Rigid automorphisms of linking systems

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Groups and Geometries
BIRS

Prelude: Automorphisms centralizing a Sylow p-subgroup

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Theorem (Glauberman, Guralnick, L., Navarro, 2019)

Gross's theorem true without assumption that $O_p(G) = 1$.

→ Uses Z_p^*-theorem, hence CFSG.

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- → (BLO 2003) $|\mathcal{L}_S^c(G)|_p^{\wedge} \simeq BG_p^{\wedge}$ and $\mathrm{Out}(\mathcal{L}_S^c(G)) \cong \mathrm{Out}(BG_p^{\wedge})$

The orbit category and cohomological obstructions to linking systems

- centric orbit category: $\mathcal{O} = \mathcal{O}(\mathcal{F}^c)$
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Theorem (Chermak (2013), Oliver (2013), Glauberman-L. (2016))

$\lim^k_{\mathcal{O}} \mathcal{Z} = 0$ *for all* $k \geq 1$ *if* p *odd and for all* $k \geq 2$ *if* $p = 2$*.*

Automorphism groups of fusion and centric linking systems

- $\mathrm{Aut}_{\mathcal{L}}(S)$ analogous to $N_G(S)$.
- $\mathrm{Aut}(\mathcal{L})$ analogous to $N_{\mathrm{Aut}(G)}(S)$.
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- Similar argument for p odd gives simpler proof of (*).
- More generally, this holds for $\mathcal{L}$ any linking locality/proper locality.

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- Depends on the Z_p^*-theorem
- Reinterprets Glauberman's work on Schreier conjecture (1966)
- Consequences for the definition of a "tame fusion system" (Andersen-Oliver-Ventura)

Centralizers of subgroups and subsystems

Centralizer $C_{\mathcal{F}}(X)$ of a subgroup $X \leq S$

- objects: $Q \leq C_S(X)$;
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- ✔ (Aschbacher, Henke, Semeraro) if $\mathcal{E}$ is normal in $\mathcal{F}$,
- ✔ (Aschbacher) if $\mathcal{E}$ is a component of $\mathcal{F}$,
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? Applications to combinatorially describing $[BH_p^\wedge, BG_p^\wedge]$???

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Problem

How to tell whether a subgroup $X \leq C_S(T)$ "acts uniquely" on $\mathcal{L}_{\mathcal{E}}$, respecting $\mathcal{E} \hookrightarrow \mathcal{F}$?

$$\begin{array}{ccccccccc} & & & & & & X & & \\ & & & & & & \downarrow{\scriptstyle 1} & & \\ 1 & \longrightarrow & \widehat{Z}^1(\mathcal{O}(\mathcal{E}^c), \mathcal{Z}_{\mathcal{E}}) & \longrightarrow & \mathrm{Aut}(\mathcal{L}_{\mathcal{E}}) & \longrightarrow & \mathrm{Aut}(\mathcal{E}) & \longrightarrow & 1 \end{array}$$

Definition

A section is a family σ of extensions $\sigma(\varphi)\colon XP \to XQ$, for each $P \xrightarrow{\varphi} Q$ in $\mathrm{Mor}(\mathcal{E}^c)$ satisfying the following conditions

- $[X, \sigma(\varphi)] \leq Z(P^{\varphi})$ for each φ,
- $\sigma(\varphi \circ c_t) = \sigma(\varphi) \circ c_t$ for each φ and each $t \in T$.

Write $\Gamma(X, \mathcal{E})$ for the collection of sections.

Rigid actions

Definition

A rigid action of X on $\mathcal{L}_{\mathcal{E}}$ (respecting $\mathcal{E} \hookrightarrow \mathcal{F}$) is a group homomorphism

$$\rho \colon X \to \widehat{Z}^1(\mathcal{O}(\mathcal{E}^c), \mathcal{Z}_{\mathcal{E}})$$

for which there is a section $\sigma \in \Gamma(X, \mathcal{E})$ such that

$$\rho(x)([\varphi]) = [x, \sigma(\varphi)]^{\varphi^{-1}}.$$

for each morphism $[\varphi]$ in the orbit category $\mathcal{O}(\mathcal{E}^c)$.

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and, on morphisms by sending $P \xrightarrow{[\varphi]} Q$ to the composite

$$K_{X,\mathcal{F}}(Q) \xrightarrow{\mathrm{res}} K_{X,\mathcal{F}}(P^{\varphi}) \xrightarrow{c_{\sigma(\varphi)}^{-1}} K_{X,\mathcal{F}}(P).$$

Obstructions to rigid actions on linking systems

Theorem (L.)

Let $\mathcal{E} \leq \mathcal{F}$ be a subsystem on $T \leq S$, and fix $X \leq C_S(T)$. Assume that $\Gamma(X, \mathcal{E})$ is nonempty. Then

(1) *there is a class $[\tau] \in \lim^2_{\mathcal{O}(\mathcal{E}^c)} K_{X,\mathcal{F}}$ such that X has a rigid action on $\mathcal{L}_{\mathcal{E}}$ if and only if $[\tau] = 0$;*

(2) *the group $\widehat{Z}^1(\mathcal{O}(\mathcal{E}^c), K_{X,\mathcal{F}})$ acts freely and transitively on the set of rigid actions when that set is nonempty.*

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Example

For $\mathcal{F} = \mathcal{F}_2(A_6 \wr X)$ with $X = \langle x \rangle$ of order 2 and $\mathcal{E} = \mathcal{F}_2(\Delta(A_6))$, one has

$$\lim^1_{\mathcal{O}(\mathcal{E}^c)} K_{X,\mathcal{F}} \cong \lim^1_{\mathcal{O}(\mathcal{E}^c)} \Omega_1 \mathcal{Z}_{\mathcal{E}} \cong C_2.$$