Odd 2-Fusion Systems

I have a program to, first, classify a large class of simple 2-fusion systems, and then, second, to use the theorem on fusion systems to give an alternate treatment of part of the CFSF, a treatment that I expect will simplify the original. I've been working on the program, on and off, for at least ten years. Today I want to give a progress report, primarily to point out that the theorem on fusion systems is reduced to a finite number of cases.

My plan is to give a rough idea of what the theorem says, and list the open cases. I'll begin with some background.

Fusion systems

Let p be a prime and S a finite p-group. A *fusion system* on S is a category $\mathcal{F}$ whose objects are the subgroups of S and, for subgroups P, Q of S the set $\hom_{\mathcal{F}}(P, Q)$ of morphisms from P to Q is a set of injective group homomorphisms from P into Q, and that set satisfies two weak axioms.

The Standard Example

Let G be a finite group and $S \in Syl_p(G)$. Then $\mathcal{F}_S(G)$ is the fusion system on S whose morphisms are those induced via conjugation in G.

A fusion system is *saturated* if it satisfies two more axioms; the Standard Example is saturated. A saturated system is *exotic* if it is the system of no finite group. There is one known class of simple exotic 2-fusion systems: The Benson-Solomon systems $\mathcal{F}_{Sol}(q)$.

The Functor

The assignment $(G, S) \mapsto \mathcal{F}_S(G)$ extends to a functor from the category of such pairs to the category of saturated p-fusion system. This functor suggests how to translate many group theoretic notions to analogous notions for fusion systems. In particular we can define *normal subsystems* and *factor systems* of a saturated $\mathcal{F}$. Then we can define *simple* and *quasisimple* systems and the *components* of $\mathcal{F}$.
For $P \leq S$ we can define the *normalizer* $N_{\mathcal{F}}(P)$ and *centralizer* $C_{\mathcal{F}}(P)$ of P in $\mathcal{F}$. P is *fully normalized* if $|N_S(P)| \geq |N_S(Q)|$ for each conjugate $Q = P\phi$ of P. If P is fully normalized, its normalizer and centralizer are saturated. Thus – specializing to the case $p = 2$ – we can consider the set $\mathfrak{C}(\mathcal{F})$ of components of centralizers of (fully centralized) involutions in $\mathcal{F}$. We say $\mathcal{F}$ is of *component type* if $\mathfrak{C}(\mathcal{F})$ is nonempty. Our theorem will determine a large subclass of the class of simple 2-fusion systems of component type.

An inductive hypothesis

We operate under an inductive hypothesis: each member of $\mathfrak{C}(\mathcal{F})$ is in the class $\tilde{\mathcal{K}}$ of "known" quasisimple systems. Here a quasisimple system is known if its simple factor system is known. The known simple systems are the 2-fusion systems of the known simple groups which are not Goldschmidt, together with the exotic Benson-Solomn systems.

The Dichotomy Theorem

In the original proof of CFSG the simple groups of component type were thought of as the groups of "odd characteristic", as almost all groups of Lie type over fields of odd characteristic are of component type; and the groups of "even characteristic" were the groups of characteristic 2-type, which include the groups of Lie type over fields of even characteristic. The Dichotomy Theorem for groups says all large simple groups are either of component type or of characteristic 2-type. There is a similar Dichotomy Theorem for fusion system.

However there are large classes of nearly simple groups of component type which are better thought of as groups of even characteristic. For one thing, the analysis of groups of component type involving such centralizers is not pleasent. Thus we want to structure our treatment of fusion systems of component type so as to include only a large natural subclass of the systems of component type, chosen in part to avoid centralizers causing the unpleasentness.

Toward that end, we partition $\tilde{\mathcal{K}}$ into two subclasses $\mathcal{K}_{sub}$ (*subintrinsic*) and $\mathcal{K}_{non}$ (*not subintrinsic*). A member $\mathcal{C}$ of $\tilde{\mathcal{K}}$ is in $\mathcal{K}_{sub}$ if there exists $\mathcal{E} \in \mathfrak{C}(\mathcal{C})$ such that $\mathcal{E}$ is a component of $C_{\mathcal{C}}(t)$ for some involution $t \in Z(\mathcal{E})$. Then $\mathcal{K}_{non} = \tilde{\mathcal{K}} - \mathcal{K}_{sub}$. For example if $\mathcal{C}$ is Benson-Solomon then $\mathcal{C}$ is subintrinsic, as for t an involution in $\mathcal{C}$, $C_{\mathcal{C}}(t)$ is the 2-fusion system of $Spin_7(q)$.

Define $\mathcal{F}$ to be of *subintrinsic component type* if some member of $\mathfrak{C}(\mathcal{F})$ is in $\mathcal{K}_{sub}$. Define $\mathcal{F}$ to be of *J-component type* if $\mathfrak{C}(\mathcal{F})$ is contained in $\mathcal{K}_{non}$ and $\mathfrak{C}_J(\mathcal{F})$ is nonempty. Here $\mathcal{C}$ is in $\mathfrak{C}_J(\mathcal{F})$ if $\mathcal{C}$ is a component of $C_{\mathcal{F}}(t)$ for some involution t with $m_2(C_S(t)) = m_2(S)$.

Define a saturated 2-fusion system $\mathcal{F}$ to be *odd* if, first, each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$, and, second, $\mathcal{F}$ is of subintrinsic component type or J-component type. This brings us to:

The Main Problem

The Main Problem

Prove each odd simple 2-fusion system is known.

There is an "ordering" on $\mathfrak{C}(\mathcal{F})$. When $\mathcal{F}$ is of subintrinsic component type, we focus on those members $\mathcal{C}$ of $\mathfrak{C}(\mathcal{F})$ that are subintrinsic in $\mathfrak{C}(\mathcal{F})$ and maximal in $\mathfrak{C}(\mathcal{F})$. Here $\mathcal{C}$ is *subintrinsic* in $\mathfrak{C}(\mathcal{F})$ if there exists $\mathcal{E} \in \mathfrak{C}(\mathcal{C})$ and an involution $z \in Z(\mathcal{E})$ such that $\mathcal{E}$ is a component of $C_{\mathcal{F}}(z)$.

The Subintrinsic Centralizer Problem

Given $\mathcal{C}$ in $\mathcal{K}_{sub}$, determine the odd simple 2-fusion systems $\mathcal{F}$ such that $\mathcal{C}$ is subintrinsic in $\mathfrak{C}(\mathcal{F})$ and maximal in $\mathfrak{C}(\mathcal{F})$.

The J Centralizer Problem

Given $\mathcal{C}$ in $\mathcal{K}_{non}$, determine the simple 2-fusion systems $\mathcal{F}$ of J-component type such that $\mathcal{C}$ is J-maximal in $\mathfrak{C}_J(\mathcal{F})$.

The centralizer problems

I call these "centralizer problems" because one proceeds by showing that for an involution t with $\mathcal{C}$ a component of $C_{\mathcal{F}}(t)$, that centralizer is dominated by $\mathcal{C}$, and in particular resembles a centralizer in a known simple system $\tilde{\mathcal{F}}$. Then one shows $\mathcal{F} \cong \tilde{\mathcal{F}}$.

We now consider the various centralizer problems, and see in particular for which $\mathcal{C} \in \tilde{\mathcal{K}}$, the centralizer problem for $\mathcal{C}$ remains open.

$\mathcal{C}$ Benson-Solomon systems

Benson-Solomon systems are subintrinsic. Henke and Lynd have solved the centralizer problem for Benson-Solomon systems; no examples occur.

In the remaining cases, $\mathcal{C}$ is the 2-fusion system of some known quasisimple group K that is not (a covering of) a Goldschmidt group. We consider the various cases

K of Lie type over a field of odd order q

Note roughly speaking the 2-fusion systems of $X(q_1)$ and $X(q_2)$ are isomorphic if the 2-shares of $q_1^2 - 1$ and $q_2^2 - 1$ are the same. I'll pick one representative.

The Ree groups and $L_2(q)$, $q \equiv \pm 3 \mod 8$ are Goldschmidt.

$\mathcal{C}$ is in $\mathcal{K}_{non}$ if K is $L_2(q)$, $q \equiv \pm 1 \mod 8$, $L_3^\epsilon(3)$, $G_2(3)$, $PSp_4(3)$, or $L_4^\epsilon(3)$. The cases $L_3(3)$, $G_2(3)$, $L_4^\epsilon(3)$ remain open.

In the remaining cases, $\mathcal{C}$ is in $\mathcal{K}_{sub}$. The cases $\Omega_6^-(3)$, $\Omega_7(3)$, and $\Omega_8^-(3)$ remain open.

Alternating groups

Suppose K is $\hat{A}_n$, with $n \geq 8$. Then $Z(K)$ is of even order so $\mathcal{C}$ is intrinsic. There are no open cases. If $n = 8$ or 10 we get the examples Mc and Ly.

Suppose next that K is A_n, with $n \geq 8$. Then $\mathcal{C} \in \mathcal{K}_{non}$. There are no open cases. A_{n+4} is an example, and S_{n+2} and $Aut(HS)$ (for $n = 8$) are near examples, failing only because $\mathcal{F} \neq O^2(\mathcal{F})$.

K simple of Lie type of even characteristic

Here $\mathcal{C}$ is in $\mathcal{K}_{non}$. The only open cases are the Tits group and $F_4(2)$.

$Z(K)$ of even order, $K/Z(K)$ sporadic or of Lie type of characteristic 2.

In this case $\mathcal{C}$ is intrinsic. There are only a small finite number of cases. The case $\hat{S}p_6(2)$ and the case $K/Z(K) \cong L_3(4)$ with $Z(K)$ of exponent 4 have been treated with examples Co_3 and O'N. The remaining cases are open.

K sporadic

In this case $\mathcal{C}$ is subinstinsic if K is Co_3, Mc, He, Ly, O'N, F_{22}, F_{23}, F_{24}, F_5, F_2, or F_1. The cases F_{22}, F_{23}, F_2 are open.

In the remaining cases $\mathcal{C} \in \mathcal{K}_{non}$. The cases M_{12}, M_{23}, HS, Ru are open.

There is strong partial information when K is $L_3(3)$, $G_2(3)$, the Tits group, $F_4(2)$, F_{22}, F_2, M_{12}, M_{22}, HS, Ru.