Shuffle Groups

Joint work with Carmen Amarra and Luke Morgan

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Perfect Shuffles

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A deck containing 2n cards:

- Cut into two piles of n cards each
- Perfectly interleave them

Out – shuffles and in - shuffles

Starting order: (0,1,2,3,4,5,6,7,8,9,10,11) (n = 6)

After an out – shuffle: (0,6,1,7,2,8,3,9,4,10,5,11) (top card stays on top)

After an in – shuffle: (6,0,7,1,8,2,9,3,10,4,11,5)

Questions (from card-players):

- how many times to regain original order?
- Can I get card 0 into any chosen position by repeated out or in shuffles?

What is a shuffle group?

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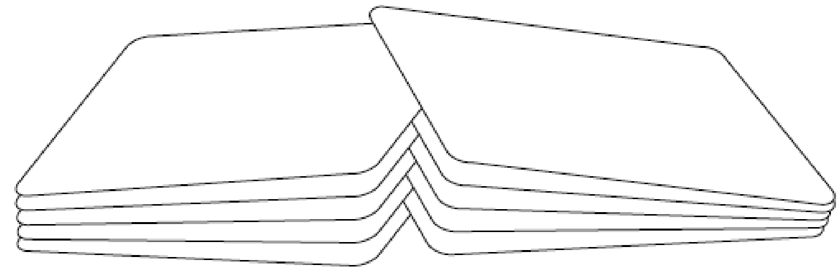

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After an out – shuffle: (0,6,1,7,2,8,3,9,4,10,5,11)

- O = (0)(1,2,4,8,5,10,9,7,3,6) (11)

After an in – shuffle: (6,0,7,1,8,2,9,3,10,4,11,5)

- I = (0,1,3,7,2,5,11,10 ,8,4,9,6)

Shuffle group is the subgroup of Sym(2n) generated by O and I.

1983 Diaconis, Graham and Kantor

“The mathematics of perfect shuffles” Advances in App. Math

- Explain they’re not the first – section 3 gives overview of earlier work:
- Alex Elimsley 1957: importance of o(2, mod 2n-1)
- Golomb 1961, deck of 2n-1 cards: Group order is (2n-1) x o(2, mod 2n-1)
- Discuss applications to parallel processing algorithms (Section 4)

And they work out the shuffle groups!

1983 Diaconis, Graham and Kantor

Write $\sigma = O$ and $\delta =$ swap the piles, so $I = \delta \circ \sigma$ and shuffle group is $\langle \sigma, \delta \rangle$,

Theorem 1.1. [8, Theorem 1] *The structure of the shuffle group* $\langle \sigma, \delta \rangle$ *on* $2n$ *points, where* $n \geqslant 2$, *is given in Table 1.*

Size of each pile n	Shuffle group $\langle \sigma, \delta \rangle$
$n = 2^f$ for some positive integer f	$C_2 \wr C_{f+1}$
$n \equiv 0 \pmod 4$, $n \geqslant 20$ and n is not a power of 2	$\ker(\mathrm{sgn}) \cap \ker(\overline{\mathrm{sgn}})$
$n \equiv 1 \pmod 4$ and $n \geqslant 5$	$\ker(\overline{\mathrm{sgn}})$
$n \equiv 2 \pmod 4$ and $n \geqslant 10$	B_n
$n \equiv 3 \pmod 4$	$\ker(\mathrm{sgn}\,\overline{\mathrm{sgn}})$
$n = 6$	$C_2^6 \rtimes \mathrm{PGL}(2,5)$
$n = 12$	$C_2^{11} \rtimes M_{12}$

TABLE 1. The shuffle group on $2n$ points

- $B_n = C_2 \wr Sym(n) \leq Sym(2n)$, for $g \in B_n$
- $sgn(g)$ sign of g on 2n points, $\overline{sgn(g)}$ sign of g on n parts of size 2
- M_{12} is the Mathieu group

“many handed shuffler”

A deck containing kn cards:

- Cut into k piles of n cards each
- “Perfectly interleave them” – What should this mean?

- The **out-shuffle** σ “picks up” top card from each pile in turn, and repeats
  - For $k = 3, n = 2$ the deck $(0,1,2,3,4,5)$ is mapped to $(0,3,5,1,4,6)$
- Allow an **arbitrary subgroup** $P \leq Sym(k)$ of the k piles to form the

Generalised shuffle group $G = Sh(P, n) \leq Sym(kn)$

Not first to study this: 1980’s

- Steve Medvedoff and Kent Morrison Math Magazine 1987
- John Cannon – early computational information.

1984 Computations: John Cannon & Kent Morrison

n \ k	12	24	36	48	60
$n \equiv 0$	6+6 A_5 acting on 2 orbits of length 6. ✓	$2^{11}.M_{12}$ ✓	36 $2^{17}.A_{18}$ ✓	48 $2^{23}.A_{24}$ ✓	60 $2^{29}.A_{30}$ ✓
$n \equiv 1$	1+6+8 Z_2 wr S_4 ✓KM.	Z_2 wr Z_3	1+18+20 (e o) $(2^8.A_9).(Z_2$ wr $S_{10})$ odd	1+24+26 (e o) $(2.M_{12}).(Z_2$ wr $A_{13})$ odd ✓	1+30+32 (e o) $(2^{14}.A_{15}).(Z_2$ wr $S_{16})$ odd ✓
$n \equiv 2$	18 Z_2 wr S_9 ✓	30 Z_2 wr S_{15} ✓	42 Z_2 wr S_{21}	54 Z_2 wr S_{27} ✓	66 Z_2 wr S_{33} ✓
$n \equiv 3$	1+10+10 $(2^4.A_5).(2^4.D_5)$ odd $(2^4.Z_5).(2^4.S_5)$ even ✓	1+16+16 $(2^7.A_8).(Z_2$ wr $S_8)$ even ✓	1+22+22 $(2^{10}.L_2(11)).(2^{10}.S_{11})$ even ✓	1+28+28 $(2^{13}.A_{14}).(Z_2$ wr $S_{14})$ even ✓	1+34+34 $(2^{16}.A_{17}).(2^{16}.S_{17})$ even ✓

Medvedov and Morrison 1987

They studied the case of $\boldsymbol{G = Sh(Sym(k), n)}$ that is $\boldsymbol{P = Sym(k)}$

Again $\boldsymbol{kn = k^f}$ ("power case") turned out to give exceptionally small G

We write the deck as $[kn] = \{0,1,\ldots,kn-1\}$

- If $kn = k^f$ then $Sh\left(Sym(k), k^{f-1}\right) = Sym(k) \wr C_f$ in product action on $[k]^f$

Showed that $Sh(Sym(k), n) \subseteq Alt(kn)$ if and only if

- either $n \equiv 0\ (mod\ 4)$ or $(k\ mod\ 4, n\ mod\ 4)$ is (0,2) or (1,2)

Explored cases k=3 and k=4 computationally for small n and

Conjectured that if $\boldsymbol{kn \neq k^f}$ and $\boldsymbol{kn \neq 4 \cdot 2^f}$ then $\boldsymbol{Sh(Sym(k), n)}$ should be $\boldsymbol{Sym(kn)}$ or $\boldsymbol{Alt(kn)}$

Amarra, Morgan and CEP

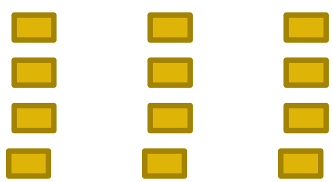
Explored $G = Sh(P, n)$ for general $P \leq Sym(k)$

- Show the “power case” where $kn = k^f$ is also special for general P
- Show certain properties of P lead to similar properties of G
- Confirm the MM-Conjecture [that G usually contains Alt(kn)] in 3 cases:
  - $k > n$
  - $k = 2^e \geq 4$ and $n \neq 2^f$ for any f
  - $k = \ell^e \neq 4$ and $n = \ell^f$ for some ℓ where e does not divide f
- We are left with several open questions

Amarra, Morgan and CEP

Suppose $P \leq Sym(k)$ is transitive. Is $G = Sh(P, n)$ transitive?

- The answer is "yes" but the converse does not hold.
- To see this use $\rho: P \to G$ where for $\tau \in Sym(k)$, $\rho(\tau)$ means "permute the piles according to τ



Label Deck as $[kn] = \{\, 0,1, \dots, kn - 1\}$

So set of piles is $[k] = \{\, 0,1, \dots, k - 1\}$

Pile 0 has cards $\{\, 0,1, \dots, n - 1\}$

In Example $k = 3, n = 4$

For $\tau = (0,1) \in Sym(3)$,
$\rho(\tau) = (0,4)(1,5)(2,6)(3,7)$

Amarra, Morgan and CEP

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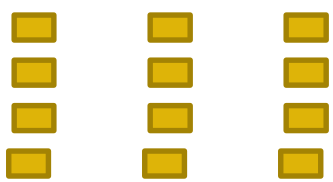
Suppose $P \leq Sym(k)$ is transitive. Is $G = Sh(P, n)$ transitive?

- If P is transitive then $\rho(P)$ has as orbits the rows: $\{ 0, n, \dots, (k-1)n\}, etc$
- We examine the `shuffle' σ and check that it "merges" all these orbits



Deck starts as
$(0,1,2, 3, 4, \dots, 11)$
σ maps this order to
$(0,4,8,1, 5 \dots, 11)$
So cards 1, 2 in row 0 are mapped to cards in row 1, 2;
And card 3 in row 3 is mapped to 1 in row 1.

But many intransitive subgroups P still have transitive shuffle groups $G = Sh(P, n)$

Amarra, Morgan and CEP

1. Suppose $P \leq Sym(k)$ is primitive but not C_p acting regularly. Then $G = Sh(P, n)$ primitive.
   - So $Sh(Sym(k), n)$ primitive if and only if $k \geq 3$
   - [so DGK case $k = 2$ is exceptional in this respect]

2. The Power case: $n = k^f$, and any $P \leq Sym(k)$ implies that $G = P \wr C_{1+f}$ [generalises DGK and MM]
3. Other interesting structure preservation happens:
   - Suppose that $k = \ell^e$, $n = \ell^f$, e does not divide f then
   - When $P = Sym(\ell) \wr Sym(e)$ in product action on $[\ell]^e$ then $G = Sym(\ell) \wr Sym(e+f)$ in product action on $[\ell]^{e+f}$
   - When $P = AGL(e, \ell)$ and ℓ is prime then $G = AGL(e+f, \ell)$
   - When $k \neq 4$, then $Sh(Sym(k), n)$ contains $Alt(k\ n)$ [proving MM conjecture for these parameters]

Amarra, Morgan and CEP

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2. Computationally if $k \leq 13$ and $k < n \leq 1000$, and n is not a power of k, then $Sh(C_k, n)$ contains $Alt(kn)$

We Conjecture: If k is an odd prime, n > k, and n is not a power of k, then $Sh(C_k, n)$ contains $Alt(kn)$

Amarra, Morgan and CEP

Suppose that k > n > 2 and that $P \leq Sym(k)$ is 2-transitive
Then $G = Sh(P, n)$ is 2-transitive.

We asked ourselves: Since finite 2-transitive groups are known can we be more specific?

First for P almost simple 2-transitive, and k > n > 2

a. Then also $Sh(P, n)$ is almost simple;
b. And if P is Alt(k) or Sym(k) then $Sh(P, n)$ contains Alt(kn) or $kn = 4 \cdot 2 = 8$ and $Sh(P, n) = AGL(3,2)$

Amarra, Morgan and CEP

Now for P affine 2-transitive, and $k = p^e > n > 2$

(1) No chance of Sh(P,n) affine unless $n = p^f$

- $n = p^f$ case covered in the “power case”:
- $Sh(P, n) \leq Sh(AGL(e, p), n) = AGL(e + f, p)$

(2) Outstanding case: $n \neq p^f$

- Clearly $Sh(P, n)$ not affine as $kn \neq p^a$
- Maybe $Sh(P, n)$ should be $Alt(kn)$ or $Sym(n)$

We proved this using the classification of 2-transitive groups
+++

Cascading shuffle groups

One last investigation, then summary and questions:

Suppose $k = 2^e \geq 4$ and $n \neq$ 2-power.

For $t \in \{ 1,2, \ldots, e \}$, the deck $[kn] = [2^t \cdot 2^{e-t} n]$ and

$G_t = Sh(C_2^t, 2^{e-t} n)$ all groups transitive on $[kn]$

How are they related? Note that G_1 is known from [DGK]

With much hard work and misgivings we proved that

$$G_1 \geq G_2 \geq \cdots \geq G_e$$

Theorem If $k = 2^e \geq 4$ and $n \neq 2 -$power, then $Sh(Sym(k), n)$ contains $Alt(kn)$

Summary and questions

MM Conjecture Open: if $kn \neq k^f$ and $kn \neq 4 \cdot 2^f$ then $Sh(Sym(k), n)$ should contain $Alt(kn)$

Our contribution to confirm it for:

- $k > n$
- $k = 2^e \geq 4$ and $n \neq 2^f$ for any f
- $k = \ell^e \neq 4$ and $n = \ell^f$ for some ℓ where e does not divide f

Our first Conjecture: If k is an odd prime, n > k, and n is not a power of k, then $Sh(C_k, n)$ contains $Alt(kn)$

More questions

Diaconis is particularly interested in $P = \langle \tau \rangle$ where τ "reverses the piles"

Not much in [MM] or our paper [AMP]

But recent computational evidence suggests some very interesting groups arise. Perhaps at last we'll be able to make sense of the computational data from John Cannon and Kent Morrison's data

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Thank you