Polynomial to Exponential transition in Ramsey theory

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(joint work with Alexander Razborov)

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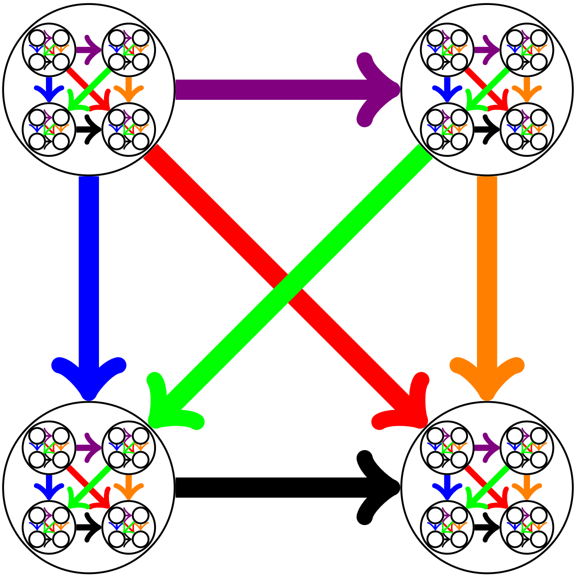

Ramsey theory for hypergraphs

Definition (Ramsey's theorem)

Given $k \geq 2$ and k-uniform hypergraphs H_1, H_2, the ramsey number

$$r(H_1, H_2)$$

is the minimum N such that every red/blue coloring of the k-sets of $[N]$ results in a red copy of H_1 or a blue copy of H_2. Write

$$r_k(s, n) := r(K_s^k, K_n^k).$$

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Observation

Note that $r_k(s, n) \leq N$ is equivalent to saying that every N-vertex K_s^k-free k-uniform hypergraph H has $\alpha(H) \geq n$.

Graphs

Theorem (Spencer 1977, Conlon 2008)

$$(1+o(1))\frac{\sqrt{2}}{e}n2^{n/2} < r_2(n,n) < \frac{4^n}{n^{c\log n/\log\log n}}$$

Theorem (Ajtai-Komlós-Szemerédi 1980, Kim 1995, sharper results by Shearer, Bohman-Keevash, Fiz Pontiveros-Griffiths-Morris)

$$r_2(3,n) = \Theta\left(\frac{n^2}{\log n}\right)$$

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Theorem

For fixed $s \geq 3$

$$n^{(s+1)/2+o(1)} < r_2(s,n) < n^{s-1+o(1)}$$

Pseudorandom Ramsey Graphs

Definition (Alon?)

An (n, d, λ) graph *is an* n*-vertex* d*-regular graph such that the absolute value of every eigenvalue of its adjacency matrix, besides the largest one, is at most* λ*.*

Conjecture (Sudakov-Szabo-Vu 2005)

For each fixed $s \geq 3$*, there exist "optimal"* K_s*-free* (n, d, λ) *graphs. I.e., graphs containing no* K_s *with*

$$d = \Omega(n^{1-\frac{1}{2s-3}}) \qquad \text{and} \qquad \lambda = O(\sqrt{d}\,).$$

Pseudorandom Ramsey Graphs

Theorem (M-Verstraëte 2019)

Let d, n, N be positive integers and $n = \lceil 2N(\log N)^2/d \rceil$. If there exists an F-free (N, d, λ)-graph and N is large enough, then

$$r(F, n) = \Omega\left(\frac{N}{\lambda}(\log N)^2\right).$$

Corollary (M-Verstraëte 2019)

If K_s-free (N, d, λ)-graphs exist with $d = \Omega(N^{1-\frac{1}{2s-3}})$ and $\lambda = O(\sqrt{d})$, then as $n \to \infty$,

$$r(s, n) = \Omega\Big(\frac{n^{s-1}}{(\log n)^{2s-4}}\Big).$$

Hypergraphs - diagonal case

Definition (tower function)

$$\mathrm{twr}_1(x) = x \qquad \textit{and} \qquad \mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}.$$

Theorem (Erdős-Hajnal-Rado 1952/1965)

$$2^{cn^2} < r_3(n,n) < 2^{2^n}$$

For fixed $k \geq 3$,

$$\mathrm{twr}_{k-1}(cn^2) < r_k(n,n) < \mathrm{twr}_k(c'n)$$

Conjecture (Erdős \$500)

$$r_3(n,n) > 2^{2^{cn}}.$$

Hypergraphs - The off-diagonal conjecture

Conjecture (Erdős-Hajnal 1972)

For fixed $s > k \geq 3$ we have $r_k(s,n) > \mathrm{twr}_{k-1}(cn)$. In particular,

$$r_k(k+1,n) > \mathrm{twr}_{k-1}(cn).$$

$$r_3(s,n) \geq r_3(4,n) > 2^{cn}$$

$$r_4(s,n) \geq r_4(5,n) > 2^{2^{cn}}$$

$$r_5(s,n) \geq r_5(6,n) > 2^{2^{2^{cn}}}$$

Hypergraphs - The off-diagonal conjecture

Theorem (Erdős-Hajnal 1972)

$r_3(4, n) > 2^{cn}$. *Consequently, the conjecture holds for* $k = 3$.

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Theorem (Erdős-Hajnal 1972)

$r_3(4,n) > 2^{cn}$. *Consequently, the conjecture holds for* $k = 3$.

Proof. Let T be a random graph tournament on N vertices and form a 3-uniform hypergraph H by making each cyclically oriented triangle a hyperedge. Then

- there is no $K_4^{(3)}$ in H (even no K_4^{3-}), and yet
- the independence number of H is $n = O(\log N)$. □

The off-diagonal conjecture - almost solved

Theorem (M-Suk 2017, Conlon-Fox-Sudakov unpublished)

The off-diagonal conjecture holds for all $s \geq k+3$:

$$r_k(k+3, n) > \mathrm{twr}_{k-1}(cn).$$

The open cases are $r_4(5, n)$ and $r_4(6, n)$ and their k-uniform counterparts.

$r_4(5, n)$ and $r_4(6, n)$

Lower bounds for $r_4(5, n)$:

- 2^{cn} (implicit in Erdős-Hajnal 1972)
- 2^{cn^2} (M-Suk 2017)
- $2^{n^{c \log \log n}}$ (M-Suk 2018)
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Lower bounds for $r_4(6, n)$:

- 2^{cn} (implicit in Erdős-Hajnal 1972)
- $2^{n^{c \log n}}$ (M-Suk 2017)
- $2^{2^{cn^{1/5}}}$ (M-Suk 2018)

The off-diagonal conjecture - almost solved

Theorem (M-Suk 2018)

$$r_4(5,n) > 2^{n^{c \log n}} \qquad r_4(6,n) > 2^{2^{cn^{1/5}}}$$

and for fixed $k \geq 4$

$$r_k(k+1,n) > \mathrm{twr}_{k-2}(n^{c \log n})$$

$$r_k(k+2,n) > \mathrm{twr}_{k-1}(cn^{1/5})$$

The Erdős-Hajnal Hypergraph Ramsey Problem

Definition (Erdős-Hajnal 1972)

For $1 \leq t \leq \binom{s}{k}$, let $r_k(s, t; n)$ be the minimum N such that every red/blue coloring of the k-sets of $[N]$ results in an s-set that contains at least t red k-subsets or an n-set all of whose k-subsets are blue (i.e., a blue K_n^k).

Example

$$r_k\left(s, \binom{s}{k}; n\right) = r_k(s, n)$$

The Erdős-Hajnal Hypergraph Ramsey Problem

Problem (Erdős-Hajnal 1972)

As t grows from 1 to $\binom{s}{k}$, there is a well-defined value $t_1 = h_1^{(k)}(s)$ at which $r_k(s, t_1 - 1; n)$ is polynomial in n while $r_k(s, t_1; n)$ is exponential in a power of n, another well-defined value $t_2 = h_2^{(k)}(s)$ at which it changes from exponential to double exponential in a power of n and so on, and finally a well-defined value $t_{k-2} = h_{k-2}^{(k)}(s) < \binom{s}{k}$ at which it changes from twr_{k-2} to twr_{k-1} in a power of n.

A Recursive Definition

Definition

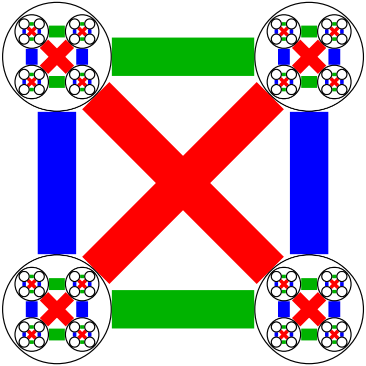
Let $g_k(s) = 0$ *for* $s < k$, $g_k(k) = 1$, *and for* $s > k$, *let* $g_k(s)$ *be the maximum of*

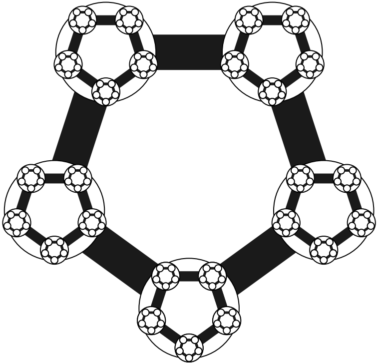
$$\sum_{i=1}^{k} g_k(s_i) + \prod_{i=1}^{k} s_i$$

where we maximize over all partitions $s = s_1 + \cdots + s_k$ *with* $s_i < s$ *for all* i.

$$g_k(s) = (1 + o(1)) \frac{k!}{k^k - k} \binom{s}{k} \qquad (k \text{ is fixed, } s \to \infty).$$

Recursion and Fractals[1]





$$g_4(s) \sim \frac{2}{21}\binom{s}{4} \qquad g_5(s) \sim \frac{1}{26}\binom{s}{5}$$

[1] Thanks to Bernard Lidický for pictures!

Polynomial to Exponential Transition

Theorem (Erdős-Hajnal)

$$h_1^{(k)}(s) \geq g_k(s) + 1 \qquad (s \geq k \geq 3).$$

In other words: every N-vertex k-uniform hypergraph H in which every s vertices span at most $g_k(s) - 1$ edges has

$$\alpha(H) > N^{\epsilon} \qquad (\epsilon = \epsilon(s,k) > 0).$$

Polynomial to Exponential Transition

Conjecture (Erdős-Hajnal 1972 $500)

$$h_1^{(k)}(s) = g_k(s) + 1 \qquad (s \geq k \geq 3).$$

In other words: there exists $C = C(k) > 0$ and, for all $N > k$, an N-vertex k-uniform hypergraph H in which every s vertices span at most $g_k(s)$ edges and

$$\alpha(H) \leq C \log N.$$

The smallest nontrivial case

$k = 3, s = 4$

Theorem (Phelps-Rödl 1986)

$$r_3(4, 2; n) < cn^2 / \log n$$

Theorem (Erdős-Hajnal 1972)

$$r_3(4, 3; n) > 2^{c'n}$$

$$h_1^{(3)}(4) = 3 = g_3(4) + 1$$

Polynomial to Exponential Transition

Theorem (Conlon-Fox-Sudakov 2010)

$h_1^{(3)}(s) = g_3(s) + 1$ *for many s values including powers of 3; also*

$$h_1^{(3)}(s) = \frac{1}{4}\binom{s}{3} + O(s \log s).$$

Proof Idea: $T(s)$ is the maximum number of directed triangles in all s-vertex tournaments. Then, if s is a power of 3,

$$h_1^{(3)}(s) - 1 \leq T(s) = \frac{1}{4}\binom{s+1}{3} = g_3(s).$$

Lucky: the maximizers for $T(s)$ are out regular tournaments, and the "recursive" tournament is just one example.

Polynomial to Exponential Transition

Theorem (M-Razborov 2019)

$$h_1^{(k)}(s) = g_k(s) + 1 \qquad (s \geq k \geq 4).$$

i.e., there exists $C = C(k) > 0$ and, for all $N > k$, an N-vertex k-uniform hypergraph H in which every s vertices span at most $g_k(s)$ edges and

$$\alpha(H) \leq C \log N.$$

Main Hurdle: The recursive definition of $g_k(s)$ – seems impossible to avoid it!!

Inducibility

Definition

Given a k-vertex graph R, the inducibility *$i(R)$ is*

$$i(R) \stackrel{\text{def}}{=} \lim_{s \to \infty} \max_{|V(H)|=s} \frac{i(R;H)}{\binom{s}{k}},$$

where $i(R;H)$ is the number of induced copies of R in an s-vertex graph H.

Golumbic-Pippenger

Conjecture (Golumbic-Pippenger 1975)

$$i(C_k) = \frac{k!}{k^k - k} \qquad (k \geq 5).$$

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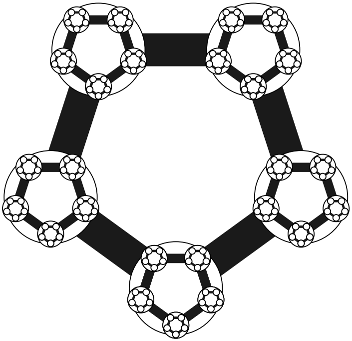
Theorem (Kral-Norin-Volec 2018)

$$i(C_k) \leq \frac{2k!}{k^k} \qquad (k \geq 5).$$

Golumbic-Pippenger

Theorem (Balogh-Hu-Lidický-Pfender 2016)

$$i(C_5) = \frac{1}{26} \qquad \left(= \frac{5!}{5^5 - 5} \right).$$

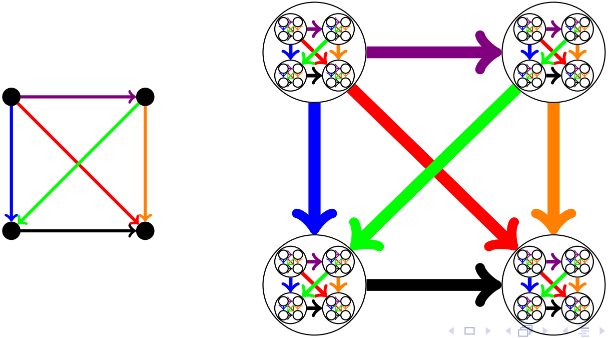


Rich Structures

Theorem (M-Razborov 2019)

Let $s \geq k \geq 4$, R be a k-vertex rainbow tournament. For any s-vertex tournament H with edges colored by the same $\binom{k}{2}$ colors,

$$i(R; H) \leq g_k(s) \qquad \left(\Longrightarrow i(R) = \frac{k!}{k^k - k}\right).$$



Proof of Erdős-Hajnal conjecture

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I.e. there exists $C = C(k) > 0$ *and, for all* $N > k$*, an* N*-vertex* k*-uniform hypergraph* H *in which every* s *vertices span at most* $g_k(s)$ *edges and* $\alpha(H) \leq C \log N$*.*

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Proof.

Fix a k-vertex rainbow tournament R. Randomly $\binom{k}{2}$-color and orient K_N (with the same colors from R). Form a k-uniform hypergraph H comprising copies of R. Then

- Every s vertices have at most $g_k(s)$ (hyper)edges
- With positive probability $\alpha(H) = O(\log N)$. □

Intuition

Question

Why might it be easier to prove inducibility results for rainbow/directed structures R than for usual graphs?

- Because of the lack of symmetries
- Research on inducibility is/was hampered by the fact that a vertex can play different roles in a copy of R. E.g. if $R = C_k$
- Previous results of inducibility of random graphs (Yuster, Fox-Huang-Lee) required trivial automorphism group and in fact even stronger "asymmetry" properties
- The rainbow tournament has the (strongest possible) asymmetry properties "for free". E.g. specifying a colored oriented edge identifies its endpoints

Thank You!!!