A local theory of localities

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From now on, let $\mathcal{F}$ be a saturated fusion system over S.

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- If $\mathcal{D}$ is a product of components, then $C_\mathcal{F}(\mathcal{D})$, $N_\mathcal{F}(\mathcal{D})$ are defined (H. preprint 2019).

Constrained fusion systems, subcentric subgroups

Definition

- A finite group G is of **characteristic p** if $C_G(O_p(G)) \leq O_p(G)$.
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Subcentric subgroups:

$$\mathcal{F}^s := \{\mathcal{F}\text{-conjugates of such subgroups } P \leq S\}$$

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"Definition" (Chermak)

A **partial group** *is a set* $\mathcal{L}$ *together with*

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- If $f \in \mathcal{L}$, $P \subseteq \mathcal{L}$, set $P^f := \{x^f : x \in P\}$ if this is defined.

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- A partial subgroup $\mathcal{N}$ is called a **partial normal subgroup** if

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(Write $\mathcal{N} \trianglelefteq \mathcal{L}$.)

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Others can be revisited or even newly proved using these one-to-one correspondences (marked in green).

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Theorem (H. 2015)

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$$\mathcal{N}_1\mathcal{N}_2 := \{\Pi(x, y) \colon x \in \mathcal{N}_1,\ y \in \mathcal{N}_2\}$$

is a partial normal subgroup of $\mathcal{L}$.

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  - $N_{\mathcal{E}}(P)$ is defined if $P \leq S$ is fully normalized (Aschbacher 2013, revisited G.-H. preprint).
- If $\mathcal{C}$ is a component of $\mathcal{F}$, then $C_{\mathcal{F}}(\mathcal{C})$, $N_{\mathcal{F}}(\mathcal{C})$ are defined (Aschbacher 2019).
- If $\mathcal{D}$ is a product of components, then $C_{\mathcal{F}}(\mathcal{D})$, $N_{\mathcal{F}}(\mathcal{D})$ are defined (H. preprint 2019).

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