alpha} \rfloor (\delta - 2) + 1$ if $n \ge \delta$, where $\alpha = 1$ if δ is even and $\alpha = 0$ if δ is odd, then G has a cycle of length $2\min(n, \delta)$.

Example 3 : For (i), fix positive integers $n_1 \ge n_2 \ge n_3$ such that $n_1 + n_2 + n_3 = n$. Let $G_3(n_1, n_2, n_3) \in \mathcal{G}(n, 3\delta - 4, \delta)$ be the bipartite graph obtained from $K_{\delta-2,n_1} \cup K_{\delta-2,n_2} \cup K_{\delta-2,n_3}$ by adding two vertices a and b that are both adjacent to every vertex in the parts of size n_1, n_2, and n_3.

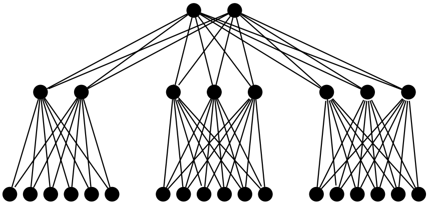

Figure: Example 3.

Our results

Theorem 2 [A. K., R. Luo and D. Zirlin]: Suppose $n \leq \delta \leq m \leq 3\delta - 5$. If $G \in \mathcal{G}(n, m, \delta)$ is 2-connected, then G contains a cycle of length $2n$, i.e., a cycle that covers X.

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Theorem 3 [A. K., R. L. and D. Z.]: Let $\delta \geq n$ and $m \leq 2\delta - 1$. If $G \in \mathcal{G}(n, m, \delta)$ does not contain a cycle of length $2n$, then either $G = G_1(n)$ in Example 1 or $G = G_2(a, b)$ for some a and b with $a + b = n$ in Example 2.

More examples

Example 4 : Let $V(\mathcal{H}) = V_1 \cup V_2$ where $|V_1| = \lfloor (n+1)/2 \rfloor$, $|V_2| = \lceil (n+1)/2 \rceil$, $V_1 \cap V_2 = \{v\}$, and let $E(\mathcal{H})$ consist of all sets of size $n/4$ contained either in V_1 or in V_2. Then this $n/4$-uniform hypergraph has an exponential in n minimum degree and no Hamiltonian cycle

Example 5 : Let $V(\mathcal{H}) = V_1 \cup V_2$ where $|V_1| = \lceil (n+2)/2 \rceil$, $|V_2| = \lfloor (n-2)/2 \rfloor$, $V_1 \cap V_2 = \emptyset$, and let $E(\mathcal{H}) = E_1 \cup E_2$, where E_1 is the set of all subsets A of $V(\mathcal{H})$ of size $\lceil n/4 \rceil$ such that $|V_1 \cap A| = 1$ (and $|V_2 \cap A| = \lceil n/4 \rceil - 1$), and $E_2 = \{V_1\}$. Then $\mathcal{H}$ has an exponential in n minimum degree, high connectivity and positive codegree of each pair of the vertices. But again, $\mathcal{H}$ has no Berge hamiltonian cycle.

Our results

Translating Theorems 2 and 3 into the language of hypergraphs, we get

Theorem 2* [A. K., R. Luo and D. Zirlin]: Suppose $n \leq \delta \leq m \leq 3\delta - 5$. If $\mathcal{H}$ is a 2-connected n-vertex hypergraph with m edges and minimum degree at least δ, then $\mathcal{H}$ has a hamiltonian Berge cycle.

Theorem 3* [A. K., R. L. and D. Z.]: Let $\delta \geq n$ and $m \leq 2\delta - 1$. If an n-vertex hypergraph $\mathcal{H}$ with m edges and minimum degree at least δ has no hamiltonian Berge cycle, then the incidence graph $I(\mathcal{H})$ is either $G_1(n)$ in Example 1 or $G_2(a, b)$ for some a and b with $a + b = n$ in Example 2.

Super-pancyclic hypergraphs

A hypergraph $\mathcal{H}$ is super-pancyclic if for every $A \subseteq V(\mathcal{H})$ with $|A| \geq 3$, $\mathcal{H}$ has a Berge cycle whose set of base vertices is A.

A bipartite graph G with partition (X, Y) is X-super-pancyclic if for every $X' \subseteq X$ with $|X'| \geq 3$, G has a cycle C with $V(C) \cap X = X'$.

Theorem 4 [Hypergraph version of Jackson's Theorem]: Suppose $\delta \geq n$ and $\delta \geq (m+2)/2$. Then every n-vertex hypergraph with m edges and minimum degree at least δ is super-pancyclic.

For a set $A \subset X$ in an X, Y-bigraph G, let $N_2(A) = \{y \in Y : |N(y) \cap A| \geq 2\}$.

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Every X-super-pancyclic bipartite graph satisfies:

$$\textit{For each } A \subseteq X \textit{ with } |A| \geq 3,\ |N_2(A)| \geq |A|; \tag{1}$$

and

$$\textit{For each } A \subseteq X \textit{ with } |A| \geq 3,\ G[A \cup N_2(A)] \textit{ is 2-connected.} \tag{2}$$

Theorem 5 [A. K., R. L. and D. Z.]: Let $\delta \geq n$ and $m \leq 3\delta - 5$. If $G \in \mathcal{G}(n, m, \delta)$ satisfies (1) and (2), then G is X-super-pancyclic.

Theorem 6 [A. K., R. L. and D. Z.]: Let $\delta \geq n$ and $m \leq 3\delta - 5$. If the incidence graph of an n-vertex hypergraph $\mathcal{H}$ with m edges and minimum degree $\delta(\mathcal{H})$ satisfies (1) and (2), then $\mathcal{H}$ is super-pancyclic.

Theorem 7 [M. Lavrov, A. K., R. L. and D. Z.]: Let $\delta \geq n$ and $n \leq 6$. If $G \in \mathcal{G}(n, m, \delta)$ satisfies (1) and (2), then G is X-super-pancyclic.