Coxeter groups, quiver mutations and hyperbolic manifolds

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(joint with Pavel Tumarkin)

"Discrete Subgroups of Lie Groups"

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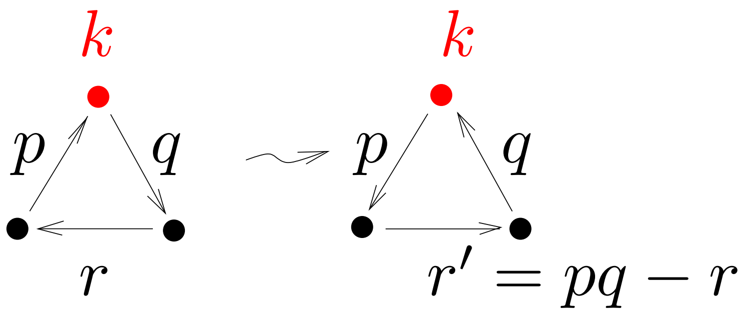
- Mutation μ_k of quivers:
  - reverse all arrows incident to k;
  - for every oriented path through k do



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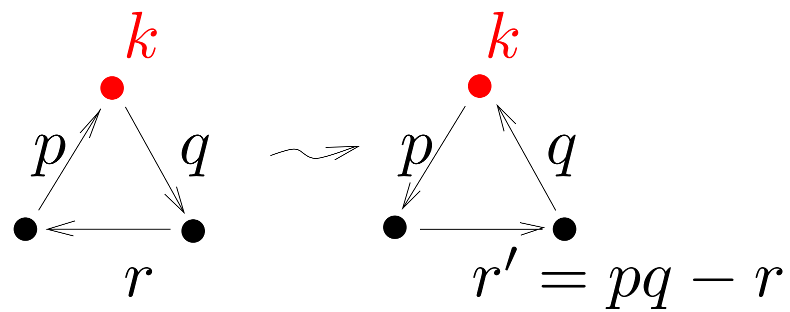

Plan:

Quiver $Q \longrightarrow$

$\longrightarrow$ (Quotient of) Coxeter group $G \longrightarrow$

$\longrightarrow$ Action of G on $X \longrightarrow$

Hyperbolic manifold X/G
with symmetry group G

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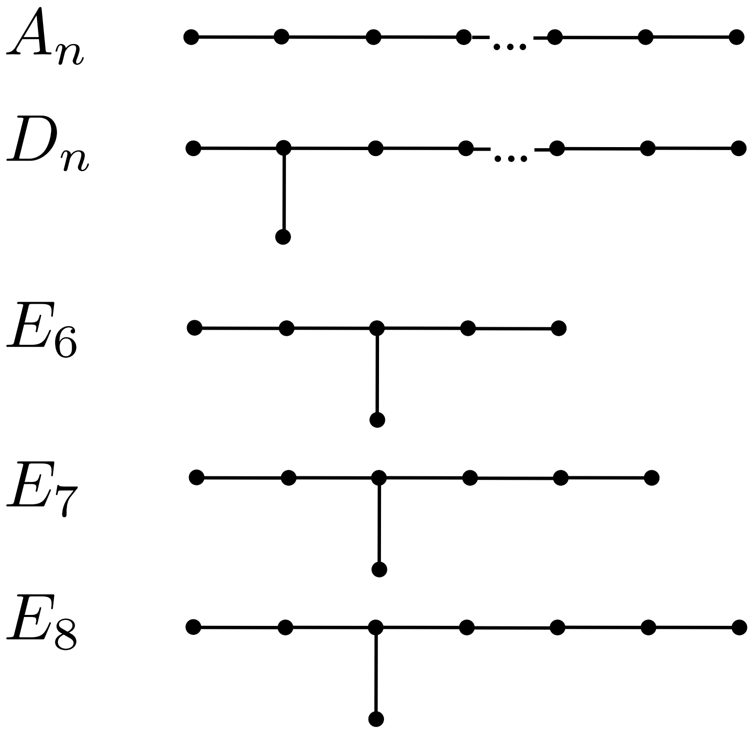

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- Generators of G – nodes of Q.

- Relations of G – (R1) $s_i^2 = e$

  (R2) $(s_i s_j)^{m_{ij}} = e$,

$$m_{ij} = \begin{cases} 2, & \bullet \qquad \bullet \\ 3, & \bullet \!\!-\!\!-\!\!-\!\!-\!\!-\!\! \bullet \\ \infty, & otherwise. \end{cases}$$

  (R3) Cycle relation:

  for each chordless cycle $1 \to 2 \to \cdots \to n \to 1$

  $(s_1 \;\; s_2 s_3 \dots s_n \dots s_3 s_2)^2 = e.$

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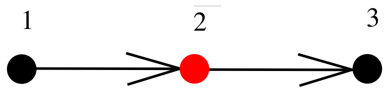
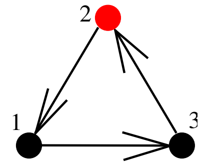
- If $Q_2 = \mu_k(Q_1)$, s_i - generators of $G(Q_1)$, t_i generators of $G(Q_2)$, then

$$t_i = \begin{cases} s_k s_i s_k, & \overset{i}{\bullet} \longrightarrow \overset{k}{\bullet} \text{ in } Q_1 \\ s_i, & \text{otherwise} \end{cases}$$

4. Geometric interpretation.

Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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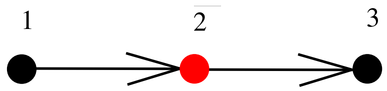
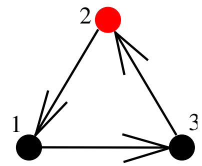
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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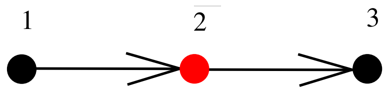
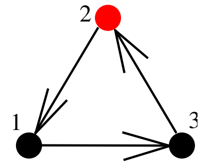
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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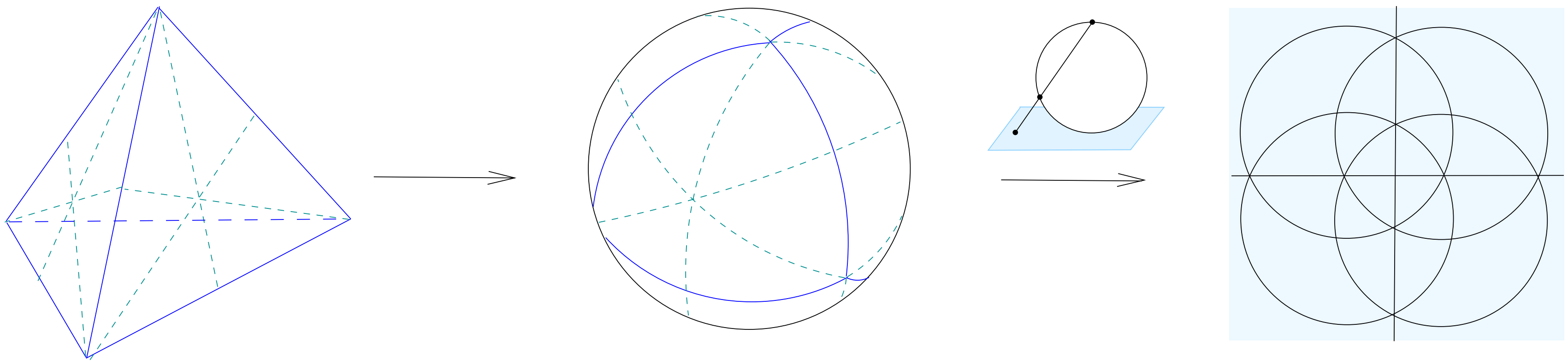
finite Coxeter group A_3, acts on S^2 by reflections, 24 elements:



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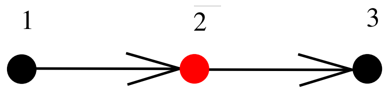
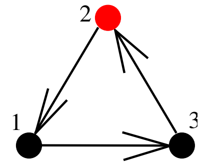
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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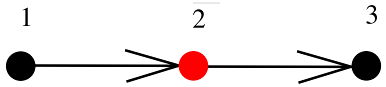
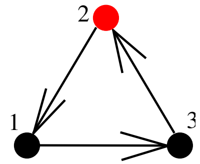
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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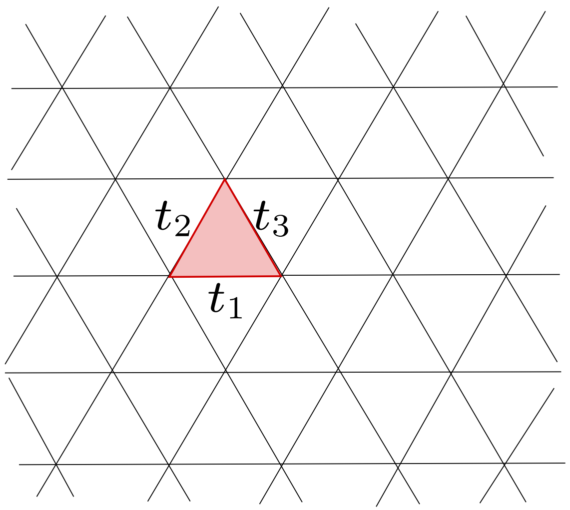

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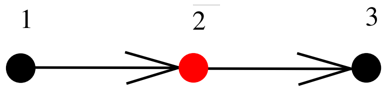
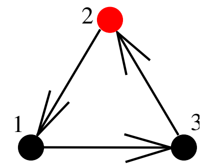
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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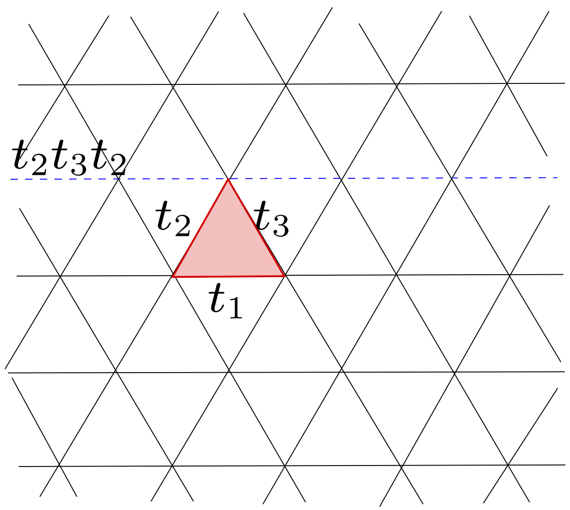

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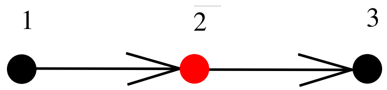
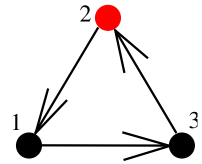
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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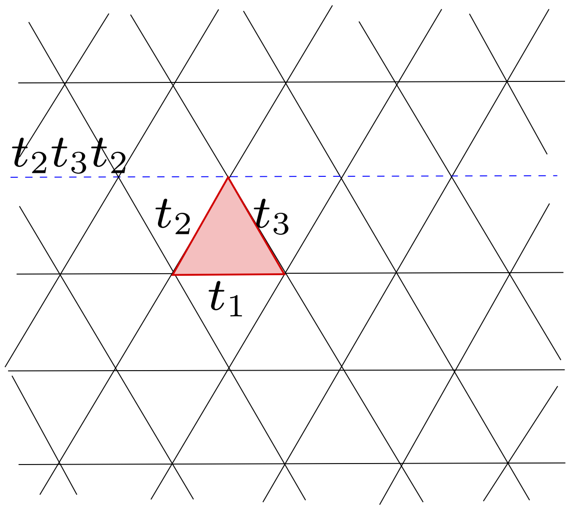

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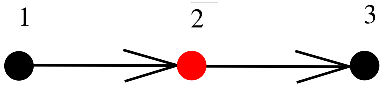
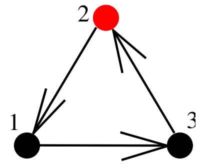
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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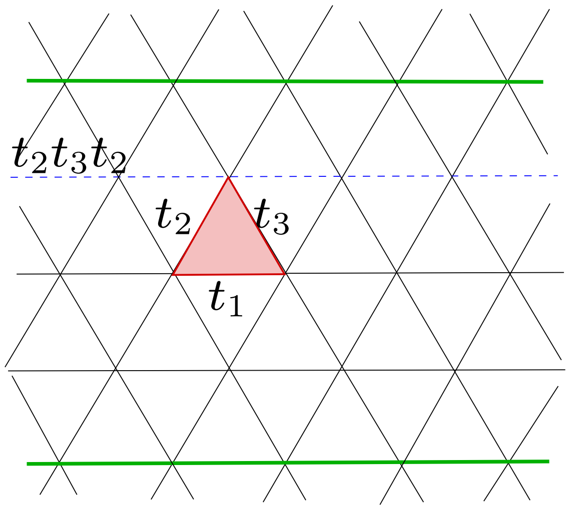

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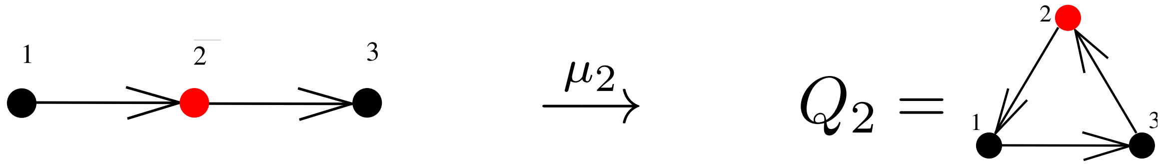
Example: $Q_1 = A_3 =$ 

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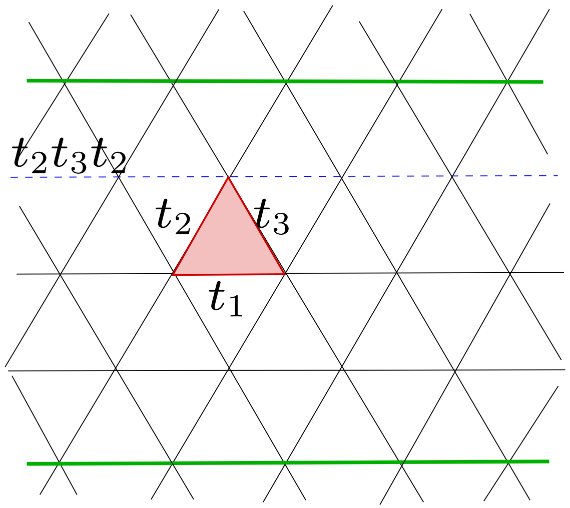

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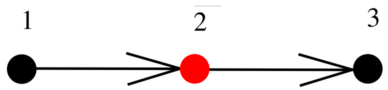
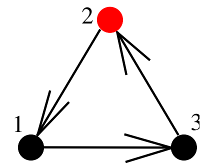
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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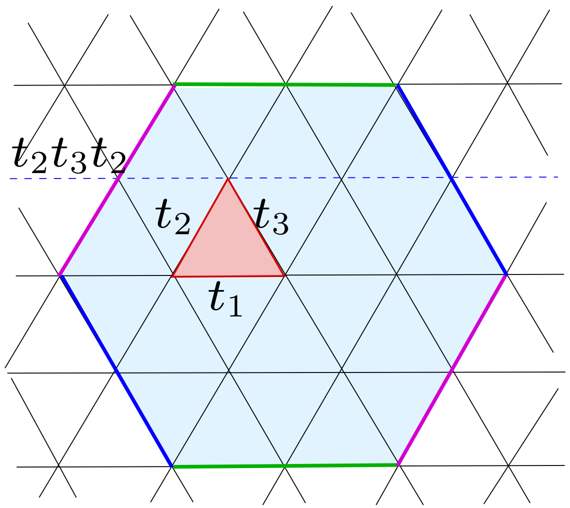

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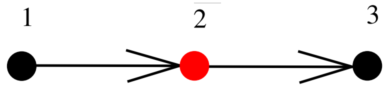
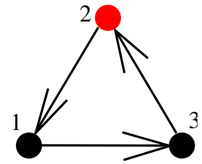
Example: $Q_1 = A_3 =$ 

$\xrightarrow{\mu_2}$ $Q_2 =$ 

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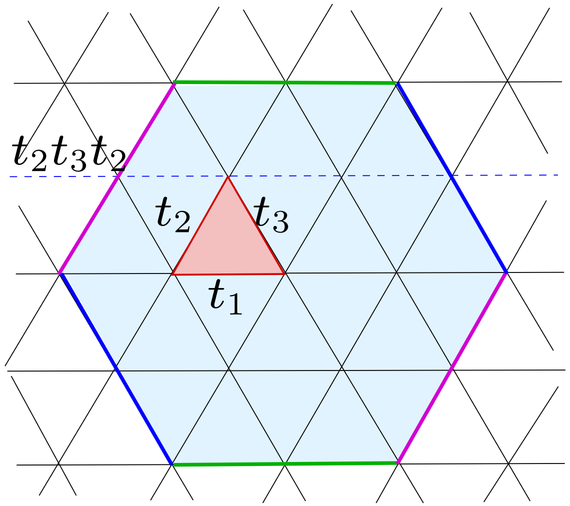

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$G = G(Q_2)$ acts on a torus T^2.

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Theorem 2 [F-Tumarkin'14] (Manifold property)
The group G_{rel} is torsion free,
i.e. if $\Sigma(G_0)$ is a manifold then X is a manifold.

Taking the quotient, we are not introducing any new singularities!

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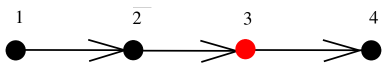
Example: $A_4 =$ 

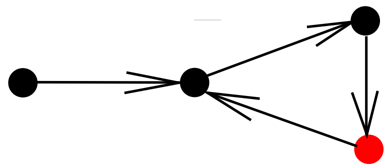
$\xrightarrow{\mu_3}$ 

diagram of hyperbolic simplex

$\Rightarrow$ Hyperbolic 3-manifold with action of the group A_4.

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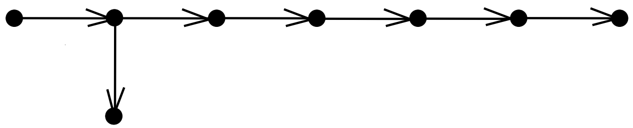
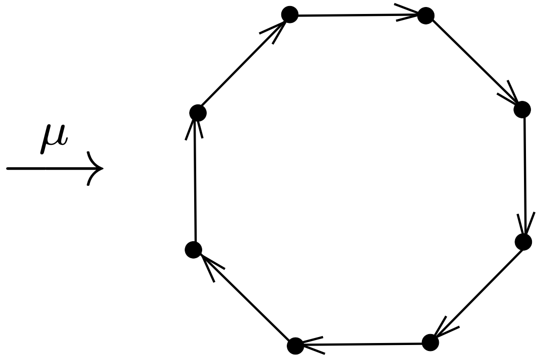
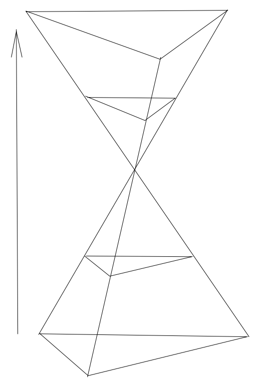
Another example:





$D_n : S^n$ $\qquad$ $\widetilde{A}_{n-1} : \mathbb{E}^{n-1}$

$$G_{rel} = NCl(\ (s_1 \ \ s_2 s_3 \ldots s_n \ldots s_3 s_2)^2)$$



$\mathbb{E}^{n-1}/(n \text{ translations}) = \mathbb{T}^{n-1}$

tiled by simplices

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More hyperbolic examples:

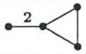
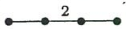
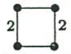
TABLE 5.1. Actions on hyperbolic manifolds.

W	Q	Q_1	$\|W\|$	$\dim X$	vol X approx.	number of cusps	$\chi(X)$
A_4			$5!$	3	$\|W\| \cdot 0.084578$	5	
D_4			$2^3 \cdot 4!$	3	$\|W\| \cdot 0.422892$	16	
D_5			$2^4 \cdot 5!$	4	$\|W\| \cdot 0.013707$	10	2
E_6			$2^7 \cdot 3^4 \cdot 5$	5	$\|W\| \cdot 0.002074$	27	
E_7			$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	6	$\|W\| \cdot 2.962092 \times 10^{-4}$	126	-52
E_8			$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	7	$\|W\| \cdot 4.110677 \times 10^{-5}$	2160	
A_7			$8!$	5		70	
D_8			$2^7 \cdot 8!$	6	$\|W\| \cdot 0.002665$	1120	-832

simplices (A_4 – E_8)

pyramids over a product of 2 simplices (A_7, D_8)

TABLE 7.1. Actions on hyperbolic manifolds, non-simply-laced case.

W	$\mathcal{G}$	$\mathcal{G}_1$	$\|W\|$	dim (X)	vol X approx.	number of cusps	$\chi(X)$ (dim X even)
B_3			$2^3 \cdot 3!$	2	8π	compact	-4
B_4			$2^4 \cdot 4!$	3	$\|W\| \cdot 0.211446$	16	
F_4			$2^7 \cdot 3^2$	3	$\|W\| \cdot 0.222228$	compact	

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- Q is of finite mutation type if $\sharp|Q' \sim_\mu Q| < \infty$.

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Connected quiver is of finite mutation type iff

(a) Q has 2 vertices, or
(b) Q arises from a triangulated surface, or
(c) Q is mutation-equivalent to one of 11 exceptional quivers:

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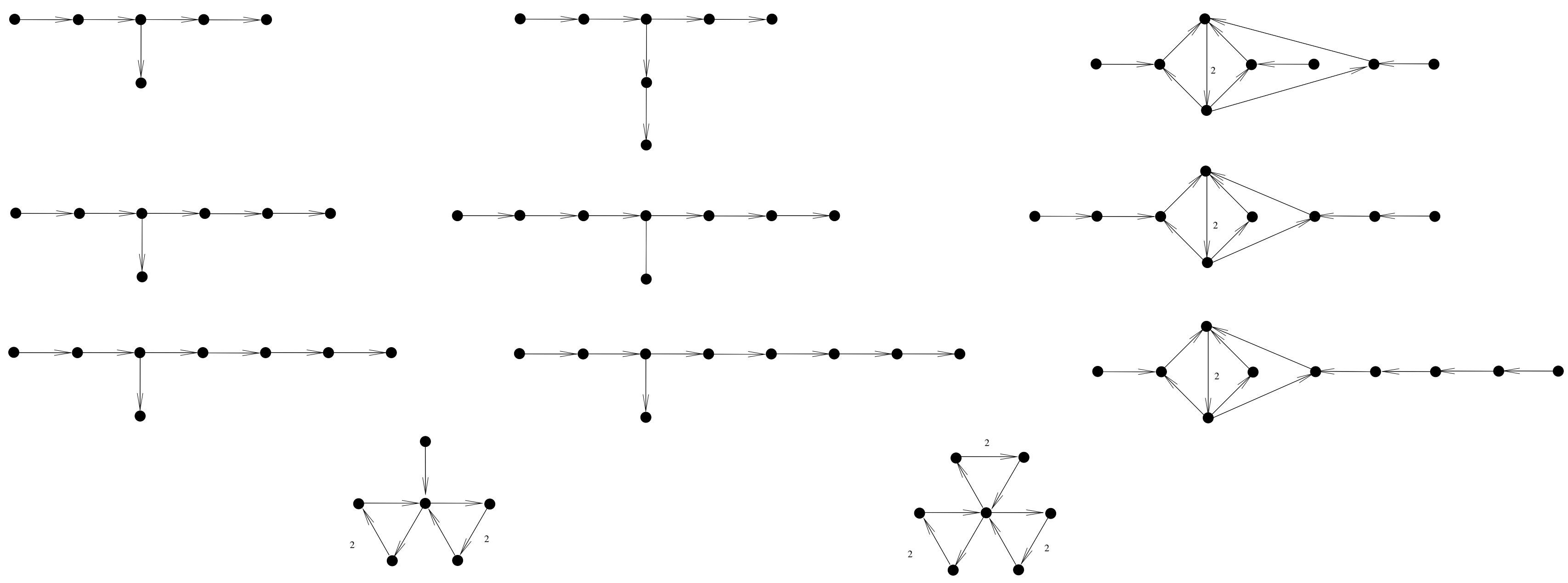

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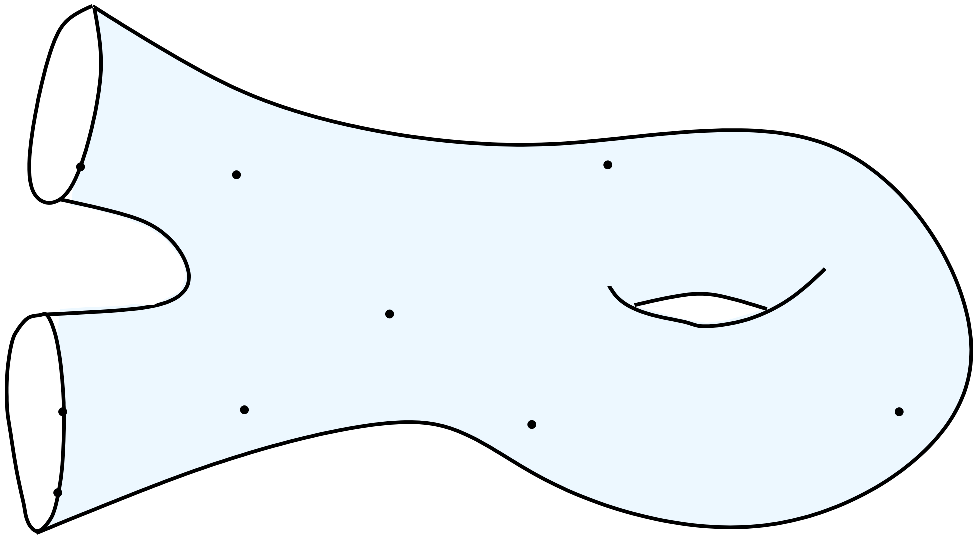
Groups $G(Q)$ for them:

(a) trivial

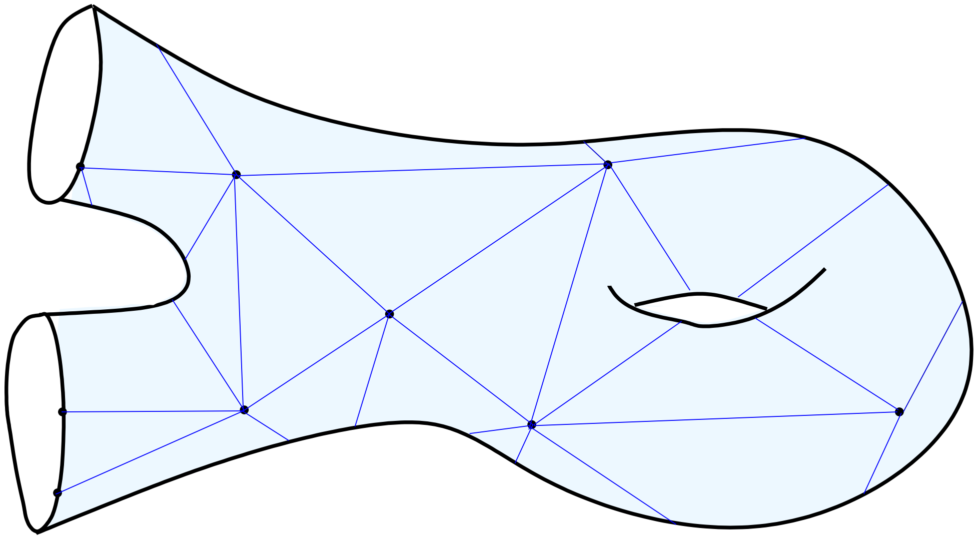
(b) ?????

(c) can construct (with some additional relations).

7. Quivers from triangulated surfaces:



7. Quivers from triangulated surfaces



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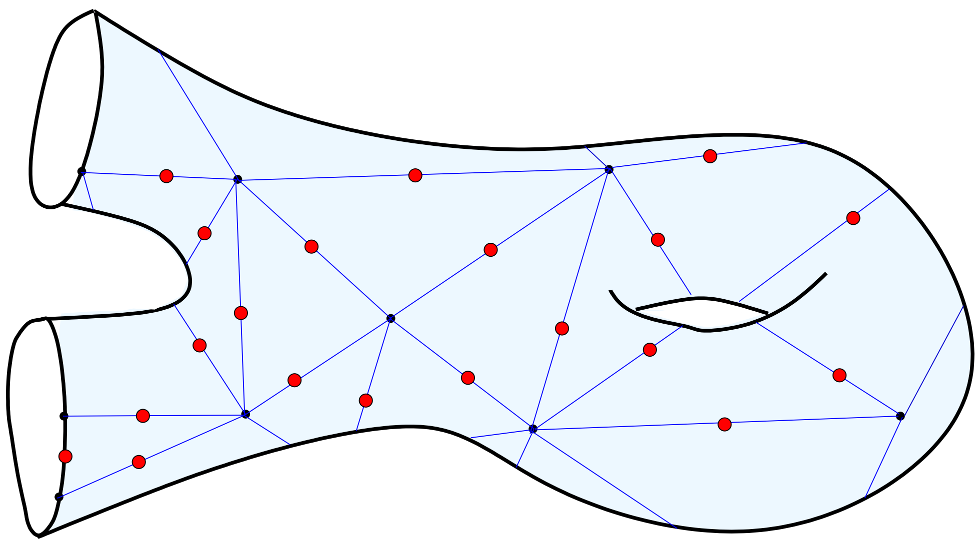
Triangulated surface $\longrightarrow$ Quiver



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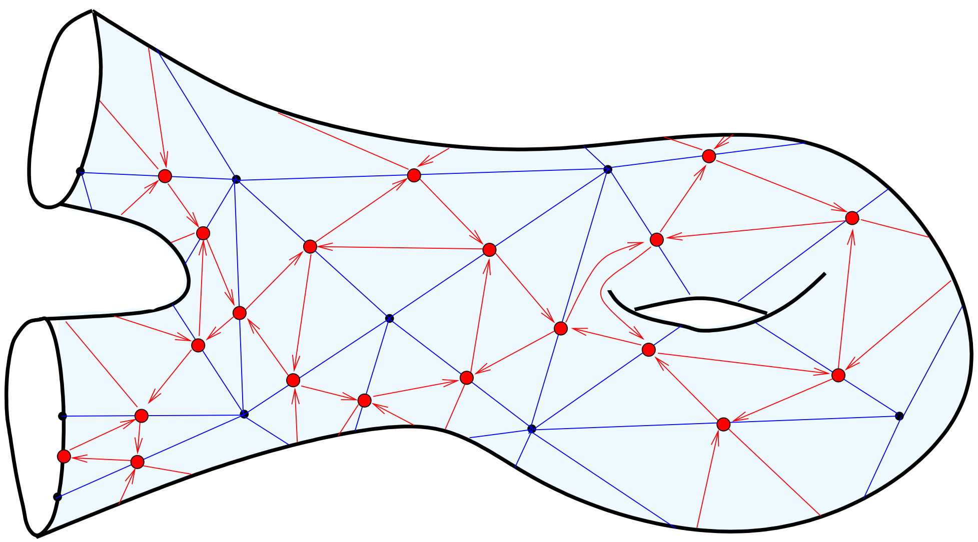
Triangulated surface	$\longrightarrow$	Quiver
edge of triangulation		vertex of quiver
two edges of one triangle		arrow of quiver



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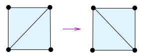
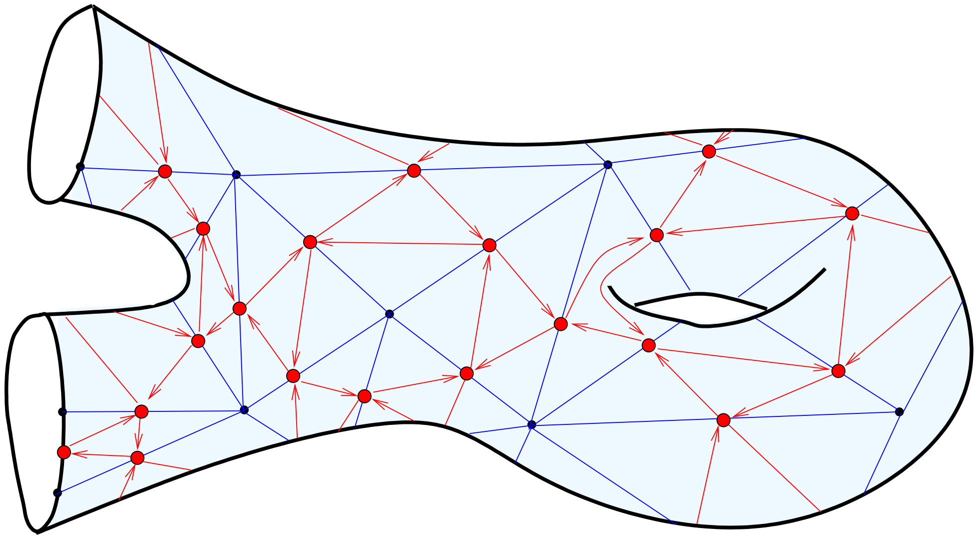
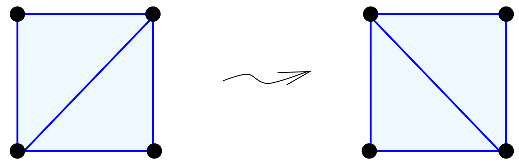
flip of triangulation 





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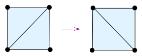
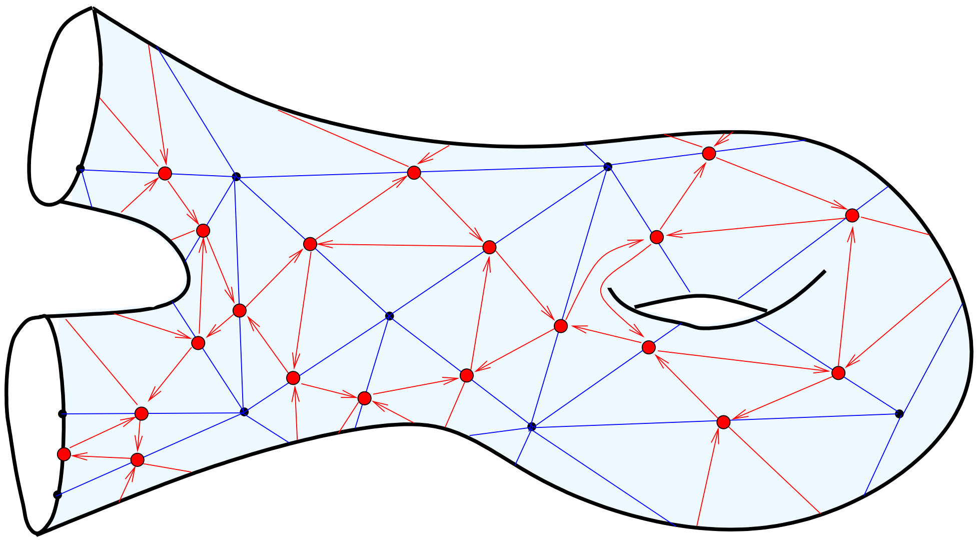
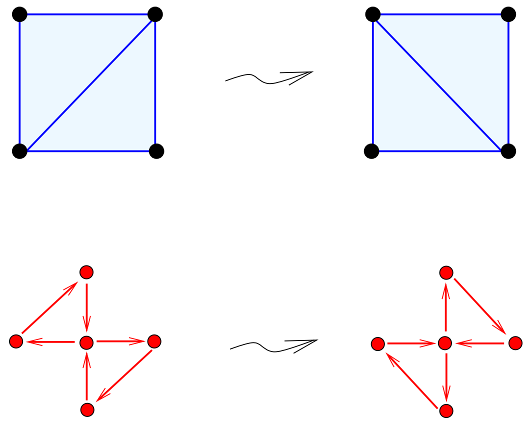
flip of triangulation 		mutation of quiver





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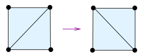
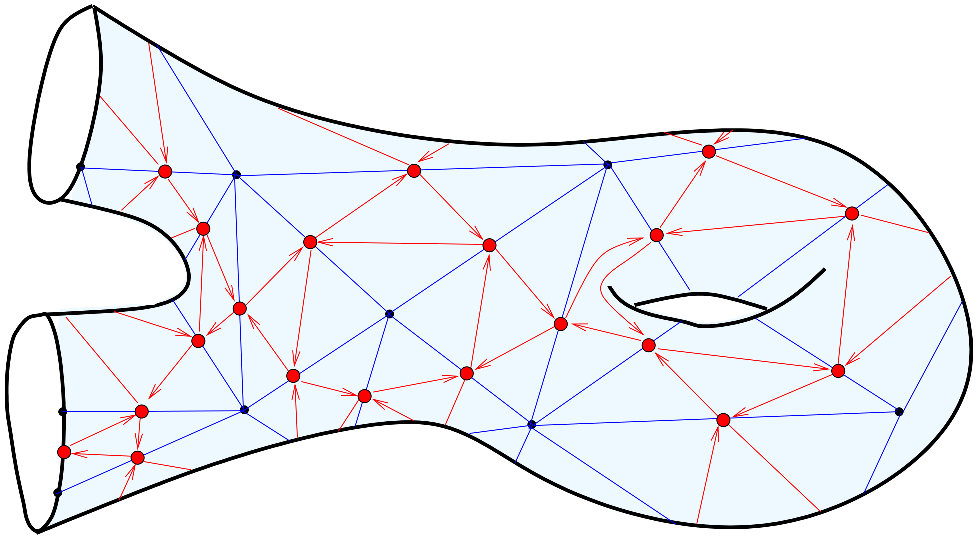
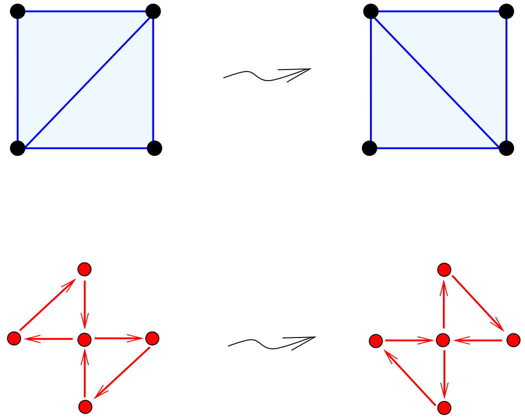
flip of triangulation 	mutation of quiver





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Want: Group G for every mut. class $Q(T)$, i.e. G for every surface.

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Construction of $G(Q)$ for unpunctured surfaces:

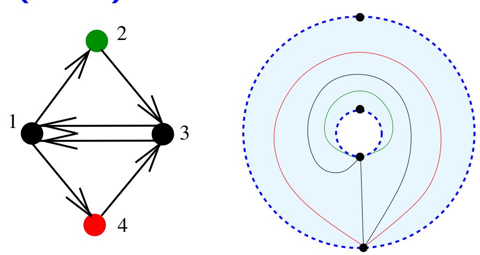
- Generators of $G \leftrightarrow$ arcs of the triangulation of Q.
- Relations of G:

(R1) $s_i = e$

(R2) $(s_i s_j)^{m_{ij}} = e$

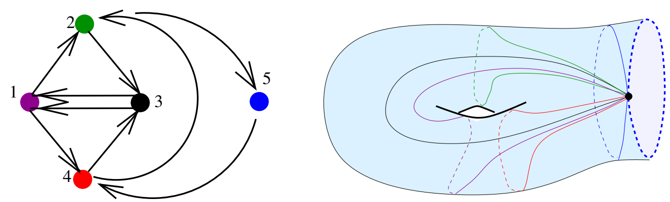
(R3) Cycle relations

(R4) $\widetilde{A}_2$-relations:



$$(s_1 \ s_2 s_3 s_4 s_3 s_2)^2 = e$$

(R5) Handle relations:



$$(s_1 \ s_2 s_3 s_4 s_5 s_4 s_3 s_2)^2 = e$$

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7. Quivers from triangulated surfaces: unpunctured case

Theorem [FT'13]
If S is an unpunctured surface, T triangulation of S, $Q = Q(T)$, $G = G(Q)$, then G is mutation invariant, i.e. G does not depend on the choice of triangulation T.

In other words, G is an invariant of a surface.

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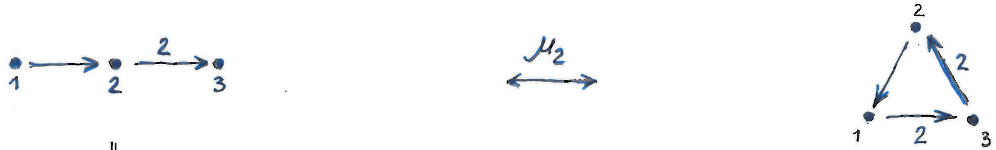
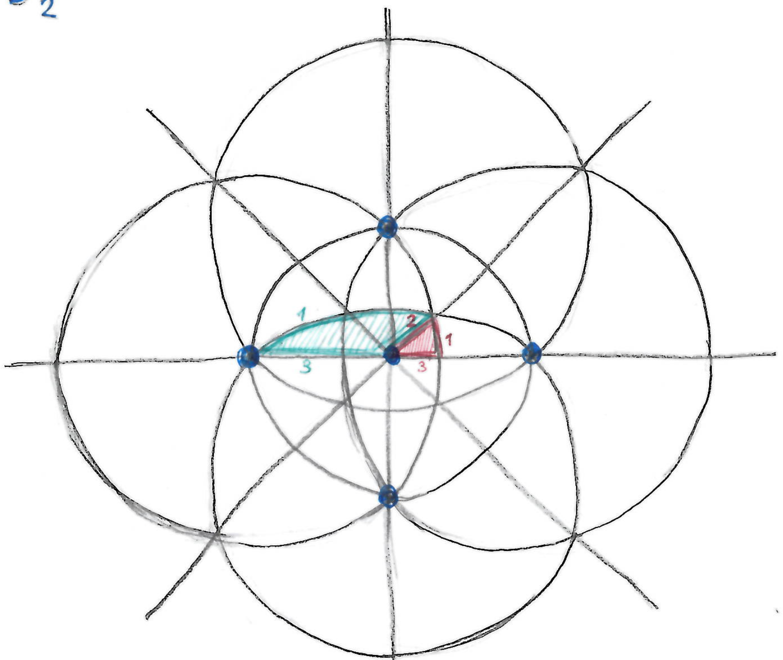
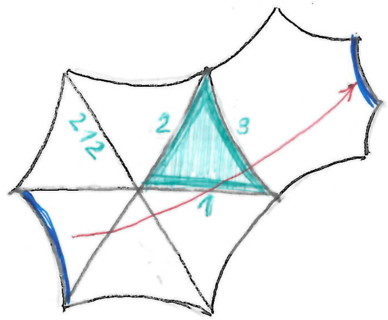
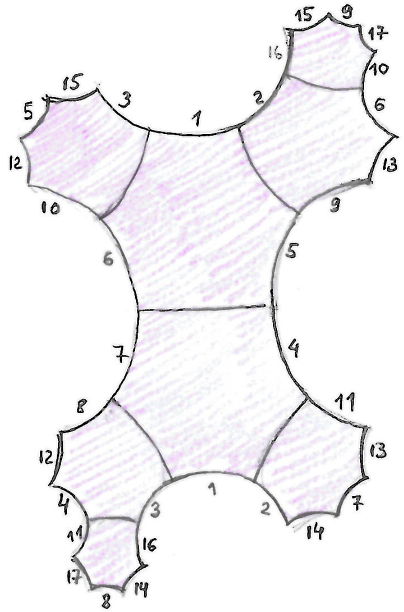
Proposition.
- G does not depend on the distribution of marked points along boundary components.
- There is a surjective homomorphism of G to an extended affine Weyl group of type A.

B_2



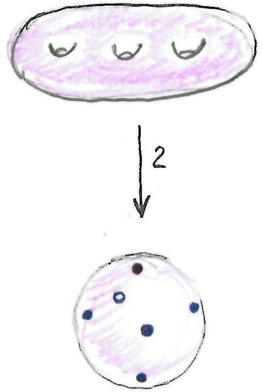






$\langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_2)^3 = (s_2 s_3)^4 = (s_3 s_1)^2 \rangle$

$\langle t_1, t_2, t_3 \mid t_i^2 = (t_1 t_2)^3 = (t_1 t_3)^4 = (t_2 t_3)^4 = (t_3\, t_2 t_1 t_2)^2 \rangle$



Thanks!