Cyclically reduced elements in Coxeter groups

Groups and Geometries — Banff 2019

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UCLouvain

August 26, 2019

The word and conjugacy problems in Coxeter groups

- Throughout this talk, (W, S) denotes a Coxeter system:

$$W = \langle s \in S \mid s^2 = 1 = (st)^{m_{st}} \text{ for all } s, t \in S \text{ with } s \neq t \rangle$$

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Solution to the word problem in W (Tits, Matsumoto, 1960's)

Assume that $\boldsymbol{w}, \boldsymbol{w}' \in \boldsymbol{W}$ represent the same element of W, and that $\boldsymbol{w}'$ is reduced. Then $\boldsymbol{w}'$ can be obtained from $\boldsymbol{w}$ by a (finite) sequence of elementary operations of the form

- *Braid relations*: $\underbrace{stst\dots}_{m_{st} \text{ letters}} \mapsto \underbrace{tsts\dots}_{m_{st} \text{ letters}}$ for distinct $s, t \in S$ with $m_{st} < \infty$.
- *ss-cancellations*: $ss \mapsto \varnothing$ for $s \in S$.

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A first step towards a better algorithm might be found by use of reductions of w of the form

$$w \mapsto sws \quad \text{whenever} \quad \ell(sws) \leq \ell(w). \tag{2}$$

We shall call w *conjugacy-reduced* if each series of reductions as in (2) starting with w leads to an element w' of W with $\ell(w') = \ell(w)$.

Conjecture 2.18 *Let C be a conjugacy class of W and put $\ell_C = \min\{\ell(w) \mid w \in C\}$. Then, for any $w \in C$, we have $\ell(w) = \ell_C$ if and only if w is conjugacy-reduced.*

By Geck and Pfeiffer [1992], the conjecture holds for Weyl groups. The authors use the result for Hecke algebra representations.

A. Cohen, *Recent results on Coxeter groups* (1994)

in NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.

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Example: $W = \langle s, t \mid s^2 = t^2 = (st)^3 = 1 \rangle = D_6$

The elements s and t are conjugate but $s \not\to t$.

Previous works

- Two elements $w, w' \in W$ are **elementarily strongly conjugate** if
  - $\ell(w') = \ell(w)$ and
  - there exists $x \in W$ with $w' = x^{-1}wx$ such that either $\ell(x^{-1}w) = \ell(x) + \ell(w)$ or $\ell(wx) = \ell(w) + \ell(x)$.

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Theorem (Geck–Pfeiffer, 1993)

Assume that W is **finite**. Let $\mathcal{O}$ be a conjugacy class in W. Then:

1. For every $w \in \mathcal{O}$ there exists w' of minimal length in $\mathcal{O}$ with $w \to w'$.
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Theorem (Geck–Kim–Pfeiffer, 2000 and He–Nie, 2012)

Assume that W is **finite**. Let $\mathcal{O}$ be a **twisted** conjugacy class in W. Then:

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- Let $\delta \in \mathrm{Aut}(W, S)$ be a diagram automorphism. Define the δ-**twisted conjugation** by $x \in W$ as $W \to W : w \mapsto x^{-1}w\delta(x)$.
  $\rightsquigarrow$ *twisted* conjugacy classes, *twisted* relations $\overset{s}{\to}$, $\overset{x}{\sim}$, etc.

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Theorem (He–Nie, 2014)

Assume that W is **affine**. Let $\mathcal{O}$ be a **twisted** conjugacy class in W. Then:

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Main result

Theorem (M., 2018)

Let (W, S) be a Coxeter system. Let $\mathcal{O}$ be a conjugacy class in W. Then:

1. For every $w \in \mathcal{O}$ there exists w' of minimal length in $\mathcal{O}$ with $w \to w'$.
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An element $w \in W$ is cyclically reduced if and only if it is of minimal length in its conjugacy class.

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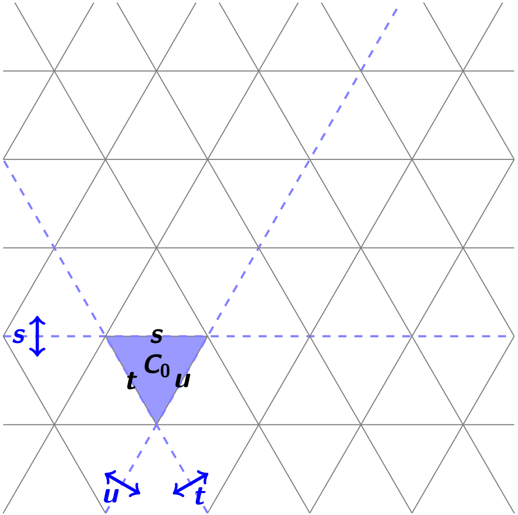
- Call $w, w' \in W$ **tightly conjugate** if w' can be obtained from w by a sequence of elem. tight conjugations. We then write $w \approx w'$.

Proof idea — The Coxeter complex Σ of (W, S)

Ex: $W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (su)^3 = (tu)^3 = 1 \rangle = \widetilde{A}_2$



$W \curvearrowright \Sigma$ simplicial complex

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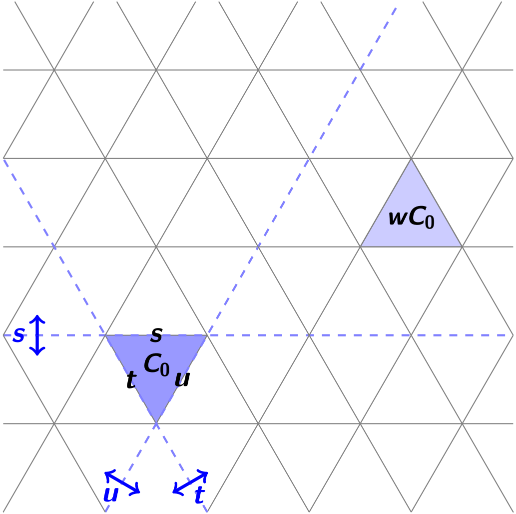

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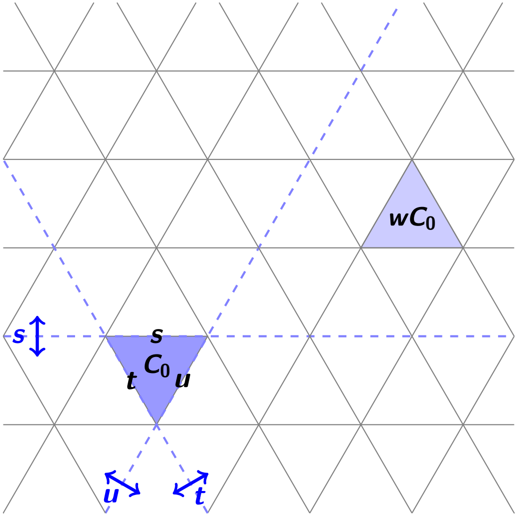

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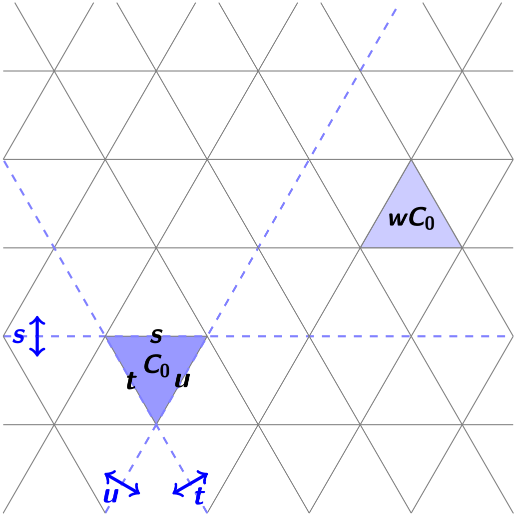

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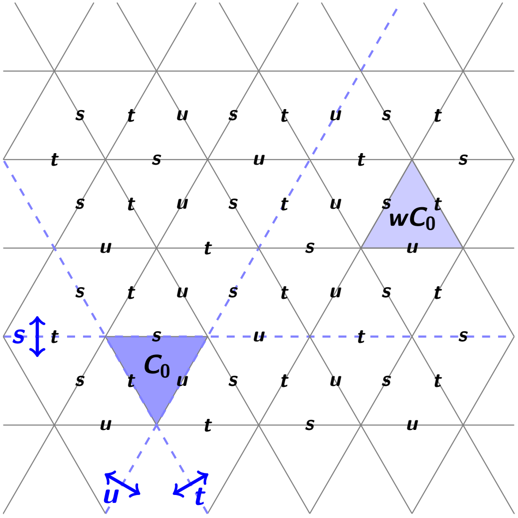

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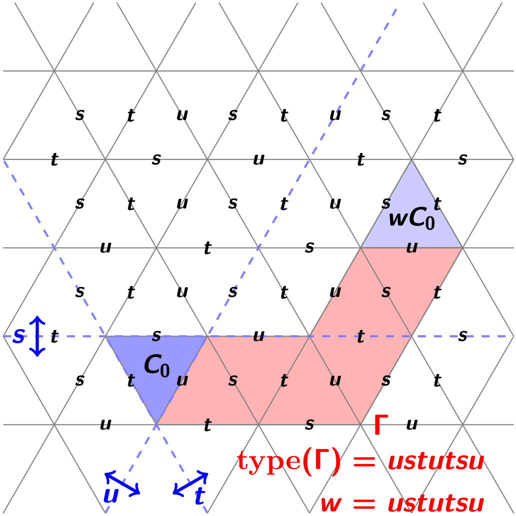

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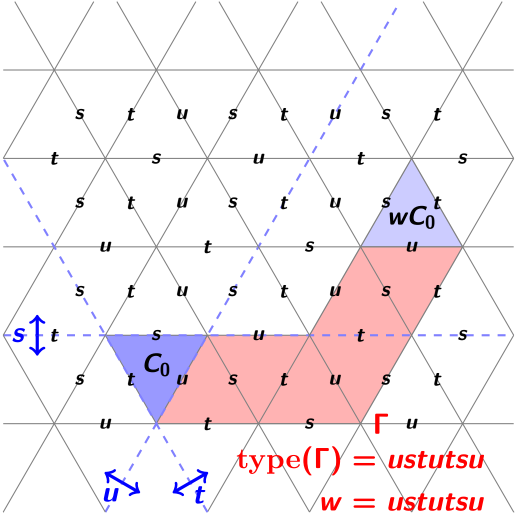

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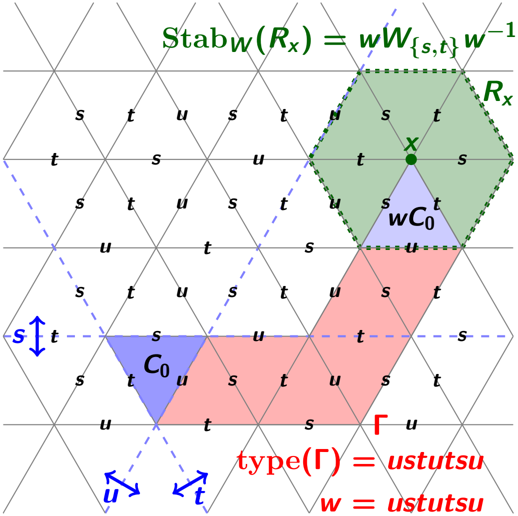

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residue $R_x = \{\text{chambers} \ni x\}$

$\mathrm{Stab}_W(R_x)$ spherical parabolic

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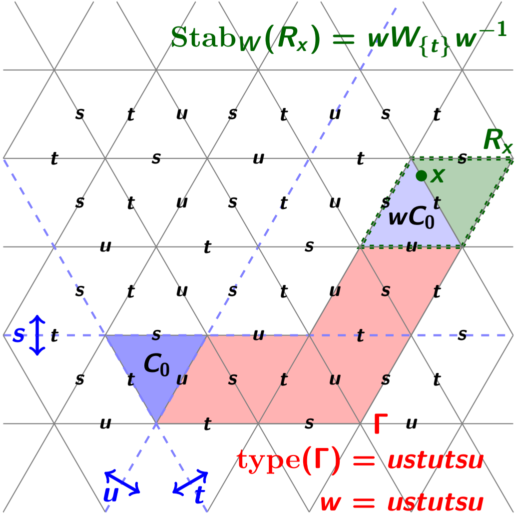


$W \curvearrowright \Sigma$ simplicial complex

chambers = max simplices

$\mathrm{Ch}(\Sigma) = \{wC_0 \mid w \in W\}$

walls = reflection hyperplanes

$\mathrm{d_{Ch}}$ chamber **distance** on $\mathrm{Ch}(\Sigma)$

$\mathrm{d_{Ch}}(C_0, wC_0) = \ell(w)$

galleries from $C_0 \leftrightarrow$ ***W***

$\Gamma = (C_0, \dots, wC_0) \mapsto \mathrm{type}(\Gamma)$

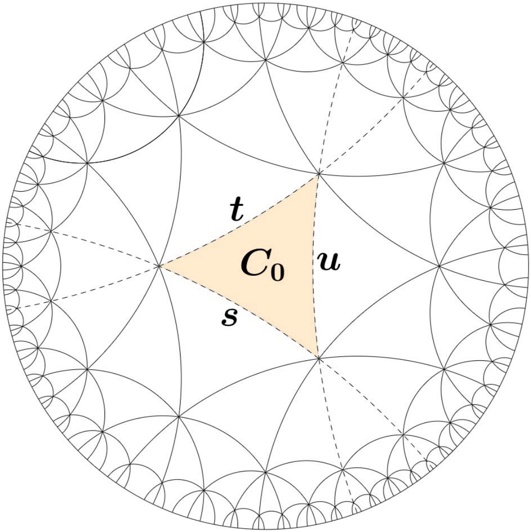
$(|\Sigma|_{\mathrm{CAT}(0)}, \mathrm{d})$ **Davis complex**

residue $R_x = \{\text{chambers} \ni x\}$

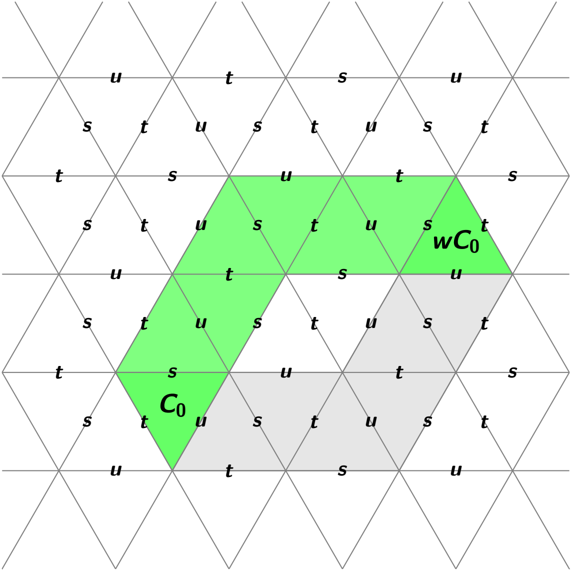
$\mathrm{Stab}_W(R_x)$ spherical parabolic

Proof idea — The Coxeter complex Σ of (W, S)

Ex: $W = \langle s, t, u \mid s^2 = t^2 = u^2 = (st)^4 = (su)^4 = (tu)^4 = 1 \rangle$



Proof idea — A geometric solution to the word problem



$w = sutstus$

$= ustutsu$

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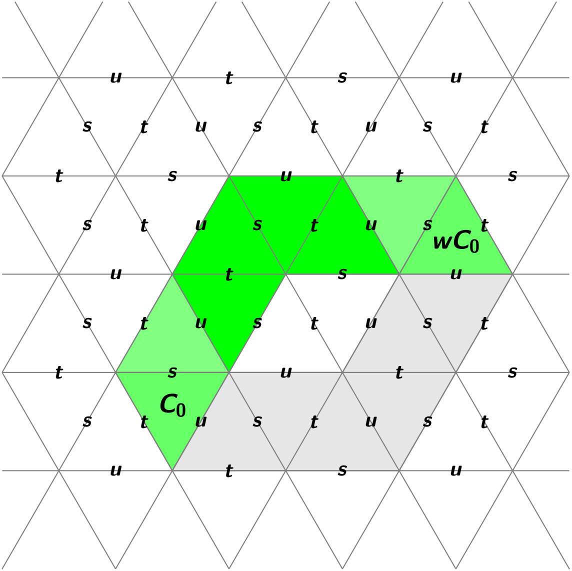

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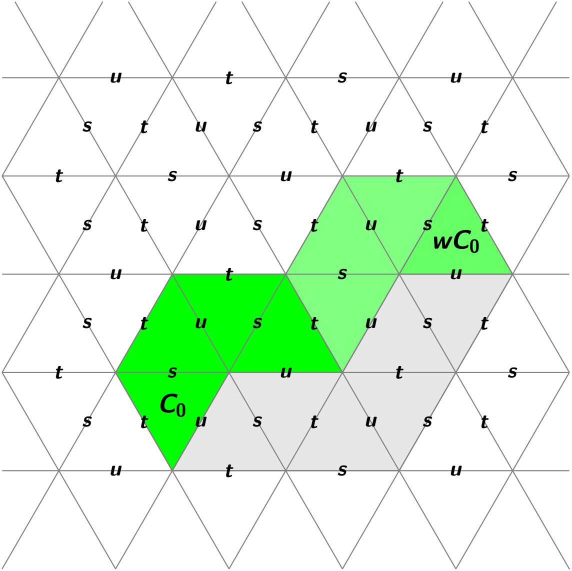

$w = su\textcolor{blue}{tst}us$

$= \textcolor{blue}{sus}tsus$

$= ustutsu$

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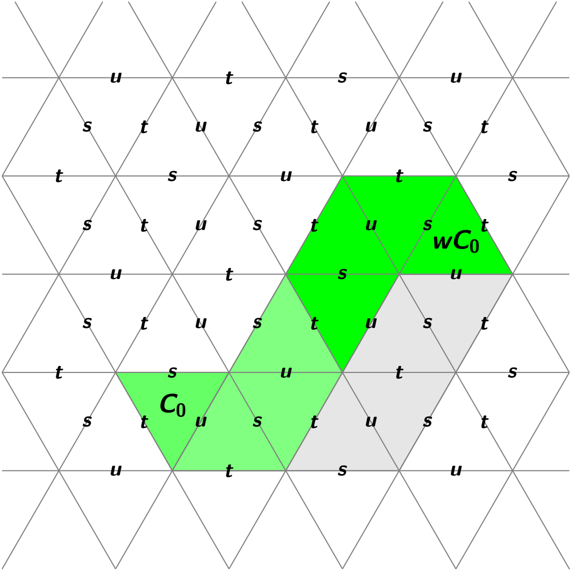

$$w = sutstus$$
$$= sustsus$$
$$= usutsus$$

$$= ustutsu$$

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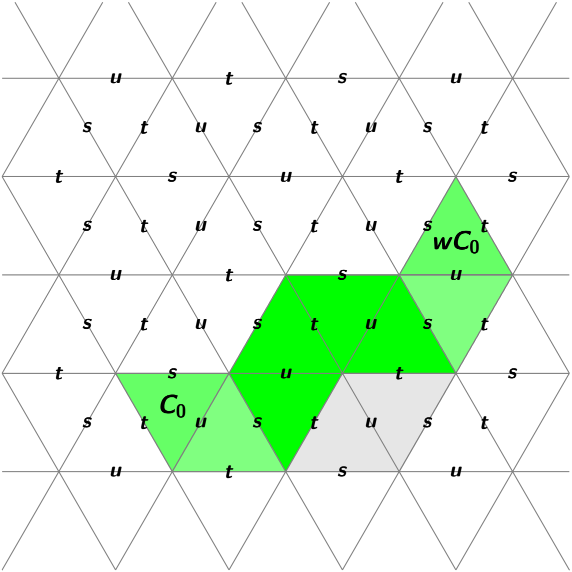

$$\begin{aligned} w &= sutstus \\ &= sustsus \\ &= usutsus \\ &= usutusu \\ &= ustutsu \end{aligned}$$

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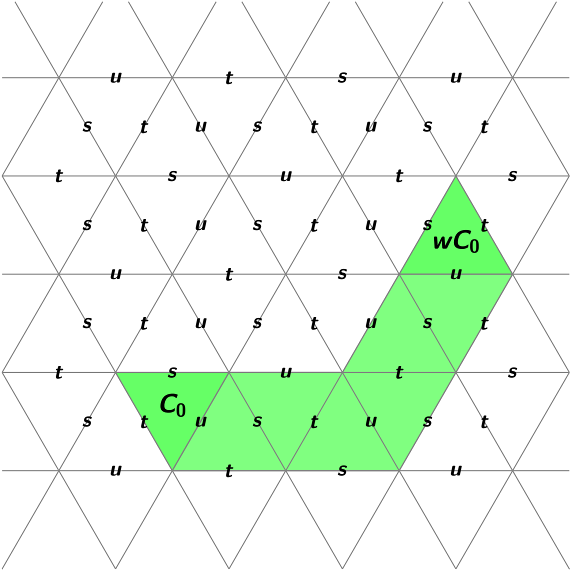

$$
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&= sustsus \\
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\end{aligned}
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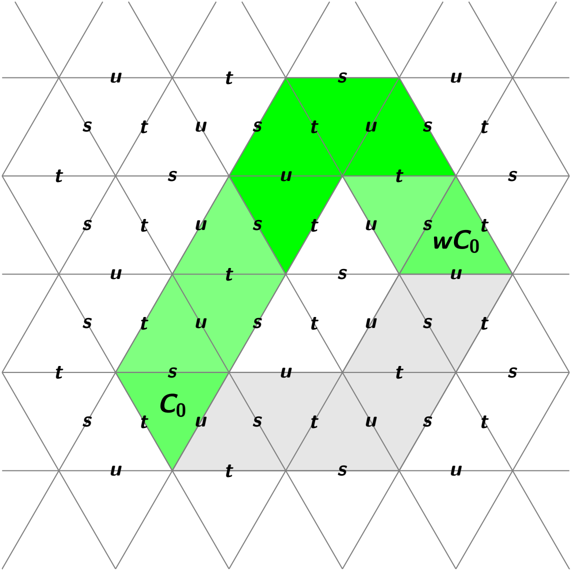

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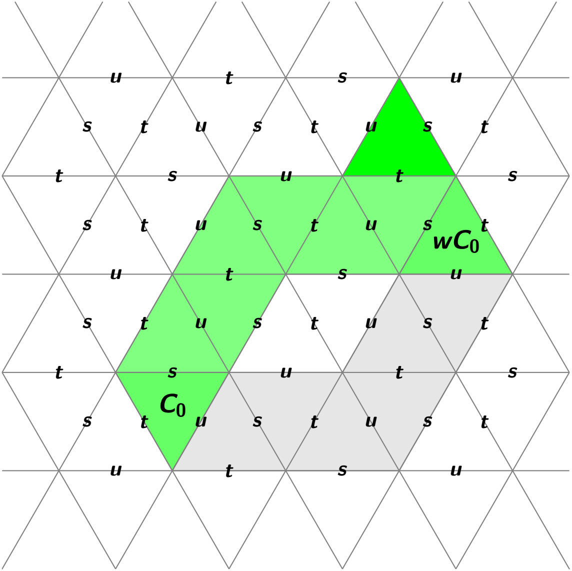

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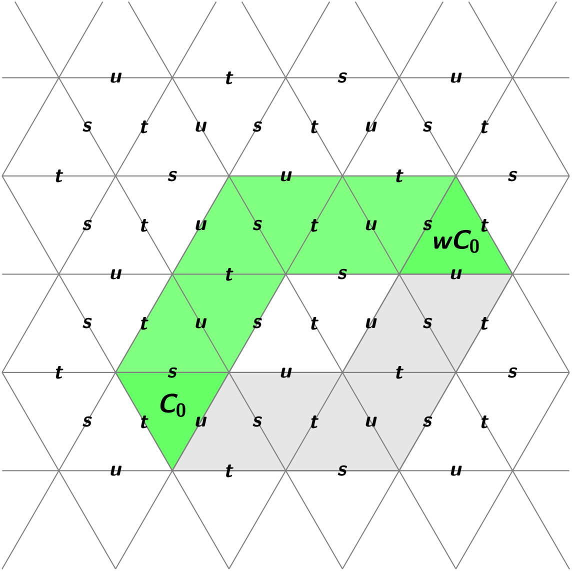

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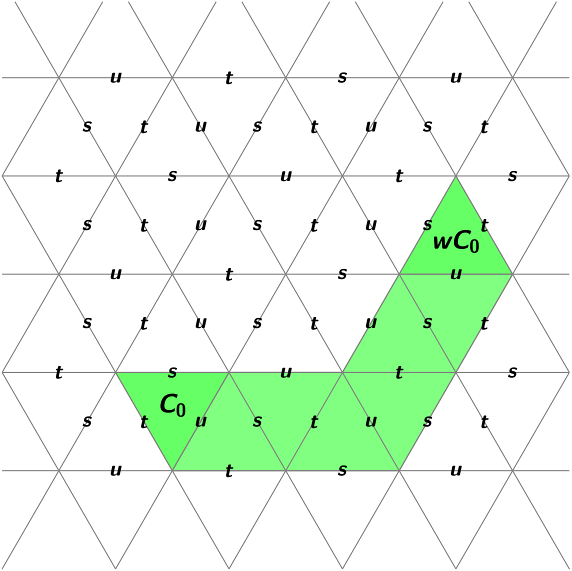

$$\begin{aligned} w &= sutsututs \\ &= sutstutts \\ &= sutstus \\ &= \dots \\ &= ustutsu \end{aligned}$$

Proof idea — Minimal displacement sets

Fix $w \in W$, and let

$$\mathcal{O}_w = \{v^{-1}wv \mid v \in W\} \quad \text{and} \quad \mathcal{O}_w^{\mathsf{min}} = \{u \in \mathcal{O}_w \mid \ell(u) \text{ minimal}\}.$$

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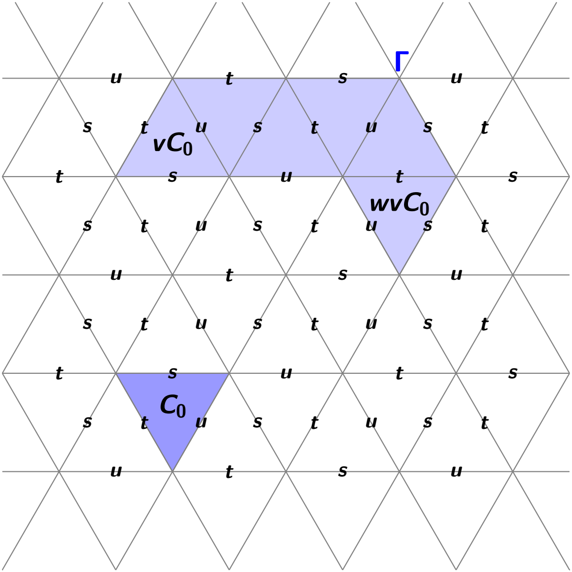

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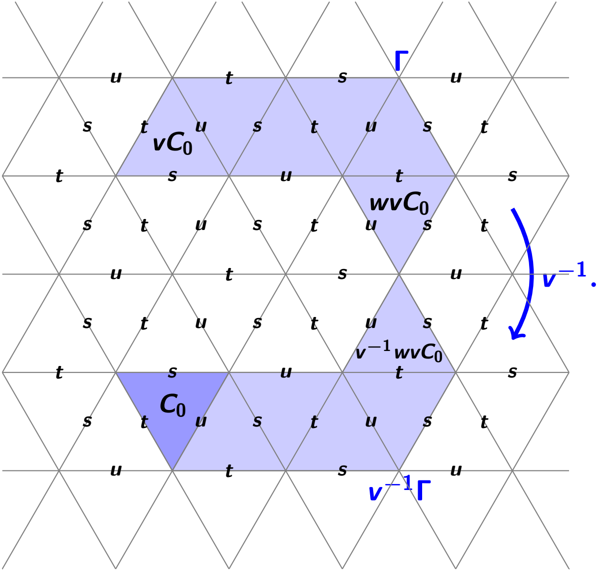

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Obs: If $\Gamma \subseteq \mathsf{CombiMin}(w)$ gallery from D to E, then $\pi(D) \to \pi(E)$.

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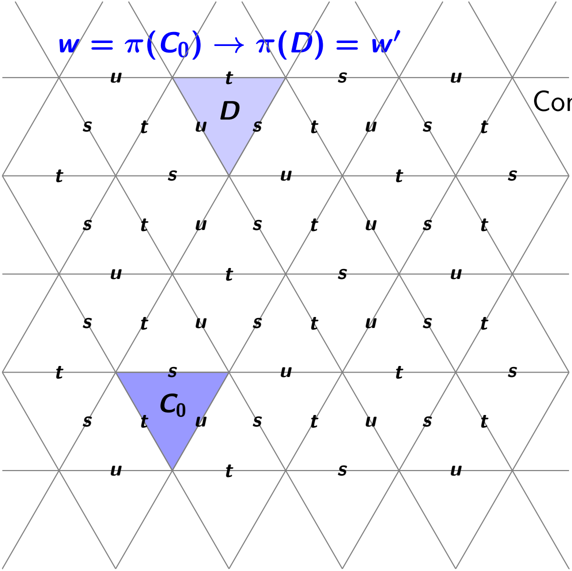
Proof: WLOG, $D = vC_0$ and $E = vsC_0$ adjacent ($v \in W, s \in S$).
$\implies \ell(v^{-1}wv) = \ell(\mathcal{O}_w^{\min}) = \ell(sv^{-1}wvs)$
$\implies \pi(D) = v^{-1}wv \xrightarrow{s} sv^{-1}wvs = \pi(E)$. □

Proof idea — Minimal displacement sets

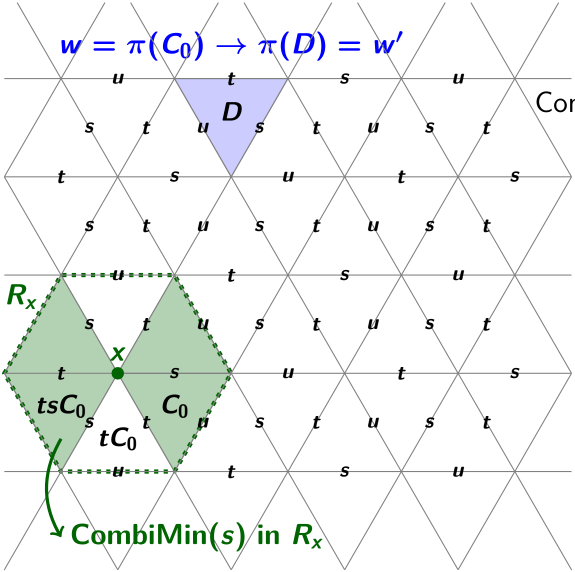


Suppose $w \in \mathcal{O}_w^{\min}$ and CombiMin(w) gallery-connected

$$\Downarrow$$

$w \to w'$ for all $w' \in \mathcal{O}_w^{\min}$

Proof idea — Minimal displacement sets



Suppose $w \in \mathcal{O}_w^{\min}$ and CombiMin(w) gallery-connected

$\Downarrow$

$w \to w'$ for all $w' \in \mathcal{O}_w^{\min}$

In general, CombiMin(w) is not connected

$\Downarrow$

needs strong conjugations!

$s \rightsquigarrow tst \rightsquigarrow ststs = t$

Proof idea — The complex $\mathcal{C}^w$

Let $\mathcal{C}^w$ be the smallest chamber subcomplex A of Σ such that

1. $C_0 \in \mathsf{Ch}(A)$;
2. If $C \in \mathsf{Ch}(A)$ and Γ minimal gallery from C to $w^{\pm 1}C$, then $\Gamma \subseteq A$;
3. Let R be a (spherical) residue such that w normalises $\mathrm{Stab}_W(R)$.
   If $C, D \in R$ and $C \in \mathsf{Ch}(A)$ and $D \in \mathsf{CombiMin}(w)$, then $D \in \mathsf{Ch}(A)$.

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Proof: This is equivalent to: If $C \in \mathrm{Ch}(\mathcal{C}^w)$, then $\pi(C_0) \to\approx \pi(C)$.

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<u>Obs</u>: If $vC_0 \in \mathcal{C}^w$ then $w \to\approx v^{-1}wv$.

<u>Proof</u>: This is equivalent to: If $C \in \mathrm{Ch}(\mathcal{C}^w)$, then $\pi(C_0) \to\approx \pi(C)$.
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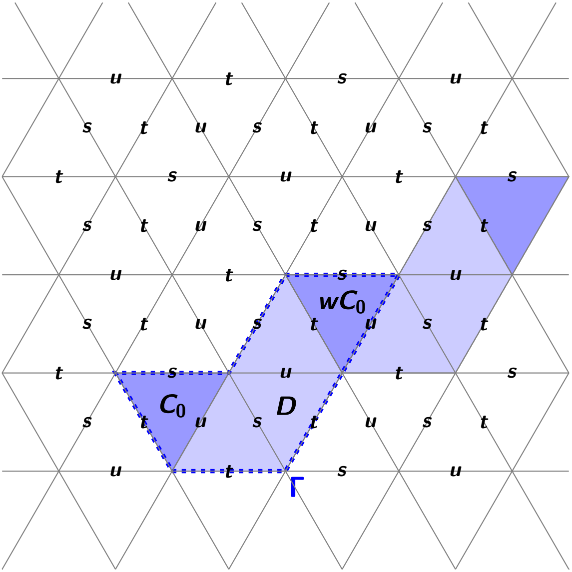
To simplify notations, say $C = C_0$. For (1), see picture; for (2), see below.

Proof idea — The complex $\mathcal{C}^w$

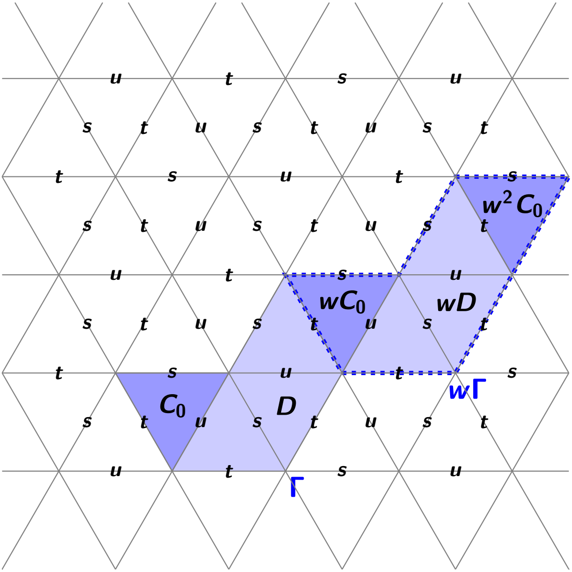


Proof of (1):

Γ from C_0 to wC_0

$\mathrm{type}(\Gamma) = usut$

Proof idea — The complex $\mathcal{C}^w$



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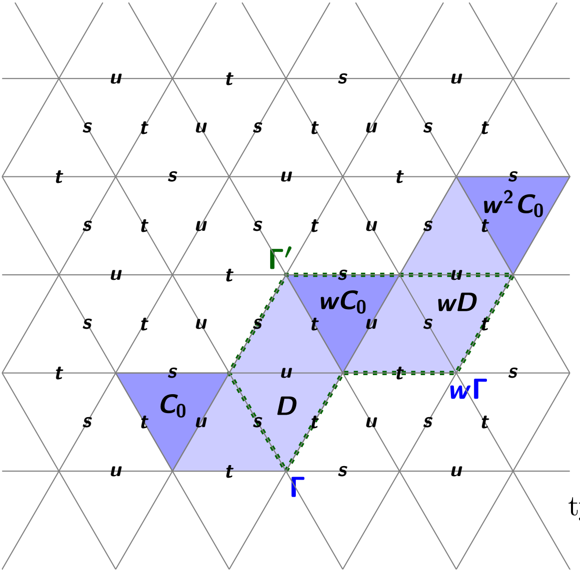
$w\Gamma$ from wC_0 to w^2C_0

$\mathrm{type}(w\Gamma) = usut$

Proof idea — The complex $\mathcal{C}^w$



Proof of (1):

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$w\Gamma$ from wC_0 to w^2C_0

$\text{type}(w\Gamma) = usut$

Γ' from D to wD

$\text{type}(\Gamma') = usutusut$

$$\text{type}(\Gamma) \xrightarrow{\text{cyclic shifts}} \text{type}(\Gamma')$$

$$\pi(C_0) \to \pi(D)$$

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Note that $\delta\colon\ W_I \to W_I : x \mapsto n_I x n_I^{-1}$ is a diagram automorphism.

Proof idea — The complex $\mathcal{C}^w$

Write $w \to\approx w'$ if there exists $w'' \in W$ with $w \to w'' \approx w'$.

Obs: If $vC_0 \in \mathcal{C}^w$ then $w \to\approx v^{-1}wv$.

Proof: This is equivalent to: If $C \in \mathrm{Ch}(\mathcal{C}^w)$, then $\pi(C_0) \to\approx \pi(C)$.
(1) If D is on a minimal gallery Γ from C to wC, then $\pi(C) \to \pi(D)$.
(2) If $C, D \in R$ and $D \in \mathrm{CombiMin}(w)$, then $\pi(C) \to\approx \pi(D)$.
To simplify notations, say $C = C_0$. For (1), see picture; for (2), see below.
Hyp: $C_0, D \in R$ and w normalises $\mathrm{Stab}_W(R) = W_I$ with $I \subseteq S$ spherical.
Lem (Lusztig '77): $N_W(W_I) = W_I \rtimes N_I$ where $N_I = \{w \in W \mid w.I = I\}$,
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As $D \in R$, we have $D = vC_0$ for some $v \in W_I$.
To prove: $w_I \cdot n_I = w \to\approx v^{-1}wv$

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By [GKP00] or [HN12] in W_I, we have $w_I \to_\delta \approx_\delta v^{-1} w_I \delta(v)$ in W_I,
and hence $w_I \cdot n_I \to\approx v^{-1} w_I \delta(v) \cdot n_I$, as desired. □

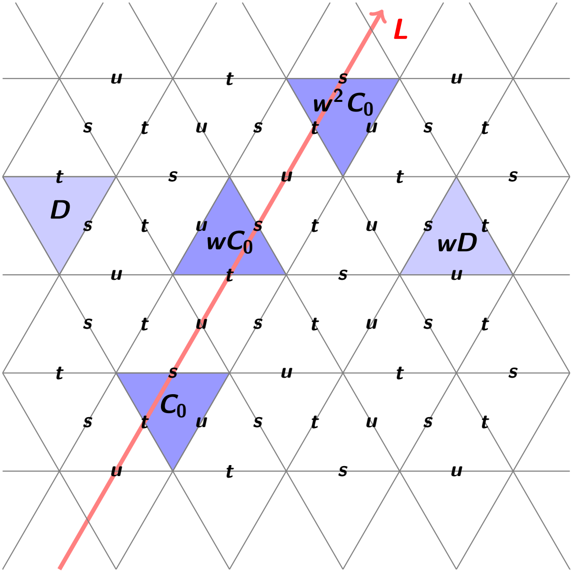
Proof idea — Here we go

Recall we have a CAT(0) metric d on $X := |\Sigma|_{\mathrm{CAT(0)}}$.
The **minimal displacement set** of w is

$$\mathrm{Min}(w) = \{x \in X \mid \mathrm{d}(x, wx) \text{ is minimal}\}.$$

If w has infinite order, then $\mathrm{Min}(w)$ is the closed convex subset of X which is the union of all w-**axes**, i.e. of all geodesic lines L stabilised by w.

Proof idea — Here we go



Example:

$$w = sut$$

$$\mathrm{Min}(w) = L$$

$$\mathrm{Min}(w^2) = |\Sigma|_{\mathrm{CAT}(0)}$$

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Reduction step: WLOG, w, w' have an axis through C_0.

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Claim: Write $w' = v^{-1}wv$ for some $v \in W$. Then $vC_0 \in \mathcal{C}^w$.
Hyp: $vC_0 \in \mathrm{CombiMin}(w)$.

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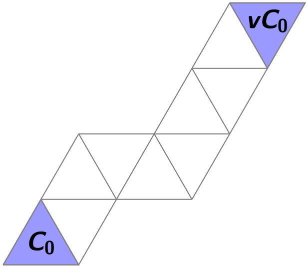

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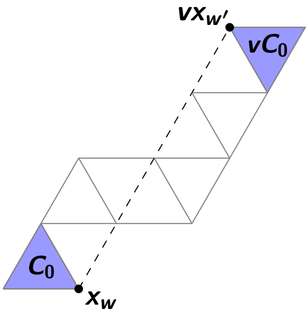

Let $x_w \in \mathrm{Min}(w)$ and $x_{w'} \in \mathrm{Min}(w')$.
$\Rightarrow vx_{w'} \in \mathrm{Min}(vw'v^{-1}) = \mathrm{Min}(w)$
$\Rightarrow [x_w, vx_{w'}] \subseteq \mathrm{Min}(w)$

Proof idea — Here we go

Theorem: Let $w' \in \mathcal{O}_w^{\min}$. Then $w \to_{\approx} w'$.

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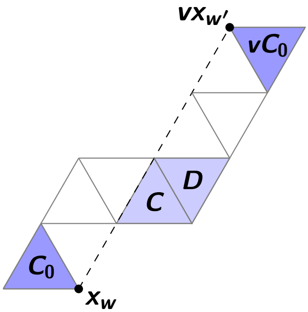

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Let $\Gamma = (C_0, \ldots, C, D, \ldots, vC_0)$ be a minimal gallery containing $[x_w, vx_{w'}]$.

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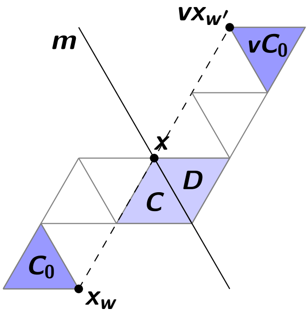

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Let m be the wall between C and D.
Let L be a w-axis through $x \in C \cap D$.

Proof idea — Here we go

Theorem: Let $w' \in \mathcal{O}_w^{\min}$. Then $w \to\approx w'$.

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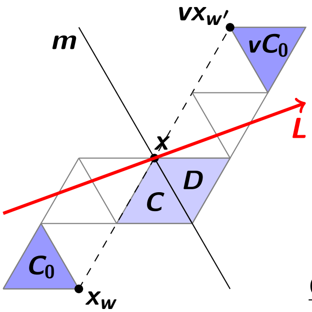

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Let m be the wall between C and D.
Let L be a w-axis through $x \in C \cap D$.

Case 1: $L \cap m = \{*\}$.

Proof idea — Here we go

Theorem: Let $w' \in \mathcal{O}_w^{\min}$. Then $w \to_{\approx} w'$.

Assume that $w \in W$ has infinite order, and let $w' \in \mathcal{O}_w^{\min}$.
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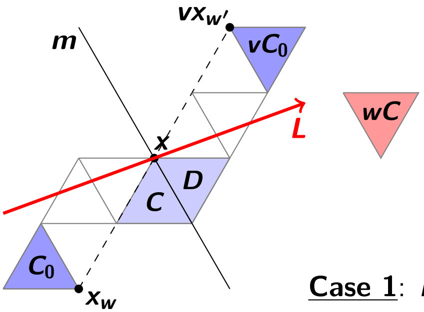

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Let m be the wall between C and D.
Let L be a w-axis through $x \in C \cap D$.

Case 1: $L \cap m = \{*\}$. Then $C \in \mathcal{C}^w \Rightarrow D \in \mathcal{C}^w$.

Proof idea — Here we go

Theorem: Let $w' \in \mathcal{O}_w^{\min}$. Then $w \to_{\approx} w'$.

Assume that $w \in W$ has infinite order, and let $w' \in \mathcal{O}_w^{\min}$.
Reduction step: WLOG, w, w' have an axis through C_0.
Claim: Write $w' = v^{-1}wv$ for some $v \in W$. Then $vC_0 \in \mathcal{C}^w$.
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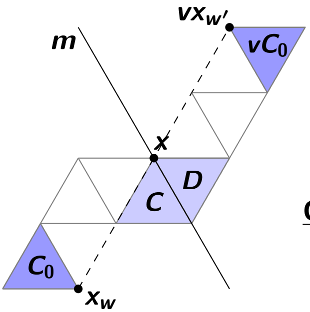

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Let m be the wall between C and D.
Let L be a w-axis through $x \in C \cap D$.

Case 2: $L \subseteq m$.

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Theorem: Let $w' \in \mathcal{O}_w^{\min}$. Then $w \to_{\approx} w'$.

Assume that $w \in W$ has infinite order, and let $w' \in \mathcal{O}_w^{\min}$.
Reduction step: WLOG, w, w' have an axis through C_0.
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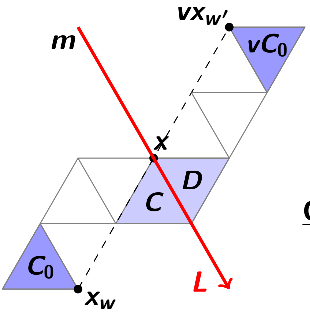

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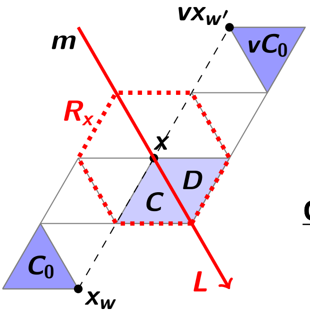

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Case 2: $L \subseteq m$. Then w normalises $\mathrm{Stab}_W(R_x)$.*

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Theorem: Let $w' \in \mathcal{O}_w^{\min}$. Then $w \to_\approx w'$.

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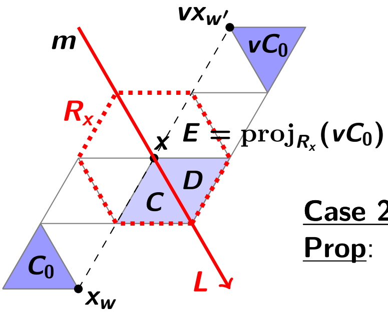

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Prop: $E = \mathrm{proj}_{R_x}(vC_0) \in \mathrm{CombiMin}(w)$

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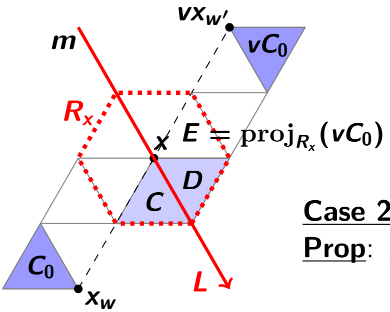

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Hence, $C \in \mathcal{C}^w \Rightarrow E \in \mathcal{C}^w$. □