Update on GLS

R. Lyons and R. Solomon and many others

Banff, August, 2019

1 References: D. GORENSTEIN, R.L., and R.S.: CFSG, Amer. Math. Soc. Surveys & Monographs 40

Published:

No. 1: Overview and Outline of Proof
No. 2: General Group Theory
No. 3: Almost Simple K-Groups
No. 4: Part II, Chapters 1–4: Uniqueness Theorems
No. 5: Part III, Chapters 1–6: The Generic Case,
No. 7: Part III, Chapters 7–11: The Generic Case, continued
No. 8: Part III, Chapters 12–17: The Generic Case, completed
No. 6: Part IV, Chapters 1–10: The Special Odd Case

In progress:

No. 9: Part V, Chapters 1–6: The Bicharacteristic and Intermediate Cases (with I. CAPDEBOSCQ)
No. 10: Part V, The Case $e(G) = 3$ (with I. CAPDEBOSCQ, K. MAGAARD, C. PARKER)
No. 11: Part II, p-Uniqueness Theorems and 2-Uniqueness Theorems (with R. FOOTE)
No. 12: Part II, The Uniqueness Case (G. STROTH)

2 A Corollary, and $\sigma_0(G)$

Corollary (of Volumes 1–8)

A minimal counterexample G to CFSG is of even type. Moreover, assuming all the uniqueness theorems in Part II, it is of special even type.

By Aschbacher-Smith Quasi Thin Theorem, $e(G) \geq 3$, so $\sigma(G) \neq \emptyset$, where

2 # A Corollary, and $\sigma_0(G)$

Corollary (of Volumes 1–8)

A minimal counterexample G to CFSG is of even type. Moreover, assuming all the uniqueness theorems in Part II, it is of special even type.

By Aschbacher-Smith Quasi Thin Theorem, $e(G) \geq 3$, so $\sigma(G) \neq \emptyset$, where

Definition

$$\sigma(G) = \begin{cases} \{\text{odd primes } p \mid m_{2,p}(G) \geq 4\} & \text{if nonempty} \\ \{\text{odd primes } p \mid m_{2,p}(G) = 3\} & \text{otherwise} \end{cases}$$

$m_{2,p}(G) =$ largest p-rank found among all 2-locals in G.
$\sigma_0(G) = \{p \in \sigma(G) \mid G \text{ has no very strong } p\text{-uniqueness subgroup}\}$

$\sigma_0(G) \neq \emptyset$: Volumes 9 and 10.
$\sigma_0(G) = \emptyset \neq \sigma(G)$: Volumes 11 and 12.

3 The foundation: Background Results

These are well-established results that we itemize in the first volume. In subsequent volumes we add a few more.
The first one is the Odd Order theorem of W. Feit and J. G. Thompson.
When M. Aschbacher and S. D. Smith published their Classification of Quasithin Groups, we added that to the list.
Another Background Result is that each sporadic simple group is determined by its "centralizer of involution pattern."
In the Generic Case we use some recent results on Phan Theory due to R. Köhl, S. Shpectorov, C. Bennett, B. Mühlherr and others, for the purpose of recognizing certain Chevalley groups.

Outline

- Case division.
- Uniqueness Theorems.
- The Generic Case.
- The Bicharacteristic Case.

5 Minimal counterexample; the finite quasisimple groups

$G =$ minimal counterexample to CFSG
G is $\mathcal{K}$-proper

$\mathcal{K} = Alt \cup Chev \cup Spor$ includes covering groups

$p =$ prime

$\mathcal{K}_p = \{K \in \mathcal{K} \mid O_{p'}(K) = 1\}$ $(O_{p'}(K) = \max\ p' \lhd K)$

$\mathcal{K}_p =$ set of possible components of $\overline{N} := N/O_{p'}(N)$,
$N = N_G(P) = p$-local subgroup in $\mathcal{K}$-proper simple group G

5 Minimal counterexample; the finite quasisimple groups

$G =$ minimal counterexample to CFSG
G is $\mathcal{K}$-proper

$\mathcal{K} = Alt \cup Chev \cup Spor$ includes covering groups

$p =$ prime

$\mathcal{K}_p = \{K \in \mathcal{K} \mid O_{p'}(K) = 1\}$ $(O_{p'}(K) = \max p' \lhd K)$
$\mathcal{K}_p =$ set of possible components of $\overline{N} := N/O_{p'}(N)$,
$N = N_G(P) = p$-local subgroup in $\mathcal{K}$-proper simple group G

Convention: For any group H, $\overline{H} := H/O_{p'}(H)$.

Notation: $m_p(X) = p$-rank of X
If $K, L \in \mathcal{K}_p$, then $K \uparrow_p L \iff$ there is $x \in Aut(L)$ of order p and $K_0 \lhd\lhd C_L(x)$ such that $\overline{K_0} \cong K$.

6 A principle

We begin by surveying the isomorphism types, for all primes p and p-local subgroups N of G, of components of $E(\overline{N})$. On the basis of this (and whether G is of even type) we select

- a prime p to guide the analysis of G, and
- a strategy for that analysis.

7 The Guiding Prime; A Basic Case Division

If G is of odd type, $p = 2$ will be the guiding prime, and

if G is of even type, an odd prime p will be the guiding prime.

Odd, 2-Special Type	Even, p-Special Type $\forall p \in \sigma_0(G)$
Odd, 2-Generic Type	Even, p-Generic Type for some $p \in \sigma_0(G)$

8 p-Component Pairs in G

$p =$ a prime; $m_p(G) \geq 3$

8 p-Component Pairs in G

$p =$ a prime; $m_p(G) \geq 3$

A p-component pair (x, K) in G consists of an element x of order p and a p-component K of $C_G(x)$.
($K \lhd\lhd C_G(x)$, $\overline{K}$ quasisimple, K minimal)

Based on $L_{p'}$-balance, $(x, K_x) < (y, K_y) \iff$ $[\overline{K}_x, y] = 1$ and K_y is a "pumpup" of K_x in $C_G(y)$:
trivial ($[\overline{K}_y, x] = 1$ and $\overline{K}_x \cong \overline{K}_y$), vertical ($\overline{K}_x \uparrow_p \overline{K}_y$), or diagonal

Information about $E(\overline{C_G(x)})$ moves along the commuting graph of elements of G of order p.

(x, K_x) is p-terminal if for all such y, $[\overline{K_y}, x] = 1$ and $\overline{K_x} \cong \overline{K_y}$ (& a further Sylow p-subgroup condition holds)

9 # p-Terminal Pairs Exist

(x, K_x) is p-terminal if $[\overline{K_y}, x] = 1$ (&…) for all $(x, K_x) < (y, K_y)$

Theorem (M. Aschbacher, R. H. Gilman, GLS)

Let (x, K) be a p-component pair. If $p > 2$, assume that K has a $Z_p \times Z_p$ subgroup disjoint from $O_{p'p}(K)$. Then a suitable series of pumpups leads to a p-terminal pair.

For a p-component pair (x, K), K itself is "terminal"
$\iff$ $K \lhd\lhd C_G(y)$ for all $y \in C_G(K)$ of order p.

Example: In $L_5(q)$, $p = 3$ dividing $q - 1$, $\omega^3 = 1 \neq \omega$,
$(diag(\omega, \omega^2, 1, 1, 1), SL_3(q))$
pumps up to $(diag(\omega^2, \omega, \omega, \omega, \omega), SL_4(q))$,
which is 3-terminal and terminal.

10 $\mathcal{K}_p = \mathcal{C}_p \cup \mathcal{T}_p \cup \mathcal{G}_p$ (a partition for each prime p)

Start with

$$\mathcal{K}_p = Chev(p) \cup \{K \mid m_p(K) = 1\} \cup \{\text{the rest}\}$$

and move some groups around the "edges".

10 $\mathcal{K}_p = \mathcal{C}_p \cup \mathcal{T}_p \cup \mathcal{G}_p$ (a partition for each prime p)

Start with

$$\mathcal{K}_p = Chev(p) \cup \{K \mid m_p(K) = 1\} \cup \{\text{the rest}\}$$

and move some groups around the "edges".

For $p \leq 11$: some groups are taken from $\mathcal{G}_p$ and put in $\mathcal{C}_p$ or $\mathcal{T}_p$.

- $\mathcal{C}_2$ contains 19 sporadic groups, $\mathcal{C}_3$ contains 16
- $\mathcal{C}_2 \cup \mathcal{T}_2$ contains $L_2(q)$ and $SL_2(q)$, q odd
- $\mathcal{T}_3$ contains A_7, $L_3(q)$, $U_3(q)$, $SL_3(q)$, $SU_3(q)$, $q \neq 3^n$

$\mathcal{K}_p = \mathcal{C}_p \cup \mathcal{T}_p \cup \mathcal{G}_p$ (a partition for each prime p)

Start with

$$\mathcal{K}_p = Chev(p) \cup \{K \mid m_p(K) = 1\} \cup \{\text{the rest}\}$$

and move some groups around the "edges".

For $p \leq 11$: some groups are taken from $\mathcal{G}_p$ and put in $\mathcal{C}_p$ or $\mathcal{T}_p$.

$\mathcal{C}_2$ contains 19 sporadic groups, $\mathcal{C}_3$ contains 16

$\mathcal{C}_2 \cup \mathcal{T}_2$ contains $L_2(q)$ and $SL_2(q)$, q odd

$\mathcal{T}_3$ contains A_7, $L_3(q)$, $U_3(q)$, $SL_3(q)$, $SU_3(q)$, $q \neq 3^n$

Properties:

- If $K, L \in \mathcal{K}_p$ and $K \uparrow_p L$, then
  - $K \in \mathcal{G}_p \implies L \in \mathcal{G}_p$
  - $L \in \mathcal{C}_p \implies K \in \mathcal{C}_p$

  and roughly speaking,
- $K \in \mathcal{C}_p \implies K$ has good balance properties for prime p
  ideally: $O_{p'}(C_{Aut(K)}(x)) = 1$ for all $x \in Aut(K)$, $x^p = 1$
- $K \in \mathcal{G}_p \implies K$ has good generation properties for p
  ideally: $Z_p \times Z_p \cong E \leq Aut(K) \implies K = \langle C_K(e) \mid e \in E^\# \rangle$

Generic type vs. Special type

Definition

Given p, G is of p-generic type if and only if there exists a p-terminal component pair (x, K_x) such that

- $\overline{K}_x \in \mathcal{G}_p$;
- $m_p(C_G(x)) \geq 3$, with strict inequality if $p > 2$.

Otherwise, G is of p-special type.

For example if $\exists$ involution $z: \; C_G(z) \cong Z_2 \times L_5(3)$, then G is of 2-generic type.

If $\exists x: \; x^3 = 1, \; C_G(x) \cong SU_6(8) \times PSp_8(3)$, then G is of 3-generic type.

Even type vs. Odd type

Definition (GLS)

G is of even type if and only if

- $m_2(G) \geq 3$,
- For every involution $z \in G$,
  - $O_{2'}(C_G(z)) = 1$
  - $K \in \mathcal{C}_2$ for every component K of $C_G(z)$.

Otherwise, G is of odd type.

If $G \in Chev(2)$, then G is of even type.
In the first-generation proof, "odd type" = "not charac. 2-type."

Most of the largest sporadic groups are of even type but not char. 2 type, e.g. $M = F_1$, $BM = F_2$, Co_1, Fi'_{24}, Fi_{23}, Fi_{22}, Suz

The Outcomes: What is G in the different cases?

Special Odd Type (Part IV)	**Special Even Type (Part V)**
SMALL CASE – $m_2(G) \le 2$: $L_2(q)$, $L_3(q)$, $U_3(q)$, q odd M_{11}, A_7	SMALL CASE – $e(G) \le 3$: $Chev(2)$, tw. rank > 1, untw. rank ≤ 3; 14 sporadics; A_{12}; some other Lie type
2-INTERMEDIATE CASE All 2-Components in $\mathcal{C}_2 \cup \mathcal{T}_2$, and $m_2(G) \ge 3$: $Chev(odd)$, untw. Lie rank ≤ 2; ${}^3D_4(q)$, some $L_4(q)$,$U_4(q)$,q odd; A_9, A_{10}, A_{11} M_{12}, Mc, Ly, $O'N$	BICHARACTERISTIC CASE All p-Components in $\mathcal{C}_p$, $e(G) \ge 4$: $p = 3$ & Co_1, Fi_{22}, Fi_{23}, Fi'_{24}, F_2, F_1 $\Omega^{\pm}_{7,8}(3)$, $U_7(2)$, ${}^2D_5(2)$, ${}^2E_6(2)$ p-INTERMEDIATE CASE Some p-Component in $\mathcal{T}_p$, none in $\mathcal{G}_p$, $e(G) \ge 4$: No Groups
p-**Generic Type (Part III)**: Large $Chev$; A_n, $n \ge 13$ $p = 2$ or $p > 2$ according as Odd or Even Type	

Uniqueness Theorems (Part II)

$Chev(2)$, twisted rank 1

i.e., groups with a strongly embedded subgroup; also J_1

14 Uniqueness Theorems when $p = 2$

- Bender-Suzuki Strongly Embedded Subgroup Theorem
- Global $C(G,T)$-Theorem (Aschbacher, Foote, Harada, Solomon)
- Terminal components K are standard if $m_2(K) > 1$ (Aschbacher, Gilman) ($[K, K^g] \neq 1$ for all $g \in G$) - rules out wreathed local structure

Theorem (Aschbacher, Gilman)

Let x be an involution of a simple group G and K a component of $C_G(x)$ which is terminal in G. If $m_2(K) > 1$, then K is standard in G, i.e., for all $g \in G$, $[K, K^g] \neq 1$.

14

Uniqueness Theorems when $p = 2$

- Bender-Suzuki Strongly Embedded Subgroup Theorem
- Global $C(G, T)$-Theorem (Aschbacher, Foote, Harada, Solomon)
- Terminal components K are standard if $m_2(K) > 1$ (Aschbacher, Gilman) ($[K, K^g] \neq 1$ for all $g \in G$) - rules out wreathed local structure

Theorem (Aschbacher, Gilman)

Let x be an involution of a simple group G and K a component of $C_G(x)$ which is terminal in G. If $m_2(K) > 1$, then K is standard in G, i.e., for all $g \in G$, $[K, K^g] \neq 1$.

GLS proves a version of this and some other uniqueness theorems for odd primes too.

A subgroup $M < G$ is strongly p-embedded in G if and only if it has order divisible by p and contains the normalizers of all its nontrivial p-subgroups.

Uniqueness Theorems in $\mathcal{K}$-proper simple G for any p

Theorem (GLS4)

Let M be a maximal subgroup of G, and K a p-component of M. Suppose that for every $y \in C_M(\overline{K})$ of order p, $C_G(y) \leq M$. Suppose that $m_p(K) > 1$, $m_p(C_M(\overline{K})) > 1$, and $m_p(M) \geq 4$. Then either M is strongly p-embedded in G or else:

1. *There is $g \in G - M$ and $E_{p^2} \cong Q \leq C_M(\overline{K})$ such that $Q^g \leq M$; and*
2. *$K \triangleleft M$.*

If $m_p(K) = 1$ and $p > 2$, we still get these conclusions, unless M is "almost strongly p-embedded in G."

Corollary

Let (x, K) be a p-component pair in G with K quasisimple and terminal in G, $m_p(K) \geq 2$, $m_p(N_G(K)) \geq 4$. Then either G has a strongly p-embedded subgroup or K is standard in G.

16 The Uniqueness Case – Volume 12

Theorem (G. Stroth – Uniqueness Case – Vol. 12, in progress)

Suppose that G is a $\mathcal{K}$-local simple group of even type with $\sigma(G) \neq \emptyset$. Assume that for each $p \in \sigma(G)$, G possesses a very strong p-uniqueness subgroup M_p. Then any such M_p is the unique maximal 2-local containing one of its Sylow 2-subgroups…

16 The Uniqueness Case – Volume 12

Theorem (G. Stroth – Uniqueness Case – Vol. 12, in progress)

Suppose that G is a $\mathcal{K}$-local simple group of even type with $\sigma(G) \neq \emptyset$. Assume that for each $p \in \sigma(G)$, G possesses a very strong p-uniqueness subgroup M_p. Then any such M_p is the unique maximal 2-local containing one of its Sylow 2-subgroups. . .

Corollary (GLS-R. Foote)

. . . and in fact is strongly embedded in G, so G does not exist.

The Uniqueness Case – Volume 12

Theorem (G. Stroth – Uniqueness Case – Vol. 12, in progress)

Suppose that G is a $\mathcal{K}$-local simple group of even type with $\sigma(G) \neq \emptyset$. Assume that for each $p \in \sigma(G)$, G possesses a very strong p-uniqueness subgroup M_p. Then any such M_p is the unique maximal 2-local containing one of its Sylow 2-subgroups...

Corollary (GLS-R. Foote)

... and in fact is strongly embedded in G, so G does not exist.

“Definition”: Let $p \in \sigma(G)$. A very strong p-uniqueness subgroup of G is a maximal subgroup M such that

- M is almost strongly p-embedded in G
- M contains a Sylow 2-subgroup of G
- For almost any $H \leq G$ such that $O_2(H) \neq 1$ and $m_p(H \cap M) > 1$, $H \leq M$
- $F^*(M) = O_2(M)E(M)$, $E(M) = 1$ or $E(M) \in Chev(2)$ of untwisted rank > 2, $m_p(C_M(E(M))) = 1$.

The Uniqueness Case – Volume 12

Theorem (G. Stroth – Uniqueness Case – Vol. 12, in progress)

Suppose that G is a $\mathcal{K}$-local simple group of even type with $\sigma(G) \neq \emptyset$. Assume that for each $p \in \sigma(G)$, G possesses a very strong p-uniqueness subgroup M_p. Then any such M_p is the unique maximal 2-local containing one of its Sylow 2-subgroups...

Corollary (GLS-R. Foote)

... and in fact is strongly embedded in G, so G does not exist.

"Definition": Let $p \in \sigma(G)$. A very strong p-uniqueness subgroup of G is a maximal subgroup M such that

- M is almost strongly p-embedded in G
- M contains a Sylow 2-subgroup of G
- For almost any $H \leq G$ such that $O_2(H) \neq 1$ and $m_p(H \cap M) > 1$, $H \leq M$
- $F^*(M) = O_2(M)E(M)$, $E(M) = 1$ or $E(M) \in Chev(2)$ of untwisted rank > 2, $m_p(C_M(E(M))) = 1$.

17 ## Bootstrapping – Volume 11

"Definition": Let $p \in \sigma(G)$. A very strong p-uniqueness subgroup of G is a maximal subgroup M such that

- M is almost strongly p-embedded in G;
- M contains a Sylow 2-subgroup of G
- For almost any $H \leq G$ such that $O_2(H) \neq 1$ and $m_p(H \cap M) > 1$, $H \leq M$.
- $F^*(M) = O_2(M)E(M)$, $E(M) = 1$ or $E(M) \in Chev(2)$, $m_p(C_M(E(M))) = 1$

Volume 11 bootstraps from the first condition to the other three.

17 Bootstrapping – Volume 11

"Definition": Let $p \in \sigma(G)$. A very strong p-uniqueness subgroup of G is a maximal subgroup M such that

- M is almost strongly p-embedded in G;
- M contains a Sylow 2-subgroup of G
- For almost any $H \leq G$ such that $O_2(H) \neq 1$ and $m_p(H \cap M) > 1$, $H \leq M$.
- $F^*(M) = O_2(M)E(M)$, $E(M) = 1$ or $E(M) \in Chev(2)$, $m_p(C_M(E(M))) = 1$

Volume 11 bootstraps from the first condition to the other three.

Hence the even type strategy is:

CHOOSE $p \in \sigma(G)$ with NO ALMOST STRONGLY p-EMBEDDED SUBGROUP IN G.
THEN SHOW $G \in \mathcal{K}$.

The Generic Case

The following conditions define this case.

- Either G is of odd type, $p = 2$, and $m_2(G) \geq 3$, or
  G is of even type, $p > 2$, and $m_p(G) \geq 4$; and
- In either case, there is a p-component pair (x, K) such that $\overline{K} \in \mathcal{G}_p$ and $m_p(C_G(x)) \geq 3$ or 4 according as $p = 2$ or $p > 2$.

p is fixed but arbitrary, satisfying these conditions (except for one possible shift).

The Generic Case

The following conditions define this case.

- Either G is of odd type, $p = 2$, and $m_2(G) \geq 3$, or
  G is of even type, $p > 2$, and $m_p(G) \geq 4$; and
- In either case, there is a p-component pair (x, K) such that $\overline{K} \in \mathcal{G}_p$ and $m_p(C_G(x)) \geq 3$ or 4 according as $p = 2$ or $p > 2$.

p is fixed but arbitrary, satisfying these conditions (except for one possible shift).

Desired conclusion: $G \cong A_n$, $n \geq 13$, or $G \in \mathit{Chev}$.

The Generic Case: Neighborhoods

A neighborhood $\mathcal{N}$ in simple group G is determined by (D, L)
$D \leq G$, $D \cong Z_p \times Z_p$, and L is a p-component of $C_G(D)$.
Then for each $1 \neq d \in D$, the subnormal closure of L in $C_G(d)$ is called L_d, and

$$\mathcal{N} = \mathcal{N}(D, L) = \{L_d \mid 1 \neq d \in D\}.$$

Example: $G = L_5(q)$, q odd, $p = 2$

- $d_1 = diag(1, -1, -1, -1, -1)$, $d_2 = diag(-1, -1, -1, -1, 1)$,
- $D = \langle d_1, d_2 \rangle$
- $C_G(D) \triangleright L \cong SL_3(q)$, $\quad L_{d_1} \cong L_{d_2} \cong SL_4(q)$, $\quad L_{d_1 d_2} = L$
- $\mathcal{N} = \{L_{d_1}, L_{d_2}, L_{d_1 d_2}\}$ and $\langle \mathcal{N} \rangle = G$

A neighborhood in $L_5(q)$ (q odd, $p = 2$)

$$\begin{array}{ccc} L_{d_1} & L_{d_2} & L_{d_1 d_2} \\ SL_4(q) & SL_4(q) & SL_3(q) \\ \backslash & | & / \\ & SL_3(q) & \end{array}$$

This example neighborhood is

- semisimple–each $O_{p'}(L_d) \leq Z(L_d)$
- nontrivial–$L_d \ncong L$ for at least two $\langle d \rangle \leq D$
- vertical–nontrivial, and each L_d is a vertical or trivial pumpup
- level–all L_d's are of Lie type with the same q.

$L_{p'}$-balance and neighborhoods

Theorem (Gorenstein-Walter)

In a neighborhood $\mathcal{N} = \mathcal{N}(D, L)$, each L_d, $d \in D$, satisfies one of the following:

- $\overline{L}_d \cong \overline{L}$ *(trivial pumpup–D centralizes* L_d*)*
- $\overline{L}_d$ *is quasisimple and* $\overline{L} \uparrow_p \overline{L}_d$ *"via"* $D/\langle d\rangle$
- $\overline{L}_d$ *is the commuting product of p covering groups of* $\overline{L}$ *permuted transitively by* $D/\langle d\rangle$*.*

Here $\overline{X}$ always means $X/O_{p'}(X)$.

$\mathcal{K}$-group observation: if $p > 2$, then the first and third possibilities never both occur in the same neighborhood.

The Generic Case – Volumes 5, 7, 8 – Flowchart

CHOOSE p, and p-TERMINAL (x, K), $\overline{K} \in \mathcal{G}_p$

$\downarrow$

CHOOSE (D, L), $x \in D \cong E_{p^2}$, AND LET $\mathcal{N} = \mathcal{N}(D, L)$

$\downarrow$

$\mathcal{N}$ IS SEMISIMPLE

$\downarrow$

$\mathcal{N}$ IS NONTRIVIAL

$\downarrow$

$\mathcal{N}$ IS VERTICAL (IF CHOSEN PROPERLY)

$\downarrow$

IF $p > 2$, $K \in Chev(2)$ AND $p | q^2 - 1$

$\downarrow$

$\mathcal{N}$ IS LEVEL AND COMES FROM SOME KNOWN G^*

$\downarrow$

$\langle \mathcal{N} \rangle \cong_c G^*$ AND $\Gamma_{D,1}(G)$ NORMALIZES $\langle \mathcal{N} \rangle$

$\downarrow$

$G = \langle \mathcal{N} \rangle$

The Generic Case: Choosing (x, K)

There is a technical "stratification" of $\mathcal{G}_p$, according to isomorphism type. We choose (x, K) so that $\overline{K}$ is in the highest stratum possible, and call it "preferred".

Roughly speaking, we prefer alternating and sporadic groups to groups of Lie type, and large rank groups of Lie type to small rank ones.

By a series of pumpups, we may choose (x, K) to be p-terminal.

Looking ahead to our neighborhood, x will lie in D, and $K = L_x$, and $L \triangleleft E(C_K(D))$.

The Generic Case: Choosing a p-Source A

We want to be able to apply signalizer functor theory to $A \cong Z_p \times Z_p \times Z_p$: need 3/2-balance for A.

We also want A to be closely related to our soon-to-be-made choice of D (recall $L \triangleleft E(C_K(D))$).

For small (rank) $\overline{K}$, choice is ad hoc. For large rank:

- $\overline{K} \cong A_n$, $n \geq 9$, $p = 2$: $A = E \times \langle x \rangle$, $E \leq K$, $\overline{E}$ is root four-group
- $\overline{K} \cong A_n$, p odd, $n > 3p$: $A = \langle x_1, x_2, x_3 \rangle$, x_i disjoint p-cycles
- $\overline{K}$ large Lie type, $p = 2$: $A = Z(SL_2(q) \times SL_2(q) \times SL_2(q))$, root $SL_2(q)$'s (Aschbacher)
- $\overline{K}$ large classical group, p odd: $A \leq K$, minimum support on natural module among all $Z_p \times Z_p \times Z_p$ subgroups

The Generic Case: Semisimple Neighborhoods

Choose $D \leq C_K(x)$ and (large) $L \triangleleft E(C_K(D))$ such that $x \in D \cong Z_p \times Z_p$ and D is "nicely related" to A.

(Best situation: $[D, A] = 1$ and $E(C_L(A)) \neq 1$.)

Let $\mathcal{N} = \{L_d \,|\, 1 \neq d \in D\}$, the neighborhood determined by D and L.

Let $W = \Theta_{3/2}(G; A)$, $M = N_G(W)$ (signalizer functor gadgets)

Signalizer functor theory implies $\Gamma_{A,2}(G) = \langle N_G(B) \,\big|\, |A : B| \leq p \rangle \leq M$.

Relationship of D and A implies Gorenstein-Walter Alternative:

1. $\mathcal{N}$ is a semisimple neighborhood ($L_d \leq E(C_G(d))$ for all $1 \neq d \in D$), OR
2. $W \neq 1$, so $\Gamma_{A,2}(G) \leq M < G$.

Option 2 $\implies$ G has a strongly p-embedded subgroup, contradiction.

Hence $\mathcal{N}$ is a semisimple neighborhood.

26 The Generic Case: More example neighborhoods $\mathcal{N}$

VERTICAL

$$
\begin{array}{ccc}
L_{d_1} & L_{d_2} & L_{d_1 d_2} \\
D_5(q) & D_5(q) & D_5(q) \\
\backslash & | & / \\
 & D_4(q) &
\end{array}
$$

NOT VERTICAL

$$
\begin{array}{ccc}
L_{d_1} & L_{d_2} & L_{d_1 d_2} \\
D_5(q) & D_5(q) & D_4(q) \times D_4(q) \\
\backslash & | & / \\
 & D_4(q) &
\end{array}
$$

The Generic Case: More example neighborhoods $\mathcal{N}$

VERTICAL

$$
\begin{array}{ccc}
L_{d_1} & L_{d_2} & L_{d_1 d_2} \\
D_5(q) & D_5(q) & D_5(q) \\
\diagdown & | & \diagup \\
 & D_4(q) &
\end{array}
$$

$$\langle \mathcal{N} \rangle = E_6(q)$$

NOT VERTICAL

$$
\begin{array}{ccc}
L_{d_1} & L_{d_2} & L_{d_1 d_2} \\
D_5(q) & D_5(q) & D_4(q) \times D_4(q) \\
\diagdown & | & \diagup \\
 & D_4(q) &
\end{array}
$$

The Generic Case: More example neighborhoods $\mathcal{N}$

VERTICAL

$$\begin{array}{ccccc} L_{d_1} & & L_{d_2} & & L_{d_1 d_2} \\ D_5(q) & & D_5(q) & & D_5(q) \\ & \backslash & | & / & \\ & & D_4(q) & & \end{array}$$

$$\langle \mathcal{N} \rangle = E_6(q)$$

NOT VERTICAL

$$\begin{array}{ccccc} L_{d_1} & & L_{d_2} & & L_{d_1 d_2} \\ D_5(q) & & D_5(q) & & D_4(q) \times D_4(q) \\ & \backslash & | & / & \\ & & D_4(q) & & \end{array}$$

$$\langle \mathcal{N} \rangle = D_5(q) \times D_5(q)$$
$$\langle \mathcal{N} \rangle D = D_5(q) \wr Z_2$$

27 The Generic Case: Volume 5

Theorem*

In the Generic Case, starting with any preferred and p-terminal (x, K), there exists $D \leq C_G(x)$, $D \cong Z_p \times Z_p$, and $L \lhd E(C_K(D))$, and a nontrivial semisimple neighborhood $\mathcal{N}(D, L)$. Moreover,

- *K is standard in G (no pumpups; $[K, K^g] = 1 \implies K = K^g$)*
- *$m_p(C_G(K)) = 1$ unless $p = 2$ and $K \cong A_n$, in which case $m_2(C_G(K)) = 2$ ...*
- *If $p > 2$, then $K \in Chev(2)$.*

The Generic Case: Nontrivial Neighborhoods

First show $\langle x \rangle$ strongly closed in $C_G(x) \implies$ almost strongly p-embedded subgroup (p odd).

Then $\exists z = x^g \in C_G(x)$, $\langle z \rangle \neq \langle x \rangle$ (Z^* theorem)

Often $[z, D] = 1$. Action of D on $K^g \triangleleft C_K(z)$ is faithful and produces a nontrivial "neighborhood in K^g" which implies that $\mathcal{N}(D, L)$ is nontrivial too.

If $[z, D] \neq 1$ find a suitable "bridge element" x_1 of order p centralizing both D and z and show
nontrivial nbhd in $K^g \implies$
nontrivial nbhd in $E(C_G(x_1)) \implies$
$\mathcal{N}(D, L)$ is nontrivial.

The Generic Case: The Ranking Function f

We have to refine our choice of (x, K) to make sure $\mathcal{N}(D, L)$ is vertical. The "bigger" the better.

If $p = 2$ and $K \cong A_n$, make sure that n is as large as possible.

If $K = {}^d\mathcal{L}_n(q) \in Chev$, maximize $f(K) = q^{n^2}$.
Subject to that, maximize the Lie type of K $(A - G)$:
$A < D < E < BC < F < G$

Call (x, K) "maximal" if it achieves these maximizations.

$f(K)$ is a crude measure of the Sylow q-subgroup of K.

Properties:

- $I \uparrow_p J \implies f(I) \leq f(J)$
- $I, J \in Chev(2)$, I involved in $J \implies f(I) \leq f(J)$.

30

The Generic Case: Vertical Neighborhoods

Theorem

If (x, K) *is maximal, then* $\mathcal{N}(D, L)$ *is vertical.*

The Generic Case: Vertical Neighborhoods

Theorem

If (x, K) is maximal, then $\mathcal{N}(D, L)$ is vertical.

Otherwise, for some $d \in D - \langle x \rangle$,

$$L_d = L_1 \times \cdots \times L_p, \quad \text{each } L_i \cong L$$

Now start with (d, L_1), with $m_p(C_{C_G(d)}(L_1)) \geq 2p - 1$. (Assuming p odd)
Take a series of vertical pumpups ending with a p-terminal pair:

$$(d, L_1) = (d_1, J_1) < \cdots < (d^*, J^*)$$

By previous theorem $m_p(C_{C_G(d^*)}(J^*)) = 1$ (in Lie type case) Hence there exists a series

$$K \downarrow_p L \sim L_1 = J_1 \uparrow_p J_2 \uparrow_p \cdots \uparrow_p J_m = J^*$$

such that $m \geq 3$, implying $f(J^*) > f(K)$ and contradicting maximality of (x, K).

The Generic Case: Volume 7

In Volume 7 our neighborhood gets refined further and the alternating group case is finished.

Theorem*

In the Generic Case, there exists a prime p and maximal pair (x, K), *and* $x \in D \cong Z_p \times Z_p$ *and* $L \lhd\lhd E(C_K(D))$, *such that*

1. $\mathcal{N} := \mathcal{N}(D, L)$ *is vertical;*
2. *If* $K \in Chev$, *then p splits K (i.e., p divides* $q^2 - 1$*);*
3. *If* $K \in Chev$, *then* $\mathcal{N}$ *is level.*

Moreover, either $K \in Chev$ *or* $G = \langle \mathcal{N} \rangle \cong A_n$, $n \geq 13$.

In the alternating case we prove $\langle \mathcal{N} \rangle \cong A_n$ and $N_G(\langle \mathcal{N} \rangle)$ is strongly embedded in G.

In this theorem, for the first time, we specify for any given K a unique choice of D and L (up to automorphisms of K) – an "acceptable subterminal pair." We do this to reduce the number of recognition results we will have to prove to recognize $\langle \mathcal{N} \rangle$.

The Generic Case: Volume 8, The Lie-type Endgame

Now that $\mathcal{N}$ looks good, we need to identify $\langle \mathcal{N} \rangle$.

Theorem

With $\mathcal{N}$ as before and $K \in Chev$,

- $\langle \mathcal{N} \rangle \in Chev$

The Generic Case: Volume 8, The Lie-type Endgame

Now that $\mathcal{N}$ looks good, we need to identify $\langle \mathcal{N} \rangle$.

Theorem

With $\mathcal{N}$ as before and $K \in Chev$,

- $\langle \mathcal{N} \rangle \in Chev$
- $\Gamma_{D,1}(G) \leq N_G(\langle \mathcal{N} \rangle)$, *i.e.,* $N_G(D_0) \leq N_G(\langle \mathcal{N} \rangle)$ *for all* $1 \neq D_0 \leq D$

The Generic Case: Volume 8, The Lie-type Endgame

Now that $\mathcal{N}$ looks good, we need to identify $\langle \mathcal{N} \rangle$.

Theorem

With $\mathcal{N}$ as before and $K \in Chev$,

- $\langle \mathcal{N} \rangle \in Chev$
- $\Gamma_{D,1}(G) \leq N_G(\langle \mathcal{N} \rangle)$, *i.e.,* $N_G(D_0) \leq N_G(\langle \mathcal{N} \rangle)$ *for all* $1 \neq D_0 \leq D$
- $N_G(\langle \mathcal{N} \rangle)$ *is almost strongly p-embedded in* G, *or* $\langle \mathcal{N} \rangle = G$.

33 The Generic Case: Recognition Criteria

Recognition theorems for groups in *Chev*:

- Curtis-Tits theorems and Phan theory (odd characteristic, some characteristic 2)
- Gilman-Griess theorem (characteristic 2, exceptional gps)
- Wong-Finkelstein-Solomon theorems for classical groups

The Generic Case: Wong-Finkelstein-Solomon Method

Illustration: recognize $Sp(V) = Sp_{2m}(q)$, $q = 2^n$, $m \geq 5$.
Standard module $V = V_1 \perp \cdots \perp V_m$, $\dim V_i = 2$
For every $I \subseteq \{1, \ldots, m\}$ let $V_I = \oplus_{i \in I} V_i$ and

$$Sp(V) \geq C_{Sp(V)}(V_{I'}) \text{"="} Sp(V_I)$$

Let $T \leq Sp(V)$, $T \cong Sym_m$ permuting $\{V_1, \ldots, V_m\}$ naturally, with $N_T(V_i) = C_T(V_i)$ for all $i = 1, \ldots, m$.

Theorem (Solomon-Wong-Finkelstein)

Let $G = \langle K, N \rangle$. Suppose
given an isomorphism $f : Sp(V_{\{1,\ldots,m-1\}}) \to K$, and
given a surjection $\lambda : T \leftarrow N$.
Then there is a surjection $g : Sp(V) \to G$,
assuming two natural conditions on f and λ:

The Generic Case: Wong-Finkelstein-Solomon Method

Illustration: recognize $Sp(V) = Sp_{2m}(q)$, $q = 2^n$, $m \geq 5$.
Standard module $V = V_1 \perp \cdots \perp V_m$, $\dim V_i = 2$
For every $I \subseteq \{1, \ldots, m\}$ let $V_I = \oplus_{i \in I} V_i$ and

$$Sp(V) \geq C_{Sp(V)}(V_{I'}) \text{"="} Sp(V_I)$$

Let $T \leq Sp(V)$, $T \cong Sym_m$ permuting $\{V_1, \ldots, V_m\}$ naturally, with $N_T(V_i) = C_T(V_i)$ for all $i = 1, \ldots, m$.

Theorem (Solomon-Wong-Finkelstein)

Let $G = \langle K, N \rangle$. Suppose
given an isomorphism $f : Sp(V_{\{1,\ldots,m-1\}}) \to K$, and
given a surjection $\lambda : T \leftarrow N$.
Then there is a surjection $g : Sp(V) \to G$,
assuming two natural conditions on f and λ:

1. $\lambda \circ f|_{T_m} = 1_{T_m}$ *and*
2. *Let $g \in N$ and $I \subseteq \{1, \ldots, m\}$ be such that $m \notin I \cup \lambda(g)(I)$. Then $f(Sp(V_I))^g = f(Sp(V_{\lambda(g)(I)}))$.*

C.Curtis, J.Tits, and K.-W.Phan

Curtis-Tits: Let $K \in \mathit{Chev}$ of twisted rank ≥ 3. Let Δ be the set of nodes of the twisted Dynkin diagram and $S := \{K_\delta \mid \delta \in \Delta\}$ the set of fundamental rank 1 subgroups. Then with respect to the generating set $\cup S$, the subgroups $\langle K_\delta, K_{\delta'} \rangle$, $\delta, \delta' \in \Delta$, together contain defining relations for the universal version of K.

Phan: Consider certain $K \in \mathit{Chev}$ of untwisted rank ≥ 3. Let Γ be the set of nodes of the untwisted Dynkin diagram. Let K_γ, $\gamma \in \Gamma$, be copies of $SL_2(q)$ and make twisted isomorphism type assumptions about $\langle K_\gamma, K_{\gamma'} \rangle$, $\gamma, \gamma' \in \Gamma$. (Most famously, $[K_\gamma, K_{\gamma'}] = 1$ if γ, γ' not connected, and $\langle K_\gamma, K_{\gamma'} \rangle \cong (S)U_3(q)$ if γ, γ' connected by a single bond.) Such subgroups can be found in K and often contain defining relations for K. Note: K itself is not necessarily a twisted group.

Volume 9: Groups of Special Even Type, $e(G) \geq 4$

Theorem ($\mathcal{C}_5$ and $\mathcal{C}_6$, in progress)

Let G be of special even type, $e(G) \geq 4$.

- *(Theorem $\mathcal{C}_5$) If $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p$, then $p = 3$ and $G \cong Co_1$, Fi_{22}, Fi_{23}, Fi'_{24}, F_2, F_1, $\Omega_7(3)$, $P\Omega_8^{\pm}(3)$, $U_7(2)$, ${}^2D_5(2)$, or ${}^2E_6(2)$.*
- *(Theorem $\mathcal{C}_6$) $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p$.*

Volume 9: Groups of Special Even Type, $e(G) \geq 4$

Theorem ($\mathcal{C}_5$ and $\mathcal{C}_6$, in progress)

Let G be of special even type, $e(G) \geq 4$.

- *(Theorem $\mathcal{C}_5$) If $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p$, then $p = 3$ and $G \cong Co_1$, Fi_{22}, Fi_{23}, Fi'_{24}, F_2, F_1, $\Omega_7(3)$, $P\Omega_8^{\pm}(3)$, $U_7(2)$, ${}^2D_5(2)$, or ${}^2E_6(2)$.*
- *(Theorem $\mathcal{C}_6$) $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p$.*

- G has even type, i.e., for all involutions $z \in G$, $O_{2'}(C_G(z)) = 1$ and all components L of $C_G(z)$ lie in $\mathcal{C}_2$. Thus, L is
  - in $Chev(2)$, or
  - one of 19 sporadic groups and their covers, or
  - $L_2(q)$, $q \in \mathcal{FM}9$, or
  - $(P)\Omega_n^{\pm}(3)$, $n = 5, 6$, or $G_2(3)$ or $L_3(3)$
- For every $x \in G$ of order p (fixed odd prime) with $m_p(C_G(x)) \geq 4$, no components of $\overline{C_G(x)}$ lie in $\mathcal{G}_p$.
- $m_{2,p}(G) \geq 4$
- There is no almost strongly p-embedded subgroup in G.

Remarks on Theorem $\mathcal{C}_5$

Setup:

- $F^*(C_G(z)) = O_2(C_G(z))E(C_G(z))$, components in $\mathcal{C}_2$ $(z^2 = 1)$
- $\mathcal{L}_p^o(G) \subseteq \mathcal{C}_p$
- $m_{2,p}(G) \geq 4$
- There is no almost strongly p-embedded subgroup in G.

1. $\mathcal{C}_3 = Chev(3) \cup 16$ sporadics $\cup \cdots$
2. K. Klinger and G. Mason (1974) proved that if $m_{2,p}(G) \geq 3$, G cannot simultaneously be of characteristic 2-type and p-type. (Subremark: Characteristic p-type includes $O_{p'}(C_G(x)) = 1$ for all x of order p. We can get by assuming $O_{p'}(C_G(x))$ has odd order for certain x.)
3. Root ideas go through Klinger-Mason back to Thompson's N-group paper. Heavy use of Thompson Dihedral Lemma.

Some tools

Lemma (Thompson Dihedral Lemma)

If $T \cong (Z_2)^n$ acts faithfully on a p-group P, p odd, then TP contains the direct product of n copies of D_{2p}. In particular $m_{2,p}(TP) \geq n-1$.

Definition

$\mathcal{E}^p(G) = \{B \leq G \mid B \cong (Z_p)^n, \text{ some } n > 0\}$.
$\mathcal{B}_*(G) =$ the set of "witnesses" to $m_{2,p}(G)$
$\{B \in \mathcal{E}^p(G) \mid \mathit{V}_G(B;2) \neq \{1\},\ m_p(B) = m_{2,p}(G)\}$,
$\mathcal{S}^p(G) =$ the set of maximal el.ab. p-groups w.r.t. inclusion

Strong Balance Theorem

Lemma (Strong Balance)

Let $B \in \mathcal{S}^p(G)$ with $m_p(B) \geq 4$. Then for all $b \in B^\#$ and every B-invariant p'-subgroup W of $C_G(b)$, $W \leq O_{p'}(C_G(b))$.
EXCEPT IF $p = 3$ and $\overline{C_G(b)}$ has a component $\overline{L} \cong L_2(3^3)$, and some $b \in B$ induces a field automorphism on $\overline{L}$ of order 3; for then, $C_{\overline{L}}(b) \cong L_2(3)$.

Theorem (Strong Balance)

G is strongly balanced with respect to any $B \in \mathcal{S}^p(G)$ such that $m_p(B) \geq 4$. (I.e., the exceptional configuration in the Strong Balance Lemma does not occur in G.)

The indirect proof constructs a p-component uniqueness subgroup and argues that it is almost strongly p-embedded.

Theorem $\mathcal{C}_5$: Stage 1

Theorem (Stage 1)

Under the hypotheses of Theorem $\mathcal{C}_5$,

1. *G is balanced with respect to any $B \in \mathcal{S}^p(G)$, $m_p(B) \geq 4$.*
2. *If $B \in \mathcal{B}_*(G)$, then there exists $B < B_1 \in \mathcal{E}^p(G)$.*

Definition: G has weak p-type if and only if for every $1 \neq b \in B \in \mathcal{E}^p(G)$,

- If $m_p(B) \geq 4$, then every component of $\overline{C_G(b)}$ lies in $\mathcal{C}_p$, and
- If $B \in \mathcal{B}_*(G)$, then $O_{p'}(C_G(b))$ has odd order.

Corollary

Under the hypotheses of Theorem $\mathcal{C}_5$, G has weak p-type (as well as even type).

Weak p-type is an analogue for $p > 2$ of even type for $p = 2$.

Groups of Symplectic Type

A 2-group T is of symplectic type $\iff$ $T = E * M$,
E extraspecial, M cyclic or of maximal class $\iff$
Every characteristic abelian subgroup of T is cyclic (P.Hall)

Proposition

Let $B \in \mathcal{B}_*(G)$. Then any B-invariant 2-subgroup $T \leq G$ such that $C_T(B) \neq 1$ is cyclic or of symplectic type.

We call such B, T a symplectic pair if T is maximal with respect to inclusion (relative to fixed B).
It is faithful if $C_B(T) = 1$, and trivial if $|T| = 2$. Examples:

In $G = F_1$, $T = F^*(C_G(z)) \cong 2^{1+24}$, $B \cong E_{3^6}$
In $G = F_2$, $|T| = 2$, $C_G(T) \cong 2^2E_6(2)2$

41 # Theorem $\mathcal{C}_5$, Stage 2: Trivial Symplectic Pairs

Theorem (Theorem $\mathcal{C}_5$, Stage 2)

Symplectic pairs exist, and $p = 3$. Also, $G \cong \Omega_7(3)$ or $P\Omega_8^{\pm}(3)$ or else

- *Every symplectic pair is faithful or trivial.*
- *Let (B, T) be a trivial symplectic pair. Let K be a component of $E(C_G(T)) \neq 1$. After replacing (B, T, K) with a possibly different triple satisfying the same conditions, there is $b \in C_G(T)$ of order 3 such that*
  - *$K > I < J$ where $I \lhd\lhd C_K(b)$, $J \lhd\lhd C_G(b)$ are 3-components;*
  - *(K, I, J), up to isomorphism, is one of a short explicit list of examples. It is called a "nonconstrained $\{2, 3\}$-neighborhood."*

A "NONCONSTRAINED (2, 3)-NEIGHBORHOOD"

$$(K, I, J)$$

$$K \lhd C_G(t) \qquad \overline{J} \lhd \overline{C_G(b)}$$

$$2\,{}^2E_6(2) \qquad Fi_{22}$$

$$\backslash \qquad /$$

$$I = 2U_6(2)$$

$$I \lhd C_G(tb)$$

$$[t, b] = 1$$

Theorem $\mathcal{C}_5$, Stage 3: Faithful Symplectic Pairs

Theorem (Klinger-Mason 90%)

Suppose faithful symplectic pairs (B, T) exist.
Among all such and all $b \in B^{\#}$, maximize $|C_T(b)|$. Then $C_G(b)$ has a 3-component J on which $BC_T(b)/\langle b \rangle$ acts faithfully.

Moreover, $\overline{J} \cong F_1, F_2, \ldots, U_6(2), \ldots$.

The configuration of $C_G(Z(T))$ and $C_G(b)$ is a "CONSTRAINED $\{2, 3\}$-NEIGHBORHOOD".

A Constrained $\{2,3\}$-Neighborhood

Here $\langle z\rangle = Z(T) \cong Z_2$, $[z,b]=1$, $b^3=1$.

$$C_G(z) \qquad\qquad\qquad\qquad \overline{J} \triangleleft \overline{N_G(\langle b\rangle)}$$

$$2^{1+24}Co_1 \qquad\qquad\qquad\qquad 3Fi'_{24}$$

$$\backslash \qquad\qquad\qquad\qquad /$$

$$C_J(z) = 2^{1+12}[3\times 3]U_4(3)2$$

Theorem $\mathcal{C}_5$, Stages 3 and 4

Theorem (Theorem $\mathcal{C}_5$, Stage 3)

Either $G \cong \Omega_7(3)$ or $\Omega_8^{\pm}(3)$, or there exists a constrained or non-constrained $\{2,3\}$-neighborhood in G matching a known group $G^* = Co_1$, Fi_{22}, Fi_{23}, Fi'_{24}, Fi_{24}, F_2, F_1, $U_7(2)$, ${}^2D_5(2)$, or ${}^2E_6(2)$.

Red: Non-constrained
Black: Constrained

45 Theorem $\mathcal{C}_5$, Stages 3 and 4

Theorem (Theorem $\mathcal{C}_5$, Stage 3)

Either $G \cong \Omega_7(3)$ or $\Omega_8^{\pm}(3)$, or there exists a constrained or non-constrained $\{2,3\}$-neighborhood in G matching a known group $G^* = Co_1$, Fi_{22}, Fi_{23}, Fi'_{24}, Fi_{24}, F_2, F_1, $U_7(2)$, ${}^2D_5(2)$, or ${}^2E_6(2)$.

Red: Non-constrained
Black: Constrained

Theorem (Theorem $\mathcal{C}_5$, Stage 4)

$G \cong G^*$.

–THANK YOU–